ON THE UNIVERSAL COEFFICIENT THEOREM OF EILENBERG AND MACLANE

BY W. S. MASSEY[1]

1. Introduction

The theorem referred to in the title was proved by Eilenberg and MacLane in [1] in 1942. Perhaps its most useful consequence is the following exact sequence, which shows that the q-dimensional singular cohomology group of a space X with coefficient group G, $H^q(X, G)$, is completely determined by the integral singular homology groups $H_q(X)$ and $H_{q-1}(X)$:

$$(1) \qquad 0 \to \operatorname{Ext}[H_{q-1}(X), G] \xrightarrow{\Psi} H^q(X, G) \xrightarrow{\Omega} \operatorname{Hom}[H_q(X), G] \to 0.$$

In this exact sequence, the definition of the homomorphism Ω and the proof that Ω is a homomorphism onto are quite straightforward. However, the definition of the homomorphism labeled "Ψ" and the rest of the proof of exactness are more involved.

The purpose of this paper is to give a more direct and straightforward treatment of these results. In the usual proof of this theorem, the group $\operatorname{Ext}(A, B)$ is defined by means of factors sets, or by means of projective resolutions (see [1] or [3]). In our treatment, we use the direct definition of $\operatorname{Ext}(A, B)$ due originally to R. Baer: An element of the group $\operatorname{Ext}(A, B)$ is an equivalence class of group extensions, and a direct definition of the addition of equivalence classes is used. Rather than define Ψ, we consider its inverse, which is always defined. A direct definition of Ψ^{-1} is given by means of the so-called "algebraic mapping cylinder" of a chain mapping. The mapping cylinder is useful for many other purposes and should be considered a standard tool in algebraic topology.

We follow what is now standard practice in proving a purely algebraic theorem which implies the existence of the exact sequence (1) as a special case. The statement and proof of this theorem is given in §5. Sections 2 and 3 contain preliminary material, most of which is rather standard, or at most a slight variation of standard material. Section 4 contains the definition of Ψ^{-1}, which we denote by Ξ.

2. Basic Definitions and Notation

In this paper, all groups considered are additive abelian groups. The following standard concepts involving such groups will be used throughout:

(1) $\operatorname{Hom}(A, B)$: the group of all homomorphisms of A into B.

(2) $\operatorname{Ext}(A, B)$: the group of all equivalence classes of extensions over A with kernel B. An extension over A with kernel B is a triple (X, φ, ψ), where X is an

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abelian group, and $\varphi:X \rightarrow A$ and $\psi:B \rightarrow X$ are homomorphisms such that the following sequence is exact:

$$0 \rightarrow B \xrightarrow{\psi} X \xrightarrow{\varphi} A \rightarrow 0.$$

Two extensions (X, φ, ψ) and (X', φ', ψ') are equivalent if there exists a homomorphism $f:X \rightarrow X'$ such that the following diagram is commutative:

$$\begin{array}{ccccc} & & X & & \\ & \nearrow & \Big| & \searrow & \\ B & & \Big| & & A \\ & \searrow & \downarrow & \nearrow & \\ & & X' & & \end{array}$$

We assume that the reader is familiar with the definition of addition of equivalence classes of extensions defined by R. Baer (see [3], chap. XIV, §1).

(3) Graded group: a group A which has been assigned a definite direct sum decomposition

$$A = \sum_n A_n, \qquad -\infty < n < +\infty.$$

(4) Homogeneous homorphism (of graded groups): a homomorphism $f:A \rightarrow B$ such that $F(A_n) \subset B_{n+q}$ for all integers n. The integer q is called the *degree* of the homogeneous homomorphism f.

(5) The group $\hom_q(A, B)$, where A and B are *graded* groups: the set of all homogeneous homomorphisms $A \rightarrow B$ of degree q. Obviously, $\hom_q(A, B)$ is a subgroup of $\mathrm{Hom}(A, B)$. Also, we let $\hom(A, B)$ denote the direct sum of the groups $\hom_q(A, B)$ for all integers q. Thus $\hom(A, B)$ is also a subgroup of $\mathrm{Hom}(A, B)$, and it is a graded group.

