THE FUNCTIONAL PONTRJAGIN COHOMOLOGY OPERATIONS

By Emery Thomas*

This note is intended as a sequel to the preceding one. Namely, we give an example of the functional operations associated with a particular set of non-additive cohomology operations. These are the generalized Pontrjagin cohomology operations, defined in [2]. We indicate here the properties of these operations needed for this note.

Let K be a complex and $L \subset K$ a subcomplex. For each prime number p we have defined a cohomology operation

$$\mathfrak{P}_p : H^{2k}(K, L; J) \to H^{2pk}(K, L; J^*),$$

where $H^q(K, L; G)$ denotes the qth cohomology group of K mod L with coefficients in the group G. Here the coefficient groups J and J^* are defined as follows: either

(i) $J = Z$ and $J^* = Z$, or

(ii) $J = Z/p^r Z$ and $J^* = Z/p^{r+1}Z$,

where $Z =$ integers and r is some fixed integer $\geqq 1$. The properties of $\mathfrak{P}_p$ required here are:

(1.1) *Let η be the natural factor homomorphism $Z/p^{r+1}Z \to Z/p^r Z$ $(r \geqq 1)$. Then, for $u \in H^{2k}(K, L; Z/p^r Z)$,*

$$\eta_* \mathfrak{P}_p(u) = u^p \qquad (p\text{-fold cup-product}).$$

(1.2) *Let $\zeta_r : Z \to Z/p^r Z$ $(r = 1, 2 \cdots)$ be the natural factor homomorphism. Then for $u \in H^{2k}(K, L; Z)$,*

(i) $\mathfrak{P}_p \zeta_{r*}(u) = \zeta_{r+1*} \mathfrak{P}_p(u)$,

(ii) $\mathfrak{P}_p(u) = u^p$ (p-fold cup-product).

Our aim in this note is to discuss the functional cohomology operation associated with each operation $\mathfrak{P}_p$. We maintain here the general notation of the preceding paper: that is, θ is a cohomology operation of type $(\Pi, n; G, q)$, X and Y are spaces, and f is a map $Y \to X$. Then, for each class $u \in H^n(X; \Pi)$ such that

$$f^*(u) = 0 \quad \text{and} \quad \theta(u) = 0,$$

we defined a class

$$\theta_f(u) \quad \text{in} \quad H^{q-1}(Y, G)/L(\theta, f),$$

where

$$L(\theta, f) = f^*(H^{q-1}(X; G)) + {}^1\theta(H^{n-1}(Y; \Pi)), \qquad ({}^1\theta = \text{suspension of } \theta).$$

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Let us now specify the operation θ to be the operation $\mathfrak{P}_p$; then, $n = 2k$, $q = 2pk$, $\Pi = J$ and $G = J^*$. We distinguish two cases as to the nature of the subgroup $L(\theta, f)$:

$$(1.3) \qquad L(\theta, f) = \begin{cases} f^*(H^{4k-1}(X; J^*)) + \mathfrak{p}(H^{2k-1}(Y; J), & \text{if } \theta = \mathfrak{P}_2 \\ f^*(H^{2pk-1}(X; J^*)), & \text{if } \theta = \mathfrak{P}_p, p > 2. \end{cases}$$

This is obtained by using the theorem of the preceding note together with [3], where it is shown that

$$^1\mathfrak{P}_2 = \mathfrak{p}, \qquad ^1\mathfrak{P}_p = 0 \qquad (p > 2),$$

the operation $\mathfrak{p}$ being the Postnikov square.

The purpose of this note is to give an example where θ_f is non-trivial, $\theta = \mathfrak{P}_p$. Namely,

THEOREM. *Let M_{p-1} be the complex projective space of $(p - 1)$ complex dimensions (p a prime). Let S^{2p-1} be the $(2p - 1)$-sphere, and $a: S^{2p-1} \to M_{p-1}$ the fibre map. Let u be a generator of $H^2(M_{p-1}; Z)$, and $\bar{u}$ its image in $H^2(M_{p-1}; J)$. Denote by θ the Pontrjagin operation $\mathfrak{P}_p$. Then,*

(i) $L(\theta, a) = 0$ *in* $H^{2p-1}(S^{2p-1}; J^*)$

(ii) $\theta_a(\bar{u})$ *generates* $H^{2p-1}(S^{2p-1}; J^*)$.

