LIAPUNOV AND STABILITY IN DYNAMICAL SYSTEMS

BY SOLOMON LEFSCHETZ

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I. INTRODUCTORY CONCEPTS. STABILITY. LIAPUNOV'S SECOND METHOD

1. The concept of stability is one of those which reach beyond the general domain of mathematics. In the commonly understood sense, a system is stable if, upon applying to it a small disturbance, it tends to return to its initial situation. In dynamics this concept acquires fundamental importance and has been made the object of profound investigations initiated by Poincaré and Liapunov and pursued with especial vigor in the Soviet Union. Above all there are certain related theorems of Liapunov which will mainly concern us here. We shall first deal with them in the context in which they were initiated by him, then broaden our point of view to what has aptly been called we believe first by G. D. Birkhoff, "topological dynamics."

2. In an Euclidean n-space E^n referred to coordinates x_i denote by x the vector with coordinates x_i and consider the real differential equation

$$\dot{x} = X(x, t), \qquad (\cdot = d/dt). \tag{2.1}$$

where X is likewise an n-vector. A distance is defined in E^n by means of a norm $\| x \|$, and we generally choose

$$\| x \| = \sup \{ \mid x_i \mid \}.$$

Let I denote the time range $-\infty < t < \infty$. Suppose that Ω is a region (connected open set) of the product space $E^n \times I$ in which X is continuous and satisfies a Lipschitz condition in every subregion Ω_1 of Ω whose closure $\bar{\Omega}_1 \subset \Omega$. There is then a standard existence theorem, proved in the texts on the subject, asserting that if $P(x^0, t_0)$ is a point of Ω there is a unique solution or *motion* $x(t, x^0, t_0)$ of (2.1) with the initial point P at time t_0, that is such that $x(t_0, x^0, t_0) = x_0$. As t varies as much as possible the set $x(t, x^0, t_0)$ generates an arc, called a *trajectory* in Ω. No two trajectories meet. The function $x(t, x^0, t_0)$ is a continuous function of (t, x^0, t_0) in the space $E^n \times I \times I$. These are the basic facts which we require. It may happen that $x(t, x^0, t_0) = x^0$ for all t in a certain range. The point x^0 is then referred to as a *critical point* of the system (2.1). In the product space $E^n \times I$ we have then a critical interval of line (an arc).

Of particular interest are the *autonomous* systems, or those such that X does not contain the time t:

$$\dot{x} = X(x). \tag{2.2}$$

The region Ω assumes then the form $\Phi \times I$, where Φ is a region of E^n. The locus of the points $x(t, x^0, t_0)$ in Φ is then called a *path* and shown to depend solely upon x^0 and not upon t_0. For this reason, in such a case, one often takes the initial time $t_0 = 0$ and denotes the solution by $x(t, x^0)$. Here then $x(t, x^0)$ and $x(t - t_0, x^0)$ represent the same path, merely differently parametrized.

3. So much for generalities. Suppose that the system (2.1) has a critical point O. At the cost of changing coordinates we may take O as the origin. For our purpose it is sufficient to confine our attention to a certain closed neighborhood of the origin whose precise size is quite immaterial. We introduce then the sets

$$\Omega(A, \tau): \| x \| \leqq A, \; t \geqq \tau$$

$$\Phi(A): \| x \| \leqq A$$

(the time has thus an infinite range) and assume that in $\Omega(A, \tau)$ the conditions for existence and uniqueness of solutions are fulfilled. At the cost of shifting the time origin we may assume that the basic set is

$$\Omega(A): \| x \| \leqq A, \; t \geqq 0.$$

We assume then explicitly that

$$X(0, t) = 0 \quad \text{for} \quad t \geqq 0.$$

As a matter of notation we shall denote by $d(x, x')$ the distance $\| x - x' \|$, by $\mathcal{S}(\varepsilon)$ the *spheroid* $\| x \| < \varepsilon$ and by $H(\varepsilon)$ its *boundary sphere* $\| x \| = \varepsilon$. Without any further remark we shall always assume below that $\varepsilon \leqq A$, so that $\mathcal{S}(\varepsilon)$, $H(\varepsilon) \subset \Phi(A)$.