(6) Graded extensions: let (X, φ, ψ) be an extension over A with kernel B, where A and B are graded. We say that (X, φ, ψ) is a *graded extension* if X is a graded group and φ and ψ are homogeneous homomorphisms. The *degree* of (X, φ, ψ) is given by

$$\text{degree } (X, \varphi, \psi) = -(\text{degree } \varphi + \text{degree } \psi).$$

Equivalence of graded extensions is defined in the obvious way. Note that equivalent graded extensions have the same degree. We let $\mathrm{ext}_n(A, B)$ denote the set of all graded extensions over A with kernel B of degree n. Obviously, $\mathrm{ext}_n(A, B)$ may be identified with the cartesian product $\prod_q \mathrm{Ext}(A_q, B_{q+n})$. By means of this identification, we may turn $\mathrm{ext}_n(A, B)$ into an abelian group. We set

$$\mathrm{ext}(A, B) = \sum_n \mathrm{ext}_n(A, B).$$

Thus $\mathrm{ext}(A, B)$ is a graded group.

(7) Graded differential groups (or chain complexes): a *graded differential group* is a pair (A, d) where A is a graded group, and $d:A \rightarrow A$ is an endomorphism which is homogeneous of degree -1 and is such that $d^2 = 0$. The homology of

(A, d), denoted by $H(A)$ or $\sum_n H_n(A)$ is defined in the usual way. If (A, d) and (A', d') are graded differential groups, a homomorphism $f:A \to A'$ is called *allowable* if (a) f is homogeneous and (b) $f \circ d = (-1)^q d' \circ f$, where q is the degree of f. As usual, an allowable homomorphism f induces a homomorphism $f_*:H(A) \to H(A')$ which is also homogeneous. Two allowable homomorphisms of degree q,

$$f, g: A \to A',$$

are *homotopic* if there exists a homomorphism $D:A \to A'$ of degree $q + 1$ such that

$$f - g = d' \circ D + (-1)^q D \circ d.$$

As usual, if f and g are homotopic, then $f_* = g_*:H(A) \to H(A')$. (These last two definitions are borrowed from Eilenberg and MacLane, [4], §2.)

3. The Algebraic Mapping Cylinder

Let (A, d) and (A', d') be graded differential groups, and let $f:A \to A'$ be an allowable homorphism of degree q. The *algebraic mapping cylinder of f* is a graded differential group $(M(f), d_f)$ defined as follows: $M(f) = A \times A'$, with the graded structure defined by

$$M(f)_n = A_{n-q-1} \times A'_n,$$

and d_f is defined by

$$d_f(a, a') = ((-1)^{q-1} d(a), d'(a') + f(a))$$

for any ordered pair $(a, a') \in A \times A'$.

It is readily verified that d_f is of degree -1, and $d_f^2 = 0$.

Let $i:A' \to M(f)$ and $j:M(f) \to A$ be defined by $i(a') = (0, a')$ and $j(a, a') = a$. It is easily seen that i and j are allowable, and that the following sequence is exact:

$$0 \to A' \xrightarrow{i} M(f) \xrightarrow{j} A \to 0.$$

This exact sequence gives rise to the following exact sequence of homology groups in the usual way:

$$\cdots \to H_n(A') \xrightarrow{i_*} H_n(M(f)) \xrightarrow{j_*} H_{n-q-1}(A) \xrightarrow{\partial} H_{n-1}(A') \to \cdots.$$

It is readily verified that the homomorphism $\partial:H(A) \to H(A')$ (of degree q) in this exact sequence is the same as the homomorphism $f_*:H(A) \to H(A')$ induced by the given map f.

This is the main purpose of the introduction of the algebraic mapping cylinder: to fit the homomorphisms $H_{n-q}(A) \to H_n(A')$ induced by f into an exact sequence *in a natural way*.[2]

[2] The facts about the mapping cylinder which we have just reviewed have been well known for several years; see [2], p. 159, exercise D, for example.

