A NOTE ON NON-STABLE COHOMOLOGY OPERATIONS

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1. Introduction

Let θ be a primary cohomology operation of type $(\Pi, n; G, q)$. Let X and Y be spaces and f a continuous map $Y \to X$. Consider classes $u \in H^n(X; \Pi)$ such that

$$f^*(u) = 0 \quad \text{and} \quad \theta(u) = 0.$$

For such elements Steenrod [2] has defined a *functional cohomology operation*, θ_f, such that $\theta_f(u)$ is a subset of $H^{q-1}(Y; G)$. If θ is an additive cohomology operation (i.e., a homomorphism), this subset is in fact a coset of the subgroup

$$L(\theta, f) = f^*(H^{q-1}(X; G)) + {}^1\theta(H^{n-1}(Y; \Pi)),$$

where ${}^1\theta$ denotes the *suspension* of θ; that is, an operation of type $(\Pi, n - 1; G, q - 1)$. Thus, if θ is additive, we may regard $\theta_f(u)$ as an element of the quotient group $H^{q-1}(Y; G)/L(\theta, f)$.

When θ is an arbitrary operation, the nature of the subset $\theta_f(u)$ is not immediately clear. In this note we show that for *all* operations θ, the subset $\theta_f(u)$ is in fact a coset of the subgroup $L(\theta, f)$ defined above. Therefore, in all cases we may regard $\theta_f(u)$ as an element of the quotient group $H^{q-1}(Y; G)/L(\theta, f)$.

We give two proofs of this result; these are based upon two different ways of defining functional operations, and hence employ quite different techniques. We include both proofs, since the techniques involved may be useful in further studies of non-stable cohomology operations.

2. The first proof

Let us recall the definition of functional cohomology operations given by Steenrod (see [2]). Let θ again be a cohomology operation of type $(\Pi, n; G, q)$. Let X, Y be spaces and f a map $Y \to X$ (f may be assumed to be an inclusion, using the mapping cylinder technique). Denote by ${}^1\theta$ the suspension of θ, and by g the inclusion map $X \subset (X, Y)$. Then the following diagram is commutative:

$$\begin{array}{ccccccccc}
 & & H^{n-1}(Y; \Pi) & \xrightarrow{\delta} & H^n(X, Y; \Pi) & \xrightarrow{g^*} & H^n(X; \Pi) & \xrightarrow{f^*} & H^n(Y; \Pi) \\
 & & \downarrow {}^1\theta & & \downarrow \theta & & \downarrow \theta & & \\
H^{q-1}(X; G) & \xrightarrow{f^*} & H^{q-1}(Y; G) & \xrightarrow{\delta} & H^q(X, Y; G) & \xrightarrow{g^*} & H^q(X; G) & &
\end{array}$$

Let u be an element of $H^n(X; \Pi)$ such that $f^*(u) = 0$ and $\theta(u) = 0$. Then, by exactness, there is an element $v \in H^n(X, Y; \Pi)$ such that $g^*(v) = u$. Since $g^*\theta(v) =$

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$\theta g^*(v) = \theta(u) = 0$, again by exactness there is an element $x \in H^{q-1}(Y; G)$ such that $\delta(x) = \theta(v)$. Consider all such pairs of the form (x, v) where $\delta(x) = \theta(v)$ and $g^*(v) = u$, for a fixed u. The totality of the elements x which occur in such pairs we denote by $\theta_f(u)$. Consider two pairs (x, v), (x', v') satisfying the above conditions. Since $g^*(v' - v) = u - u = 0$, there is an element $w \in H^{n-1}(Y; \Pi)$ such that $v' = v + \delta w$. If θ is *additive*, then,

$$\theta(v') = \theta(v) + \theta(\delta w) = \theta(v) + \delta\, {}^1\theta(w).$$

Thus,

$$\delta(x' - x) = \theta(v') - \theta(v) = \delta\, {}^1\theta(w).$$

That is, $\delta(x' - x - {}^1\theta(w)) = 0$. Therefore, because of exactness, there is a class $y \in H^{q-1}(X; G)$ such that

$$x' - x = {}^1\theta(w) + f^*(y).$$

Therefore, $x' - x$ belongs to $L(\theta, f)$, and it is easily seen that $\theta_f(u)$ is a coset of $L(\theta, f)$.