4. The question that now arises is this: If a motion starts at time $t_0 (\geqq 0)$ not too far from the origin, will it remain near the origin, perhaps even tend to it, or else tend to leave the origin? In the first two cases we have stability, in the third instability. With Liapunov these concepts are given precision in the following:

DEFINITIONS. Given any $\varepsilon \leqq A$ and $t_0 \geqq 0$, suppose that there corresponds to them an $\eta(\varepsilon, t_0) > 0$ such that whenever the initial point x^0, at time t_0 of a solution $x(t, t_0, x^0)$ is not as far as η from the origin then $\| x \| < \varepsilon$ for $t \geqq t_0$. Then the origin is *stable*. If η may be chosen independently of t_0 the origin is *uniformly stable*.

The origin is *asymptotically stable* whenever it is stable and in addition $x(t) \to 0$ as $t \to +\infty$. Finally it is *unstable* whenever given any $0 < \varepsilon < A$ and no matter how small η some solution starting within $\mathcal{S}(\eta)$ reaches the boundary $H(\varepsilon)$ of $\mathcal{S}(\varepsilon)$.

Regarding these various types of stability there are four very general theorems due to Liapunov and which are now generally referred to as constituting *Liapunov's second method.* His first method refers to establishing the stability of (2.1) in certain cases by means of explicit power series solutions given by him. Remarkably enough the second method has been inverted by a number of savants: Persidskii, Massera, Kurzweil, Krasovskii, Zubov. We shall only deal for the equation (2.1) with the *direct* results of Liapunov: sufficiency conditions, and shall deal with the *inversion*: necessary conditions, in connection with the stability of invariant sets.

5. The theorems of Liapunov rest upon the properties of certain functions which must first be defined.

Let a function $V(x, t)$ be defined and of class C^1 in a certain region $\Omega(A)$ and let $V(0, t) = 0$. We will say that:

V is of *fixed sign, positive* or *negative,* whenever $V \geqq 0$ or $\leqq 0$ in $\Omega(A)$:

V is *positive definite* whenever there exists a continuous function $W(x)$ defined in $\Phi(A)$, zero for $x = 0$ and only then, positive there otherwise, and such that $V \geqq W$ under the same conditions;

V is *negative definite* whenever $-V$ is positive definite.

The time derivative $\dot{V}$ as to t along the trajectories of (2.1) is given by

$$\dot{V} = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \cdot X_i . \tag{5.1}$$

A function of the general type of V is called a *Liapunov function.*

6. We are now ready for the theorems of Liapunov. They all refer to the system (2.1) and the stability of the origin.

(6.1) Stability Theorem. *If there exists a positive definite function V, in a suitable $\Omega(A)$, whose $\dot{V}$ is of fixed negative sign in Ω then the origin is stable.*

Let W be the same as in the definition of positive definiteness for V. Take now the trajectory γ with initial point $P(x^0, \tau)$, $\tau \geqq 0$, $x^0 \in \Phi \neq 0$. Upon integrating $\dot{V}$ along γ, and setting $V_0 = V(x^0, \tau) > 0$, we have

$$V - V_0 = \int_\tau^t \dot{V}\, dt \leqq 0. \tag{6.2}$$

Hence $V \leqq V_0$ on γ beyond the initial point.

Take now any ε positive and $\leqq A$. Since $H(\varepsilon)$ is compact, W has a positive lower bound ω on it. Since $V(x, \tau)$ is continuous and $\rightarrow 0$ with x we can find an $\eta(\varepsilon, \tau)$ such that $V(x, \tau) < \omega$ for $\| x \| < \eta$. Hence if $\| x^0 \| < \eta$, we will have $V_0 < \omega$, and so V on γ will never reach ω, hence x will never reach $H(\varepsilon)$ and so we have stability.

To have *uniform stability* it will be sufficient to have $V(x, t) \rightarrow 0$ with x uniformly in t for $t \geqq \tau$.