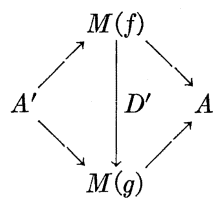
Let f, $g: A \to A'$ be allowable homomorphisms of degree q which are homotopic. Then each homotopy $D: A \to A'$ such that

$$f - g = d' \circ D + (-1)^q D \circ d$$

defines a homomorphism

$$D': M(f) \to M(g)$$

by the formula $D'(a, a') = (a, a' + Da)$. It is readily verified that D' is homogeneous of degree 0, and that $D' \circ d_f = d_g \circ D'$, i.e., D' is an allowable homomorphism $(M(f), d_f) \to (M(g), d_g)$. Moreover, the following diagram is commutative:



It follows that the exact sequences of the mapping cylinders $M(f)$ and $M(g)$ are isomorphic (use the five lemma!). This statement may be formulated as follows:

LEMMA 1. *The exact sequence of the mapping cylinder $M(f)$ depends only on the homotopy class of f.*

4. The Homology of hom (A, B), Where A and B Are Graded Differential Groups

Let (A, d) and (B, d') be graded differential groups. Define a homomorphism δ:hom$(A, B) \to$ hom(A, B) by

$$\delta(f) = f \circ d - (-1)^n d' \circ f$$

for any element of $f \in \hom_n(A, B)$ (i.e., any homogeneous homomorphism f of degree n). It is readily verified that δ is homogeneous of degree -1, and that $\delta^2 = 0$. Therefore (hom(A, B), δ) is a graded, differential group.

An element $f \in \hom_n(A, B)$ is a cycle (i.e., $\delta(f) = 0$) if and only if f is allowable according to the definition of §2. Moreover, two cycles f, $g \in \hom_n(A, B)$ are homologous if and only if there exists an element of $D \in \hom_{n+1}(A, B)$ such that

$$\delta(D) = f - g.$$

This is precisely the condition that f and g should be homotopic. It follows that assigning to each cycle $f \in \hom_n(A, B)$ its induced homomorphism $f_*: H(A) \to H(B)$ defines a map $\Omega: H(\hom(A, B)) \to \hom(H(A), H(B))$. Ω is a homogeneous homomorphism of degree 0. Moreover, Ω is "natural."

LEMMA 2. *If A is free, then Ω is a homomorphism onto.*

Although this is a well-known result, we will outline the proof for the sake of completeness.

Since A is free, the subgroups $\mathcal{Z}(A)$ (the cycles of A) and $\mathcal{B}(A)$ (the bounding cycles of A) are also free. Moreover, $\mathcal{Z}(A)$ is a direct summand of A. For each integer q choose a direct sum decomposition

$$A_q = \mathcal{Z}_q(A) + Q_q ,$$

and let $Q = \sum_q Q_q$.

Now let $h \in \hom_n(H(A), H(B))$. Since $\mathcal{Z}(A)$ is free, one can find a homogeneous homomorphism $g : \mathcal{Z}(A) \to \mathcal{Z}(B)$ of degree n such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{Z}(A) & \xrightarrow{g} & \mathcal{Z}(B) \\ \downarrow & & \downarrow \\ H(A) & \xrightarrow{h} & H(B) \end{array}$$

(The vertical arrows denote the homomorphisms which assign to each cycle its homology class.) Then commutativity implies that g maps $\mathcal{B}(A)$ into $\mathcal{B}(B)$. Hence one can find a homomorphism $f: Q \to B$ of degree n such that $d'f(x) = (-1)^n g\, d(x)$ for any $x \in Q$, i.e., so that the following diagram is commutative (use the fact that Q is free):

$$\begin{array}{ccc} Q & \xrightarrow{(-1)^n f} & B \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d'} \\ \mathcal{Z}(A) & \longrightarrow & \mathcal{Z}(B) \end{array}$$

Now define $F: A \to B$ by

$$F(x) = \begin{cases} g(x) & \text{for } x \in \mathcal{Z}(A), \\ f(x) & \text{for } x \in Q. \end{cases}$$

Then F is allowable, and $F_* = h$, as desired.