In general θ is not additive. In the above characterization of $\theta_f(u)$, we only used the additivity of θ to assert that $\theta(v + \delta w) = \theta(v) + \theta(\delta w)$. However, by the following lemma, this is true for *any* operation θ; hence, the subset $\theta_f(u)$ will continue to be a coset of the subgroup $L(\theta, f)$ as was asserted.

LEMMA. *Let θ be a cohomology operation of type* $(\Pi, n; G, q)$. *Let X be a space and $Y \subset X$ a subspace. Let $v \in H^n(X, Y; \Pi)$ and $w \in H^{n-1}(Y; \Pi)$. Then*

$$\theta(v + \delta w) = \theta(v) + \theta(\delta w).$$

PROOF. Let K be an Eilenberg-MacLane complex of type (Π, n), and let $e \in K$ be a base point. Denote by ι the fundamental class of $H^n(K, e; \Pi)$. Then, to θ there corresponds an element y_θ in $H^q(K, e; G)$ given by $y_\theta = \theta(\iota)$. Let (X_1, Y_1) be any pair and let $u_1 \in H^n(X_1, Y_1; \Pi)$. Using the notation of Eilenberg-MacLane in (7.1′) of [3], we have

$$\theta(u_1) = y_\theta \vdash (u_1).$$

Let (X_1, Y_2) be a second pair, and let $u_2 \in H^n(X_1, Y_2; \Pi)$. Eilenberg-MacLane denote by $y_\theta \vdash (u_1, u_2)$ the function of two variables obtained from y_θ. If $Y_1 = Y_2$, then by 10.2 of [3], we have:

$$y_\theta \vdash (u_1, u_2) = \theta(u_1 + u_2) - \theta(u_1) - \theta(u_2).$$

In particular, if $u_2 = 0$, then

$$y_\theta \vdash (u_1, 0) = \theta(u_1) - \theta(u_1) - \theta(0) = 0,$$

since $\theta(0) = 0$ for any cohomology operation θ.

To prove the lemma, set $X_1 = X$, $Y_1 = Y$, $u_1 = v$, and $u_2 = \delta w$. Let g denote the inclusion map $X \subset (X, Y)$. Then, using (7.6) of [3], we have:

$$y_\theta \vdash (v, \delta w) = y_\theta \vdash (v, g^*\delta w) = y_\theta \vdash (v, 0) = 0,$$

since $g^*\delta = 0$ by the exactness of the cohomology sequence of (X, Y). Hence,

$$\theta(v + \delta w) - \theta(v) - \theta(\delta w) = 0,$$

as was asserted.

3. The second proof

Peterson [1] has given an alternative definition of functional cohomology operations. We now prove our result using this definition. As before let θ be a cohomology operation of type $(\Pi, n; G, q)$ now regarded as an element of $H^q(\Pi, n; G)$. Let Z be a space with two non-vanishing homotopy groups—Π in dimension n and G in dimension $(q - 1)$—with θ as its Eilenberg-MacLane invariant. In particular we may construct Z as a fibre space over $K(\Pi, n)$ with fibre $K(G, q - 1)$; denote by ρ the projection $Z \to K(\Pi, n)$.

For any two spaces A and B let $\pi(A; B)$ denote the set of homotopy classes of maps from A to B. In particular, suppose that B is an Eilenberg-MacLane space of type (Λ, r). Then, $\pi(A; B)$ may be identified in a natural fashion with $H^r(A; \Lambda)$. In what follows we always assume this identification has been made.