(6.3) Asymptotic Stability Theorem. *If there exists a $V \rightarrow 0$ with x uni-*

formly in t for $t \geqq \tau$, and such that V is positive definite and $\dot{V}$ negative definite then the origin is asymptotically stable.

Let W_1 be the W for $-\dot{V}$. At all events the origin is stable and so there exists an $\eta > 0$ such that if $\| x(\tau) \| < \eta$ then $\| x \| < A$. That is, if the initial point P is nearer than η to the origin then x does not reach $\| x \| = A$.

Now if $V \to 0$ along γ so does x. For suppose that $\| x \| \geqq \alpha > 0$. Now in $\alpha \leqq \| x \| \leqq A$, W and hence V has a positive lower bound ω, and this contradicts $V \to 0$. Thus all that is necessary is to show that $V \to 0$ along γ.

Suppose that $V \geqq \omega$ along γ. Then $\| x \| \geqq \beta > 0$. Let $\zeta = \inf W_1$ for $\beta \leqq \| x \| \leqq A$. Then $-\dot{V} \geqq \zeta$ in that region and if x along γ remains in it, then by (6.2) $V \leqq V_0 - \zeta(t - t_0)$, which with increasing time becomes <0. Since this contradicts $V \geqq 0$, sometime $\| x \| < \beta$, a contradiction which proves that $V \to 0$, hence $x \to 0$, along γ, and the theorem follows.

(6.4) First Instability Theorem. *Suppose that:*

(a) *$V(x, t) \to 0$ with x uniformly in t for $t \geqq \tau$;*

(b) *$\dot{V}$ is positive definite in $\Omega(A)$;*

(c) *no matter how small $\varepsilon > 0$ the function V assumes the sign $+$ somewhere in $\| x \| < \varepsilon$.*

Then the origin is unstable.

Because of the continuity of V and its $\to 0$ uniformly in x, we may assume A and τ such that $| V |$ is bounded in $\Omega(A)$ for $t \geqq \tau$.

Take any small ε and an initial point $\| x^0 \| < \varepsilon$ such that $V_0 > 0$. Then by (6.2) and since $\dot{V} > 0$, $V > V_0$ along γ. Since $V_0 > 0$ the least distance from the origin of a point of γ beyond $P(x^0, \tau)$ is positive; for if it were zero we would have $V_0 = 0$. Let $0 < \lambda < \mu$. Let $\alpha = \inf W_1$ for $\lambda \leqq \| x \| \leqq \varepsilon$, and let S denote that region. Now if γ remained in S we would have $V \geqq V_0 + \alpha(t - \tau)$ which $\to +\infty$ with t. Since $V \to 0$ with x uniformly in t for $t \geqq \tau$, in S: $\sup V = \beta$ is finite. Since V crosses β, γ cannot remain in S and since it cannot leave through $H(\lambda)$ it must leave through $H(\varepsilon)$, and so we have instability.

(6.5) Second Instability Theorem. *Let there exist a function V with the following properties:*

(a) *$| V |$ is bounded in Ω;*

(b) *in Ω: $\dot{V} = \lambda V + W$, where λ is a positive constant and W is of fixed non-negative sign (it may be identically zero);*

(c) *for τ sufficiently large and every $t \geqq \tau$ and in $\mathcal{S}(\varepsilon)$, ε sufficiently small, V assumes somewhere a positive value.*

Then the origin is unstable.

If $W = 0$ the assertion is a consequence of the preceding theorem. Suppose then $W \neq 0$.

Let the initial point x^0 at time τ be in $\mathcal{S}(\varepsilon)$ and such that $V_0 = V(x^0, \tau) > 0$. Suppose that the correspondlng trajectory γ remains in $\mathcal{S}(\varepsilon)$ so that on it $| V |$ remains bounded. Thus along γ for $t \geqq \tau$:

$$\dot{V} - \lambda V \geqq 0$$

and hence

$$V \geqq V_0 e^{\lambda(t-\tau)}$$

which $\to +\infty$ with t. Since this contradicts the boundedness of V in $\mathcal{S}(\varepsilon)$, X reaches the boundary $H(\varepsilon)$ of $\mathcal{S}(\varepsilon)$ and so we have instability.