Next, we will define a homomorphism

$$\Xi : \Omega^{-1}(0) \to \operatorname{ext}(H(A), H(B))$$

as follows. Let φ be an element of $\Omega^{-1}(0)$ of degree n, that is, $\varphi \in H_n(\hom\,(A, B))$ and $\Omega(\varphi) = 0$. Let $f \in \hom_n(A, B)$ be a representative cycle for φ; the condition $\Omega(\varphi) = 0$ means that $f_* = 0$. Consider the exact sequence of the mapping cylinder $M(f)$:

$$\cdots \to H_{q-n}(A) \xrightarrow{f_*} H_q(B) \xrightarrow{i_*} H_q(M(f)) \xrightarrow{j_*} H_{q-n-1}(A) \xrightarrow{f_*} \cdots .$$

Since $f_* = 0$, we have the following exact sequence of length three,

$$0 \to H(B) \xrightarrow{i_*} H(M(f)) \xrightarrow{j_*} H(A) \to 0$$

where i_* and j_* are homogeneous of degrees 0 and $-(n + 1)$ respectively. Thus $(H(M(f)), j_* , i_*)$ is a graded extension over $H(A)$ with kernal $H(B)$ of degree

$n + 1$. It follows from Lemma 1 that if one replaces f by a homologous cycle, one obtains an equivalent graded extension over $H(A)$ with kernel $H(B)$. Therefore this process defines a function

$$\Xi : \Omega^{-1}(0) \to \text{ext}(H(A), H(B))$$

which is homogeneous of degree $+1$.

LEMMA 3. *Ξ is a homomorphism.*

PROOF. Let $\varphi, \psi \in H_n(\text{hom }(A, B))$ be such that $\Omega(\varphi) = \Omega(\psi) = 0$. We will prove that

$$\Xi(\varphi + \psi) = \Xi(\varphi) + \Xi(\psi).$$

Choose representative cycles $f, g \in \text{hom}_n(A, B)$ for φ and ψ respectively. Then $f + g$ is a representative cycle for $\varphi + \psi$. If we consider the mapping cylinders of f, g, and $f + g$, we obtain exact sequences as follows (for all values of q):

$$0 \to H_q(B) \xrightarrow{i_1} H_q(M(f)) \xrightarrow{j_1} H_{q-n-1}(A) \to 0,$$

$$0 \to H_q(B) \xrightarrow{i_2} H_q(M(g)) \xrightarrow{j_2} H_{q-n-1}(A) \to 0,$$

$$0 \to H_q(B) \xrightarrow{i} H_q(M(f + g)) \xrightarrow{j} H_{q-n-1}(A) \to 0.$$

Next, we define a graded extension

$$0 \to H_q(B) \xrightarrow{\alpha} X_q \xrightarrow{\beta} H_{q-n-1}(A) \to 0,$$

which is the Baer sum of the extensions $(H(M(f)), j_1, i_1)$ and $(H(M(g)), j_2, i_2)$, as follows. Let Y_q denote the set of all pairs (x, x') such that $x \in H_q(M(f))$, $x' \in H_q(M(g))$, and $j_1(x) = j_2(x')$. Let K_q denote the subgroup of Y_q consisting of all pairs of the form $(i_1(y), -i_2(y))$ for $y \in H_q(B)$. Define $X_q = Y_q/K_q$ for any integer q. The homomorphism $\alpha : H_q(B) \to X_q$ is defined by $\alpha(y) =$ coset of $(i_1(y), 0)$ mod K_q = coset of $(0, i_2(y))$ mod K_q ; finally, the homomorphism $\beta : X_q \to H_{n+q-1}(A)$ is defined by

$$\beta[(x, x') + K_q] = j_1(x) = j_2(x')$$

for any coset, $(x, x') + K_q$.

In order to prove the lemma, it clearly suffices to define a homomorphism

$$\mu : X_q \to H_q(M(f + g))$$

for each q such that the following diagram is commutative:

$$\begin{array}{ccccc} & & X_q & & \\ & {}^{\alpha}\nearrow & \big\downarrow{\scriptstyle \mu} & \searrow^{\beta} & \\ H_q(B) & & & & H_{q-n-1}(A) \\ & {}_{i}\searrow & & \nearrow_{j} & \\ & & H_q(M(f + g)) & & \end{array}$$

In order to define μ, we will define a homomorphism $\nu: Y_q \to H_q(M(f + g))$, and prove that K_q is contained in the kernel of ν. Then μ will be the homomorphism induced by ν.