PROOF. The fact that $L(\theta, a) = 0$ follows at once from (1.3). For, $H^1(S^3, G) = 0$, as does $H^{2p-1}(M_{p-1}; G)$, for all primes p and any coefficient group G. Thus, $\theta_a(\bar{u})$ belongs to $H^{2p-1}(S^{2p-1}; J^*)$, a cyclic group isomorphic to J^*.

The proof of part (ii) is in two steps. We first prove the result for $J = J^* = Z$; using this result, we then prove the case $J = Z/p^rZ$, $J^* = Z/p^{r+1}Z$.

Let n be a fixed integer >1. We define a cohomology operation $\tau(\equiv\tau(n))$ of type $(Z, 2; Z, 2n)$ by

$$\tau(u) = u^n \qquad (n\text{-fold cup-product})$$

for $u \in H^2(K, L; Z)$.

Now, let X, Y be spaces and f a map $Y \to X$. Suppose that $u \in H^2(X, Z)$ is a class such that

$$f^*(u) = 0 \quad \text{and} \quad \tau(u) = 0.$$

We then have the functional operation $\tau_f(u)$ defined, which belongs to $H^{2n-1}(Y; Z)/L(\tau, f)$. In this case,

$$(1.4) \qquad L(\tau, f) = f^*(H^{2n-1}(X; Z)),$$

since the suspension of τ (i.e. $^1\tau$) is zero.

Since $\tau(u) = u^n = 0$, we immediately have $u \smile u^{n-1} = 0$, by the associativity of the cup-product. Set $v = u^{n-1}$. Then, following Steenrod's original treatment of functional operations (see [1]), we have a functional cup-product defined:

namely,

$$u \smile_f v \in H^{2n-1}(Y; Z)/L(\smile v, f),$$

where

$$L(\smile v, f) = f^*(H^{2n-1}(X; Z)) + \smile v(H^1(Y; Z)).$$

Denote by $\bar{\phi}$ the natural factor homomorphism

$$H^{2n-1}(Y; Z)/L(\tau, f) \rightarrow H^{2n-1}(Y; Z)/L(\smile v, f).$$

We then assert:

(1.5) LEMMA $$\bar{\phi}\tau_f(u) = u \smile_f v.$$

(1.6) COROLLARY. *If* $H^1(Y; Z) = 0$, then

$$\tau_f(u) = u \smile_f v.$$

PROOF. Using the mapping cylinder technique, we may assume that $Y \subset X$ and that f is the inclusion map. Let g be the inclusion $X \subset (X, Y)$. Let $w \in H^{2n-1}(Y; Z)$ be a representative for $\tau_f(u)$. That is, there is a class x in $H^2(X, Y; Z)$ such that

$$\delta w = \tau(x), \qquad g^*(x) = u.$$

But,

$$\tau(x) = x^n = x \smile x^{n-1} = x \smile g^*(x^{n-1}) = x \smile (g^*x)^{n-1} = x \smile u^{n-1}$$
$$= x \smile v.$$

Hence,

$$\delta w = x \smile v, \qquad g^*(x) = u.$$

Thus, w may also be taken as a representative for $u \smile_f v$, which completes the proof.

Let us now return to the hypotheses of the theorem: that is, set $X = M_{p-1}$, $Y = S^{2p-1}$, and $f = a: S^{2p-1} \rightarrow M_{p-1}$; again, let u denote a generator of $H^2(M_{p-1}; Z)$. Then, by (1.2) (ii) we have

$$\mathfrak{P}_p(u) = \tau(u); \qquad (n = p)$$

hence, by (1.6),

(1.7) $$\theta_a(u) = u \smile_a v,$$

where $\theta = \mathfrak{P}_p$.

We proceed to show that $\theta_a(u)$ in fact is a generator for $H^{2p-1}(S^{2p-1}; Z)$. By hypothesis, u generates $H^2(M_{p-1}; Z)$; hence, $v(=u^{p-1})$ generates $H^{2p-2}(M_{p-1}; Z)$. Using (7.3) in [1], but restated in terms of the sphere-bundle situation of [1; §11], we have:

$H^2(M_{p-1}; Z)$ and $H^{2p-2}(M_{p-1}; Z)$ are paired in a completely orthogonal fashion to $H^{2p-1}(S^{2p-1}; Z)$ by $u_1 \smile_a u_2$, for $u_1 \in H^2(M_{p-1}; Z)$, $u_2 \in H^{2p-2}(M_{p-1}; Z)$.