REMARK. *If the basic system* (2.1) *is autonomous, stability is always uniform* (since one may always shift the time by any constant amount without changing anything).

GEOMETRIC INTERPRETATION. To simplify matters take an autonomous system. The functions V are then functions of x alone and the loci V = const. represent, if V is positive, definite ovals (or ovaloids) surrounding the origin. Ordinary stability means more or less that paths enter arbitrarily small ovals and wander within them. When stability is asymptotic the paths actually $\to 0$. For "strong" instability the paths emerging from the interior of an oval reach that oval.

7. *Some simple examples*: (a) Consider the system

$$\dot{x} = Px + q(x, t) \tag{7.1}$$

where x, q are n-vectors, P is a constant matrix and $q(0, t) = 0$ for $t \geqq 0$. Thus the origin is a critical point. We suppose q such that, in a certain Ω, conditions for the existence and uniqueness of solutions of (7.1) are satisfied. Finally it is assumed that Px is the "principal part" of the right hand side, i.e., that $q = o(\| x \|)$, uniformly[1] in Ω.

To discuss the stability of the origin let us limit our attention to the simple case where the characteristic roots λ_i of P are distinct and have negative real parts. Suppose first that they are all real. We may choose a coordinate system in which $P = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$. Take then

$$V(x) = \sum_{h=1}^{h} x_h^2 .$$

We have then

$$\dot{V} = \sum_{h=1}^{n} \lambda_h x_h^2 + o(\| x \|^2).$$

Thus in a suitably small Ω the function $\dot{V}$ is negative definite. Since V is positive definite and $\to 0$ with x uniformly in t, the origin is asymptotically stable.

Suppose now that $2m$ of the λ_h are complex, $0 \leqq 2m \leqq n$. For a complex pair say λ_k, $\bar{\lambda}_k$ we introduce coordinates in which $\bar{x}_k = x_{k+1}$, and so that P is still diagonal. The Liapunov function is now

$$V = \sum_{k=1}^{n-m} | x_k |^2.$$

(one term for each complex conjugate pair) and the rest is as before.

[1] Specifically, assume that, for all $(x, t) \in \Omega$, $\| q \| (x, t) \| \leqq \varphi(\| x \|)$, where $[\varphi(r)/r] \to 0$ as $r \to 0$.

(b) *Lagrange's theorem on the stability of static equilibrium.* Lagrange stated the following property: If the potential energy of a system has a *minimum* at a position P, then P is a stable position of static equilibrium. We shall derive this result from the Liapunov theorem on stability of the origin under very general conditions.

Generally speaking a dynamical system Σ depends upon n positional coordinates $q_1, \cdots, q_n$ and n kinematic coordinates $p_1, \cdots, p_n$. Its potential energy is a function $V(q)$ of q (the vector with components q_i). Under our assumptions we may suppose that the minimum occurs for $q = 0$ and that $V(0) = 0$. To simplify matters we shall assume that V is analytical and then it is positive definite near $q = 0$.

The kinetic energy on the other hand, again assumed analytical, is a positive definite function of the p_j for small q, near $p = 0$. We have then a hamiltonian function

$$H = K + V$$

and the equations of motion are

$$\dot{q}_s = \frac{\partial H}{\partial p_s}, \qquad \dot{p}_s = -\frac{\partial H}{\partial q_s}, \qquad (s = 1, 2, \cdots, n).$$

Therefore

$$\dot{H} = \sum_{s=1}^{n} \left(\frac{\partial H}{\partial q_s} \frac{\partial H}{\partial p_s} - \frac{\partial H}{\partial p_s} \frac{\partial H}{\partial q_s} \right) \equiv 0.$$

Thus H satisfies all the conditions for stability of $p = 0$, $q = 0$ and Lagrange's theorem follows.

II. Topological Dynamics

8. In the theory of ordinary differential equations two streams of thought are mixed: analytical and geometrical (or rather topological). To this mixture one may trace many difficulties of this theory and also much of its attractiveness.