The definition of ν is based on the following fact: *If $(x, x') \in Y_q$, then it is possible to choose representative cycles $(a, b) \in M(f)_q$ and $(a', b') \in M(g)_q$ for x and x' respectively such that $a = a'$.* To prove this fact, let (a, b) and (a_0, b_0) be any representative cycles for x and x'. Since (a, b) and (a_0, b_0) are cycles, both a and a_0 are cycles of A. Since $j_1(x) = j_2(x')$, a and a_0 must be homologous cycles of A, i.e., there exists an element $t \in A$ such that

$$a_0 = a + d(t).$$

Hence

$$\begin{aligned}(a_0, b_0) &= (a + d(t), b_0)\\ &= (a, b_0 + (-1)^n g(t)) + (-1)^{n-1} d_g(t, 0).\end{aligned}$$

Therefore if we choose $a' = a$, $b' = b_0 + (-1)^n g(t)$, then (a', b') is a representative cycle for x' having the required property.

In order to define $\nu(x, x')$ for $(x, x') \in Y_q$, choose representative cycles (a, b) and (a, b') for x and x' respectively. Then a computation shows that

$$d_{f+g}(a, b + b') = 0.$$

Hence $(a, b + b')$ is a cycle in $M(f + g)$. Define $\nu(x, x')$ to be its homology class.

To justify this definition, we must show that $\nu(x, x')$ is independent of the choice of the representative cycles (a, b) and (a, b'). This means we must show that if $(a_0, b_0) \sim (a, b)$ in $M(f)$ and $(a_0, b_0') \sim (a, b')$ in $M(g)$, then $(a_0, b_0 + b_0') \sim (a, b + b')$ in $M(f + g)$. This may be shown as follows. Let $(s, t) \in M(f)$ and $(s', t') \in M(g)$ be elements such that

(1) $$d_f(s, t) = (a_0, b_0) - (a, b),$$

(2) $$d_g(s', t') = (a_0, b_0') - (a, b').$$

These equations imply that

$$\begin{aligned}(-1)^{n-1} d(s) &= a_0 - a,\\ (-1)^{n-1} d(s') &= a_0 - a,\end{aligned}$$

from which it follows that $d(s - s') = 0$, i.e., $s - s'$ is a cycle of A. Since the induced homomorphism $g_* = 0$, it follows that there exists an element $r \in B$ such that

$$g(s - s') = d'(r),$$

and hence

(3) $$d_g(s', t') = d_g(s, t' - r).$$

Therefore if we set $t'' = t' - r$, (2) and (3) give

$$d_g(s, t'') = (a_0, b_0') - (a, b'). \tag{4}$$

An easy calculation using (1) and (4) now shows that

$$d_{f+g}(s, t + t'') = (a_0, b_0 + b_0') - (a, b + b')$$

as desired.

Thus the definition of ν is justified. It is obvious that ν is a homomorphism. The proof that K_q is contained in the kernel of ν is straightforward, and left to the reader. Likewise, the proof of the commutativity relations $\mu \circ \alpha = i$ and $j \circ \mu = \beta$ is also left to the reader.

This completes the proof of Lemma 3. It should be noted that the homomorphism Ξ is natural.

5. Statement and Proof of the Main Theorem

The purpose of this section is to prove the following:

THEOREM. *Let A and B be graded differential groups. If A is free, then the homomorphism Ξ is an isomorphism of the kernel of Ω onto* $\text{ext}(H(A), H(B))$.

In order to prove that the kernel of Ξ is zero, we need the following two lemmas:

LEMMA 4. *Let (A, d) and (A', d') be graded differential groups with A and A' free. If $\varphi: A' \to A'$ is an allowable homomorphism such that $\varphi_*: H(A) \to H(A')$ is an isomorphism onto, then there exists an allowable homomorphism $\psi: A' \to A$ such that $\psi \circ \varphi$ and $\varphi \circ \psi$ are homotopic to the identity maps of A and A' respectively.*

For the proof, see [2], p. 154, Theorem 13.3.