Since u and v are each generators of their respective cohomology groups, this says that $u \smile_a v$ is a generator for $H^{2p-1}(S^{2p-1}; Z)$; but by (1.7), $u \smile_a v = \theta_a(u)$, where $\theta = \mathfrak{P}_p$. Thus, the theorem is proved for the case $J = J^* = Z$.

In order to prove the case $J = Z/p^r Z$, $J^* = Z/p^{r+1}Z$, we must digress to make some general remarks about functional cohomology operations. Suppose that θ is a cohomology operation of type $(\Pi, n; G, q)$. Let Π', G' be abelian groups and let

$$\nu : \Pi' \to \Pi, \qquad \mu : G \to G'$$

each be homomorphisms. Denote by μ_*, ν_* the homomorphisms of cohomology groups induced by μ and ν. Then, we have the composite cohomology operations:

$$\mu_*\theta \text{ of type } (\Pi, n; G', q)$$

and,

$$\theta\nu_* \text{ of type } (\Pi', n; G, q).$$

For any cohomology operation θ, continue to denote by ${}^1\theta$ the *suspension* of θ.

(1.8) Lemma. ${}^1(\mu_*\theta) = \mu_*{}^1\theta$; ${}^1(\theta\nu_*) = ({}^1\theta)\nu_*$.

The proof follows at once from the fact that μ_* and ν_* commute with the relative coboundary operator.

Now, let X, Y be spaces and f a map $Y \to X$. Suppose that $u \in H^n(X; \Pi)$ is a class such that

$$f^*(u) = 0 \quad \text{and} \quad \theta(u) = 0.$$

Then, clearly $\mu_*\theta(u) = 0$. Thus, we have defined functional operations

$$\theta_f(u) \quad \text{and} \quad (\mu_*\theta)_f(u).$$

These are defined relative to two subgroups:

$$L(\theta, f) = f^*(H^{q-1}(X; G)) + {}^1\theta(H^{n-1}(Y; \Pi)) \tag{1.9}$$

$$L(\mu_*\theta, f) = f^*(H^{q-1}(X; G')) + {}^1(\mu_*\theta)(H^{n-1}(Y, \Pi)). \tag{1.10}$$

Since μ_* commutes with f^*, using Lemma 1.8 we have:

$$\mu_*(L(\theta, f)) \subset L(\mu_*\theta, f)$$

Denote by $\bar{\mu}$ the factor homomorphism of $H^{q-1}(Y; G)/L(\theta, f)$ to $H^{q-1}(Y; G')/L(\mu_*\theta, f)$ induced by μ_*. We assert:

(1.11) Lemma. $\bar{\mu}\theta_f(u) = (\mu_*\theta)_f(u)$.

The proof follows at once from the definitions involved.

Similarly, let $u' \in H^n(X; \Pi')$ be a class such that

$$f^*(u') = 0 \quad \text{and} \quad (\theta\nu_*)(u') = 0.$$

Then, $\theta(\nu_*(u')) = 0$ also. Hence, we have defined functional operations

$$(\theta\nu_*)_f(u') \quad \text{and} \quad \theta_f(\nu_* u').$$

By Lemma 1.8 we see that

$$^1(\theta\nu_*)(H^{n-1}(Y; \Pi')) \subset {}^1\theta(H^{n-1}(Y, \Pi)).$$

Thus, we may define a factor homomorphism $\bar{\nu}$ mapping $H^{q-1}(Y; G)/L(\theta\nu_*, f)$ to $H^{q-1}(Y; G)/L(\theta, f)$. Thus, corresponding to Lemma 1.11 we have:

(1.12) LEMMA. $\bar{\nu}[(\theta\nu_*)_f(u')] = \theta_f(\nu_* u')$.