Now differential equations, more than any other part of mathematics, receive their impulsion from physics, understood in the largest sense possible. So often therefore, the solution of differential equations must come down to explicit and even numerical expressions. However all too often this can only be accomplished under very restrictive approximations. The problem therefore arises to obtain at least some qualitative, i.e., in the last analysis *topological* information about elusive solutions. Frequently also the requirements of the physicist are not for an exact, isolated solution, but for the behavior of a whole family of solutions. And this leads again to the topological behavior of the solutions.

This general point of view has led in recent years to an endeavor to isolate if possible the topological from the analytical study of differential equations and has given rise to *topological dynamics*. Its full fledged attack first appeared in G. D. Birkhoff's book *Dynamical Systems* (New York, 1926); then later in

Chapter V of Niemitzkii-Stepanov: *Qualitative Theory of Differential Equations* (1st ed. 1947, Russian), which attributes the basic definitions to Birkhoff and A. A. Markov (Comptes Rendus 123, (1931), 823–825. The theory has been extended to include n-parameter flows as well as the 1-parameter flows defined by ordinary differential equations and an extremely general treatment has been published by Gottschalk and Hedlund (*Topological Dynamics*, A.M.S. Colloq. vol. 36, 1955). This is the subject then that we propose to discuss here, with particular emphasis on results described in a very recent and interesting book by Zubov: *The Methods of A. M. Liapunov and Their Application* (Moscow, 1957). It is not necessary to emphasize that most of the available literature is in Russian. A forthcoming fairly exhaustive survey on this subject by Henry Antosiewicz appears in *Contributions to Nonlinear Oscillations*, IV, Ann. of Math. Study, 41, (1958) 147–166. It may be added that many of the Russian notions concerning general stability theory have been found recently, independently, by Robert Bass.

9. Let then R be a metric space and t a real variable whose range is the infinite interval $I:(-\infty, +\infty)$. Let $f(p, t)$ (p any point of R) be a mapping $R \times I \rightarrow R$ with the following properties:

(a) for every fixed t the mapping $f(p, t)$ is a topological mapping φ_t of R onto itself;

(b) $f(p, 0) = p$ (the mapping φ_0 is the identity);

(c) $f[f(p, t_1), t_2] = f(p, t_1 + t_2)$ whatever $t_1, t_2 \in I$.

One verifies then the associative law for φ_t and since $f[f(p, t), -t] = f(p, 0) = p$, $\varphi_t^{-1} = f(p, -t)$, and so φ_t has all the properties of the basic operation in an abelian group. That is, the operation $f(p, t)$ determines a *one-parameter group of transformations acting upon R.*

The analogy of $f(p, t)$ with the solution of an *autonomous system* of ordinary differential equations is obvious. We will refer therefore to the locus $f(p, t)$ (p fixed, all t) as a *path*, and to the locus parametrized by t as a *motion*. The collection of all the motions is a *dynamical system.*

10. Regarding a definite motion there are three possibilities:

(a) $f(p, t) = p$, whatever p. That is, the point p is fixed under the transformation φ_t. We refer to p as a *critical point* of the dynamical system.

(b) Corresponding to p and for some t there is a $\tau \neq 0$ such that $f(p, t + \tau) = f(p, t)$. Then by (9c) this holds for every t. The number τ is a *period* of the motion. It is not known *a priori* whether this period is unique. At all events if $\tau_1, \cdots, \tau_k$ are periods then so is every linear combination $m_1\tau_1 + \cdots + m_k\tau_k$, where the m_j are integers not all zero. We have now two possibilities. First there is an arbitrarily small positive period: given $\delta > 0$ there is always a period τ_0 such that $0 < \tau_0 < \delta$. Because of the continuity of $f(p, t)$, given $\varepsilon > 0$ we may choose

$\delta > 0$ such that $d[f(p, t), p] < \varepsilon$, for $t \in [0, \delta]$ and hence for $t \in [0, \tau_0]$. Given any t we may find a $t' \in [0, \tau_0]$ such that $t = m\tau_0 + t'$, m an integer. Then since τ_0 is a period $f(p, t) = f(p, t')$, $t' \in [0, \tau_0]$ and so $d[f(p, t), p] < \varepsilon$. Hence $d[f(p, t), p] = 0$, $f(p, t) = p$ and since t is arbitrary, p is a fixed point. Thus this case is ruled out.