LEMMA 5. *Let (A, d), (A', d'), and (A'', d'') be graded differential groups and let $f_0, f_1: A \to A'$, $g_0, g_1: A' \to A''$ be allowable homomorphisms. If f_0 is homotopic to f_1 and g_0 is homotopic to g_1, then $g_0 \circ f_0$ is homotopic to $g_1 \circ f_1$.*

The proof is routine and is left to the reader.

Now let $\varphi \in H_n(\text{hom}(A, B))$ and assume that $\Omega(\varphi) = 0$, $\Xi(\varphi) = 0$; we will prove that $\varphi = 0$. Choose a representative cycle $f \in \text{hom}_n(A, B)$ for φ; since $\Omega(\varphi) = 0$, $f_* = 0$. Since $\Xi(\varphi) = 0$, it follows that the extension

$$0 \to H_q(B) \xrightarrow{i_*} H_q(M(f)) \xrightarrow{j_*} H_{q-n-1}(A) \to 0$$

is a split extension. Hence there exists a homogeneous homomorphism $\eta: H(A) \to H(M(f))$ of degree $n + 1$ such that $j_* \circ \eta$ is the identity map of $H(A)$. Since A is free, it follows from Lemma 2 that there exists an allowable homomorphism $r: A \to M_f$ such that $r_* = \eta$. Let $r_1: A \to A$ and $r_2: A \to B$ be the unique homogeneous homomorphisms such that

$$r(a) = (r_1 a, r_2 a)$$

for any $a \in A$. The fact that r is allowable implies that

$$(1) \qquad r_1 \circ d = d \circ r_1 ,$$

$$(2) \qquad (-1)^{n+1} r_2 \circ d = d' \circ r_2 + f \circ r_1 .$$

From equation (1), it follows that r_1 is allowable. Since $j \circ r = r_1$, it follows that r_{1*} is the identity map of $H(A)$. Equation (2) implies that $f \circ r_1$ is homotopic to 0. By Lemma 4, there exists an allowable homomorphism $g:A \to A$ such that $r_1 \circ g$ is homotopic to the identity map of A. By use of Lemma 5, we obtain the following sequence of homotopies:

$$f \cong f \circ (r_1 \circ g) = (f \circ r_1) \circ g \cong 0 \circ g = 0.$$

Therefore $f \cong 0$, i.e., f is a bounding cycle in $\hom_n(A, B)$, and hence $\varphi = 0$ as desired.

Next, we will prove that Ξ is onto. Let (X, φ, ψ) be a graded extension of degree $n + 1$ over $H(A)$ with kernel $H(B)$. This means we have an exact sequence

$$0 \to H_q(B) \xrightarrow{\psi} X_q \xrightarrow{\varphi} H_{q-n-1}(A) \to 0$$

for each integer q. We must exhibit an allowable homomorphism of degree n,

$$f:A \to B$$

such that $f_* = 0$ and such that the extension determined by the mapping cylinder of f,

$$0 \to H_q(B) \xrightarrow{i_*} H_q(M(f)) \xrightarrow{j_*} H_{q-n-1}(A) \to 0$$

is equivalent to the given extension. For this purpose, we consider the following exact sequences associated with the graded differential groups A and B:

$$0 \to \mathcal{B}_{q-n-1}(A) \xrightarrow{\alpha} \mathcal{Z}_{q-n-1}(A) \xrightarrow{\beta} H_{q-n-1}(A) \to 0,$$

$$0 \to \mathcal{B}_q(B) \xrightarrow{\alpha'} \mathcal{Z}_q(B) \xrightarrow{\beta'} H_q(B) \to 0,$$

where α and α' are the inclusion homomorphisms of the group of bounding cycles in the group of cycles, and β and β' are the maps which assign to each cycle its homology class. Since $\mathcal{Z}_{q-n-1}(A)$ is free, we can find a homomorphism $\sigma:\mathcal{Z}_{q-n-1}(A) \to X_q$ such that the following diagram is commutative:

$$\begin{array}{ccc} & & \mathcal{Z}_{q-n-1}(A) \\ & \swarrow^{\sigma} & \downarrow \beta \\ X_q & \xrightarrow{\varphi} & H_{q-n-1}(A) \end{array}$$