We now apply Lemmas 1.11 and 1.12 to conclude the proof of our theorem. Recall the factor map $\zeta_r: Z \to Z/p^r Z$ $(r = 1, 2, \cdots)$ defined initially. Then, by 1.2, we have

$$\theta\zeta_{r*} = \zeta_{r+1*}\theta \qquad (\theta = \mathfrak{P}_p). \tag{1.13}$$

Now, let u be a generator for $H^2(M_{p-1}; Z)$; then $\bar{u} = \zeta_{r*}(u)$ is a generator for $H^2(M_{p-1}; Z/p^r Z)$. Our aim is to show that

$$\theta_a(\bar{u}) \quad \text{generates} \quad H^{2p-1}(S^{2p-1}; Z/p^{r+1}Z),$$

where $a: S^{2p-1} \to M_{p-1}$ is the factor map. Now by Lemma 1.12,

$$\theta_a(\bar{u}) = \theta_a(\zeta_{r*}u) = \bar{\zeta}_r[(\theta\zeta_{r*})_a(u)],$$

where we set $\Pi' = Z$, $\Pi = Z/p^r Z$, $\zeta_r = \nu: Z \to Z/p^r Z$. But $\bar{\zeta}_r$ = identity, since $L(\theta\zeta_{r*}, a) = 0$. Hence

$$\theta_a(\bar{u}) = (\theta\zeta_{r*})_a(u) = (\zeta_{r+1*}\theta)_a(u) = \bar{\zeta}_{r+1}\theta_a(u),$$

by Lemma 1.11 and (1.13). Now $L(\zeta_{r+1*}\theta, a) = 0$; hence, $\bar{\zeta}_{r+1}\theta_a(u) = \zeta_{r+1*}\theta_a(u)$. Therefore,

$$\theta_a(\bar{u}) = \zeta_{r+1*}\theta_a(u).$$

But, we have already shown that $\theta_a(u)$ is a generator for $H^{2p-1}(S^{2p-1}; Z)$. Hence, $\zeta_{r+1*}\theta_a(u)$ is a generator for $H^{2p-1}(S^{2p-1}; Z/p^{r+1}Z)$, which concludes the proof of the theorem.

Although the theorem shows the non-triviality of the operation θ_f $(\theta = \mathfrak{P}_p)$, it does not really give us any new information; since the results obtained can all be expressed in terms of the functional cup-product. However, it is easy to alter the above example to produce one where the operation θ_f $(\theta = \mathfrak{P}_p)$ does yield new information.

Let a again be the fibre map $S^{2p-1} \to M_{p-1}$, and let α denote the homotopy class of a. That is, α belongs to $\pi_{2p-1}(M_{p-1})$, a cyclic infinite group. Let p be a prime and r an integer $\geqq 1$. Let $f: S^{2p-1} \to M_{p-1}$ be a map which represents $p^r\alpha$.

As is well known M_{p-1} can be given a (CW) cell-decomposition

$$M_{p-1} = S^2 \cup e^4 \cup \cdots \cup e^{2p-2},$$

where each cell e^{2i} has dimension $2i$. Let M^*_{p-1} be the complex obtained from M_{p-1} by attaching a 3-cell to S^2 by a map of degree p^r. Let g be the inclusion map $M_{p-1} \subset M^*_{p-1}$, and set

$$h = g \circ f : S^{2p-1} \to M^*_{p-1} .$$

Let θ be the Pontrjagin operation $\mathfrak{P}_p$ and τ the p-fold cup-product, with Z/p^rZ as coefficients. Let $\bar{u}$ be a generator of $H^2(M^*_{p-1}; Z/p^rZ)$ (notice that $H^2(M^*_{p-1}; Z) = 0$). Then,

$$(1.14) \qquad \begin{aligned} \tau_h(\bar{u}) &= 0 \quad \text{in} \quad H^{2p-1}(S^{2p-1}; Z/p^rZ); \\ \theta_h(\bar{u}) &\neq 0 \quad \text{in} \quad H^{2p-1}(S^{2p-1}; Z/p^{r+1}Z). \end{aligned}$$

Thus, the operation θ_h shows that the map h is essential, whereas the operation τ_h fails to do so. We omit the proof; it is a simple consequence of the naturality of the functional operations.

UNIVERSITY OF CALIFORNIA (BERKELEY)

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