Let X and Y be spaces and f a continuous map $Y \to X$. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & H^{q-1}(X; G) & \xrightarrow{\omega_\#} & \pi(X; Z) & \xrightarrow{\rho_\#} & H^n(X; \Pi) & \xrightarrow{\theta} & H^q(X; G) \\
 & & \downarrow f^* & & \downarrow f^\# & & \downarrow f^* & & \\
H^{n-1}(Y; \Pi) & \xrightarrow{{}^1\theta} & H^{q-1}(Y; G) & \xrightarrow{\omega_\#} & \pi(Y; Z) & \xrightarrow{\rho_\#} & H^n(Y; \Pi) & &
\end{array}$$

The rows are exact as sets with distinguished elements (see [1] for details).

Again let u be an element of $H^n(X; \Pi)$ such that

$$f^*(u) = 0 \quad \text{and} \quad \theta(u) = 0.$$

Then, by exactness, there exists an element $v \in \pi(X; Z)$ such that $\rho_\#(v) = u$. Further, by commutativity and exactness, there is an element $x \in H^{q-1}(Y; G)$ such that $\omega_\#(x) = f^\#(v)$. The collection of all x such that

$$\omega_\#(x) = f^\#(v) \quad \text{and} \quad \rho_\#(v) = u,$$

for a fixed u, is defined to be $\theta_f(u)$. If θ is a suspension—that is, $\theta = {}^1\psi$, where $\psi \in H^{q+1}(\Pi, n + 1; G)$—then Z may be considered to be a space of loops. Thus, in the above diagram, all the sets are groups and all the functions are group homomorphisms. In this case the same proof as before shows that $\theta_f(u)$ is a coset of $L(\theta, f) = \operatorname{Im} {}^1\theta + \operatorname{Im} f^*$.

When θ is not a suspension, Z is not a space of loops and $\pi(X; Z)$ is not necessarily a group. However, we do have the following algebraic structure. The fibre space $\rho: Z \to K(\Pi, n)$ is a principal fibre space with the monoide $K(G, q - 1)$ as fibre (see §4 for details). Hence, there is a map $\phi: K(G, q - 1) \times Z \to Z$, and thus a natural map $\phi_{\#}:H^{q-1}(X; G) \times \pi(X; Z) \to \pi(X; Z)$. In §4, we prove that if $v, v' \in \pi(X; Z)$, then

$$(3.1) \qquad \rho_{\#}(v) = \rho_{\#}(v')$$

if and only if there exists an element $w \in H^{q-1}(X; G)$ such that

$$\phi_{\#}(w, v) = v'.$$

Notice that ${}^1\theta$ is a homomorphism; thus $H^{q-1}(Y; G)/\mathrm{Im}\,{}^1\theta \approx \mathrm{Im}\,\omega_{\#} \subset \pi(Y; Z)$ can be given a group structure such that $\omega_{\#}$ is a homomorphism. Property (1) in the definition of a principal fibre space (see §4) shows that

$$(3.2) \qquad \phi_{\#}(y, \omega_{\#}(y')) = \omega_{\#}(y + y'),$$

for $y, y' \in H^{q-1}(Y; G)$.

Now, let $v, v' \in \pi(X; Z)$ be elements such that $\rho_{\#}(v') = \rho_{\#}(v) = u$. Then, by (3.1), there is a class $w \in H^{q-1}(X; G)$ such that $\phi_{\#}(w, v) = v'$. Let $x, x' \in H^{q-1}(Y; G)$ be elements such that $\omega_{\#}(x) = f^{\#}(v)$ and $\omega_{\#}(x') = f^{\#}(v')$. Then

$$\omega_{\#}(x') = f^{\#}(v') = f^{\#}\phi_{\#}(w, v) = \phi_{\#}(f^{*}w, f^{\#}v) = \phi_{\#}(f^{*}w, \omega_{\#}(x)).$$

But by (3.2),

$$\phi_{\#}(f^{*}w, \omega_{\#}(x)) = \omega_{\#}(f^{*}w + x).$$

Hence,

$$\omega_{\#}(x' - x - f^{*}w) = 0.$$

Thus, by exactness,

$$x' - x = f^{*}w + {}^1\theta(y),$$

for some class $y \in H^{n-1}(Y; \Pi)$, which completes the proof.

4. Principal fibre spaces

In this section we give a definition of principal fibre spaces and prove the results needed in §3.