Define a homomorphism $\pi:\mathcal{Z}_{q-n-1}(A) \times \mathcal{Z}_q(B) \to X_q$ by

$$\pi(a, b) = \sigma(a) + \psi\beta'(b)$$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{B}_q(B) & \xrightarrow{\ \mu\ } & K_q & \xrightarrow{\ \nu\ } & \mathcal{B}_{q-n-1}(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \alpha} & & \\
0 & \longrightarrow & \mathcal{Z}_q(B) & \xrightarrow{\ \xi\ } & \mathcal{Z}_{q-n-1}(A) \times \mathcal{Z}_q(B) & \xrightarrow{\ \eta\ } & \mathcal{Z}_{q-n-1}(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \beta} & & \\
0 & \longrightarrow & H_q(B) & \xrightarrow{\ \psi\ } & X_q & \xrightarrow{\ \phi\ } & H_{q-n-1}(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

FIGURE 1

for $a \in \mathcal{Z}_{q-n-1}(A)$ and $b \in \mathcal{Z}_q(B)$. This leads to the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{Z}_q(B) & \xrightarrow{\ \xi\ } & \mathcal{Z}_{q-n-1}(A) \times \mathcal{Z}_q(B) & \xrightarrow{\ \eta\ } & \mathcal{Z}_{q-n-1}(A) & \to & 0 \\
 & & \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \beta} & & \\
0 & \to & H_q(B) & \xrightarrow{\ \psi\ } & X_q & \xrightarrow{\ \phi\ } & H_{q-n-1}(A) & \to & 0
\end{array}$$

Both the horizontal sequences are exact. This diagram may now be enlarged to the diagram shown in Figure 1. In this diagram, K_q is the kernel of the homomorphism π and ρ is the inclusion map. The homomorphisms μ and ν are induced by ξ and η respectively. All rows and all columns of this diagram are exact, and commutativity holds around every square. Since A is free, $\mathcal{B}(A)$ is free, and hence we can choose a homomorphism

$$r: \mathcal{B}_{q-n-1}(A) \longrightarrow K_q$$

such that $\nu \circ r$ = identity on $\mathcal{B}_{q-n-1}(A)$. Let $k: \mathcal{B}_{q-n-1}(A) \longrightarrow \mathcal{Z}_{q-n-1}(A)$ and $h: \mathcal{B}_{q-n-1}(A) \longrightarrow \mathcal{Z}_q(B)$ be the unique homomorphisms such that

$$\rho r(x) = (k(x), h(x))$$

for any element $x \in \mathcal{B}_{q-n-1}(A)$. From the commutativity around the upper right-hand square of the diagram, one readily finds that $k = \alpha$; hence $\rho r(x) = (x, h(x))$ for any $x \in \mathcal{B}_{q-n-1}(A)$.

Since $\mathcal{B}(A)$ is free, $\mathcal{Z}(A)$ is a direct summand of A. For each integer q choose a subgroup Q_q of A_q such that we have a direct sum decomposition

$$A_q = \mathcal{Z}_q(A) + Q_q .$$

Note that the differential operator d maps Q_q isomorphically onto $\mathcal{B}_{q-1}(B)$. Define a homogeneous homomorphism

$$f: A \longrightarrow B$$

of degree n by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathcal{Z}(A), \\ (-1)^{n-1}h\, d(x) & \text{if } x \in Q. \end{cases}$$

It is readily verified that f is allowable, and that $f_* = 0$. If $(M(f), d_f)$ denotes the mapping cylinder of f, one finds that

$$\mathcal{Z}(Mf) = \mathcal{Z}(A) \times \mathcal{Z}(B),$$

i.e., an element $(a, b) \in M(f)$ is a cycle if and only if $a \in A$ and $b \in B$ are cycles. Hence in Figure 1 we may re-label the group $\mathcal{Z}_{q-n-1}(A) \times \mathcal{Z}_q(B)$ in the middle of the diagram, and call it $\mathcal{Z}_q(M(f))$.