Let $p:E \to B$ be a fibre space (in the sense of Serre) with fibre $F = p^{-1}(b_0)$, $b_0 \in B$. Let $i:F \to E$ be the inclusion, and set

$$E^{*} = \{(e_1, e_2) \mid e_i \in E \ (i = 1, 2),\ p(e_1) = p(e_2)\}.$$

Let $p_j:E^{*} \to E$, $(j = 1, 2)$, be the projection maps. Then, (E, p, B) is called a *principal fibre space* if F is a monoide (i.e., space with a multiplication) with unit and homotopy inverse, and if there exist maps

$$\phi:F \times E \to E \quad \text{and} \quad h:E^{*} \to F$$

subject to the following conditions:

(1) The following diagram is commutative, where ψ is the multiplication in F,

$$\begin{array}{ccc} F \times F & \xrightarrow{\psi} & F \\ \Big\downarrow 1 \times i & & \Big\downarrow i \\ F \times E & \xrightarrow{\phi} & E \end{array}$$

(2) The following diagram is commutative, where the map $F \times B \to B$ is the natural projection;

$$\begin{array}{ccc} F \times E & \xrightarrow{\phi} & E \\ \Big\downarrow 1 \times p & & \Big\downarrow p \\ F \times B & \longrightarrow & B \end{array}$$

(3) $\phi(h, p_1) : E^* \to E$ is homotopic to p_2.

LEMMA 4.1: *Let (E, p, B) be a principal fibre space. Let X be a space, and let $v, v' \in \pi(X; E)$. Then,*

$$p_{\#}(v) = p_{\#}(v')$$

if and only if there exists a map $w \in \pi(X; F)$ such that

$$\phi_{\#}(w, v) = v'.$$

PROOF. Suppose that $\phi_{\#}(w, v) = v'$. Then, by condition (2) above,

$$p_{\#}(v') = p_{\#}\phi_{\#}(w, v) = p_{\#}(v).$$

Suppose conversely that $p_{\#}(v) = p_{\#}(v')$. Let $\tilde{v}$, $\tilde{v}'$ represent v and v'. We may assume that $p\tilde{v} = p\tilde{v}'$ by the covering homotopy theorem. Let $w = [h(\tilde{v}, \tilde{v}')] \in \pi(X; F)$. Then, by condition (3),

$$\phi_{\#}(w, v) = \phi_{\#}([h \circ (\tilde{v}, \tilde{v}')], v) = v'.$$

This completes the proof.

We now must show that the fibre space considered in §3 is a principal fibre space. We first show that the space of paths on B is a principal fibre space over B.

Let $b_0 \in B$. Let E be the space of all paths (f, r), where $f : [0, r] \to B$, $r \geqq 0$, and $f(0) = b_0$. Let $p : E \to B$ be defined by $p(f, r) = f(r)$. Then $F = p^{-1}(b_0)$ is the space of loops on B. Define $\phi : F \times E \to E$ by

$$\phi[(f, r), (g, s)](t) = \begin{cases} f(t), & 0 \leqq t \leqq r \\ g(t - r), & r \leqq t \leqq r + s. \end{cases}$$

Define $h : E^* \to F$ by

$$h[(f, r), (g, s)](t) = \begin{cases} g(t), & 0 \leqq t \leqq s \\ f(r + s - t), & s \leqq t \leqq r + s. \end{cases}$$

This makes (E, p, B) into a principal fibre space.

LEMMA 4.2. *Let (E, p, B) be a principal fibre space. Let $(\tilde{E}, \tilde{p}, \tilde{B})$ be the fibre space induced by a map $\tilde{B} \to B$. Then, $(\tilde{E}, \tilde{p}, \tilde{B})$ is a principal fibre space.*

The proof is entirely straightforward and is left to the reader.

Now the fibre space considered in §3 can be obtained as an induced fibre space from $\theta: K(\Pi, n) \to K(G, q)$ and the path space over $K(G, q)$. Hence, it is a principal fibre space by Lemma 4.2, and thus Lemma 4.1 may be used.

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