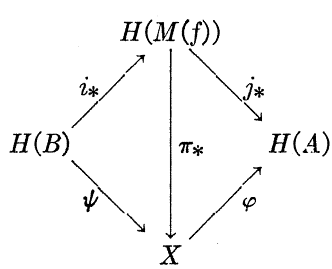
We now make the following assertion: *The subgroup $\mathcal{B}_q(M(f))$ is contained in the kernel of π.* To prove this, it suffices to prove that $\pi\, d_f(a, b) = 0$ for any $(a, b) \in M(f)$. Now

$$\begin{aligned} \pi\, d_f(a, b) &= \pi((-1)^{n-1}\, d(a), d'(b) + f(a)) \\ &= (-1)^{n-1}\sigma\, d(a) + \psi\beta' d'(b) + \psi\beta' f(a). \end{aligned}$$

But $\psi\beta' d'(b) = 0$ because $\beta' \circ d' = 0$. Let $a = x + y$, where $x \in \mathcal{Z}(A)$ and $y \in Q$. Then $d(x) = 0$, $f(x) = 0$, and $f(y) = (-1)^{n-1}h\, d(y)$; hence

$$\begin{aligned} \pi\, d_f(a, b) &= (-1)^{n-1}[\sigma\, d(y) + \psi\beta' h\, d(y)] \\ &= (-1)^{n-1}\pi(dy, hdy) \\ &= (-1)^{n-1}\pi\rho r(dy) = 0 \end{aligned}$$

since $\pi \circ \rho = 0$. Thus the assertion is proved. Therefore $\pi{:}\mathcal{Z}(M(f)) \to X$ induces a homomorphism $\pi_*{:}H(M(f)) \to X$, and it is obvious that the following diagram is commutative:



Thus the two extensions are equivalent, as was to be proved.

6. The Universal Coefficient Theorem for Cohomology

In this section, we show how the theorem of the last section implies the universal coefficient theorem stated in the introduction.

Consider the case where A and B are graded, differential groups, but B has only elements of degree 0, i.e., $B_q = 0$, for $q \neq 0$. This automatically implies that the differential operator is 0 on B. In this case, $H(B) = B$, and

$$\mathrm{hom}_m(H(A), H(B)) = \mathrm{Hom}(H_{-m}(A), B),$$

$$\mathrm{ext}_n(H(A), H(B)) = \mathrm{Ext}(H_{-n}(A), B).$$

In case $(A, d) = (C(X), \partial)$, the group of singular chains of the space X, then $\mathrm{hom}_n(C(X), B) = C^{-n}(X, B)$, (the group of singular cochains of X of dimension $-n$ with coefficients in B) and $H_n(\mathrm{hom}(C(X), B)) = H^{-n}(X, B)$. Making these substitutions in the main theorem, one obtains the result mentioned in the introduction. In this case, one has the following further result which is well known:

LEMMA 6. *If A is free and B has only elements of degree 0, then the kernel of* $\Omega: H(\mathrm{hom}(A, B)) \rightarrow \mathrm{hom}(H(A), B)$ *is a direct summand of* $H(\mathrm{hom}(A, B))$.

The reader may either work out a proof for himself or consult [3], p. 66, Theorem 6.2a.

BROWN UNIVERSITY

BIBLIOGRAPHY

[1] S. EILENBERG AND S. MACLANE, *Group Extensions and Homology*, Ann. of Math., 43 (1942), 757–831.

[2] S. EILENBERG AND N. STEENROD, Foundations of Algebraic Topology. Princeton University Press, 1952.

[3] S. EILENBERG AND H. CARTAN, Homological Algebra. Princeton University Press, 1956.

[4] S. EILENBERG AND S. MACLANE, *On the Groups $H(\Pi, n)$*, III, Ann. of Math., 60 (1954), 513–557.