We have then a least period τ (in absolute value), and since $-\tau$ is also a period we may assume that $\tau > 0$. Thus τ is the least positive period of $f(p, t)$, and it is this τ which is generally designated as the period of the motion. The latter is then said to be *periodic*. The corresponding path γ is then a continuous one-one image of a circle, and as the circle is compact, γ is the topological image of a circle, that is, it is a Jordan curve.

(c) Corresponding to a given point p and for distinct t, t' we always have $f(p, t) \neq f(p, t')$. In this case the path γ_p is the continuous one-one image[2] of the line I.

11. We have now some simple theorems and definitions.

(11.1) *Continuity of $f(p, t)$ in the initial position p.* (Consequence of the definition).

Explicity, given ε, $T > 0$ there exists a $\delta > 0$ such that $d(p, q) < \delta \Rightarrow d[f(p, t), f(q, t)] < \varepsilon$ for $t \in [0, T]$.

(11.2) *The path γ_p of p is also the path of every point of γ_p. Hence if q is not in γ_p, γ_p and γ_q do not intersect.* (Immediate consequence of 10b).

Let p designate now any point of a path γ and let γ_p^+, γ_p^- denote respectively the part of γ described *beyond* p, i.e., for $t \geqq 0$, and *before* p, i.e. for $t \leqq 0$. Then the sets $\Omega(\gamma) = \bigcap_p \overline{\gamma_p^+}$, $A(\gamma) = \bigcap_p \overline{\gamma_p^-}$ are known after G. D. Birkhoff as the Ω and A (capital α) sets of γ, or (more carelessly) of p. An ω- (α-) point of γ is a point of $\Omega(\gamma)$ (or of $A(\gamma)$).

From point-set topology we infer at once:

(11.3) *The Ω and A sets are closed and if R is compact they are connected.* (However if R is not compact they may not exist).

An *invariant set M* is a union of paths. Hence if it contains a point p then it contains the whole path of p.

(11.4) *The closure $\bar{M}$ of an invariant set M is an invariant set.*

Let $p \in \bar{M}$ and $q \in \gamma_p$. Given $\varepsilon > 0$ there is a $\delta > 0$ such that if r is not as far as δ from p then γ_r has a point nearer than ε to q. Since there are points r in M and so with γ_r in M, there are points of M nearer than ε to q. Hence $q \in \bar{M}$ and so $\gamma_p \subset \bar{M}$. Hence $\bar{M}$ is invariant.

[2] If the inverse mapping is continuous, then (see Definitions in (11.2)) γ_p^+ does not intersect Ω_p, hence γ_p^+ is called asymptotic; otherwise γ_p^+ is contained in Ω_p and γ_p^+ is called *recurrent*.

The same argument shows that

(11.5) *The boundary of an invariant set is an invariant set.*

(11.6) *If M, N are invariant sets so are their union $M \cup N$, intersection $M \cap N$ and difference $M - N$. Hence $\Omega(\gamma)$ and $\mathrm{A}(\gamma)$ are invariant sets.* (Proof immediate).

12. *Some examples.*

(a) An autonomous system in E^n constitutes a dynamical system, provided that the general solution $x(t, x^0)$ exists for $-\infty < t < +\infty$, i.e. is defined on $I \times E^n$. This is true if, e.g. (A. Wintner) there are constants $\alpha > 0$, $\beta > 0$ such that, on E^n, $\| X(x) \| \leqq \alpha + \beta \| x \|$; or if e.g. (F. Brauer & S. Sternberg; cf. *American J. Math.* vol. 80 (1958), pp. 421–430) the path through the point at infinity is unique.

(b) Consider a non-autonomous system in E_n

(12.1) $$\dot{x} = X(x, t),\ t \geqq 0,$$

(arbitrary x and $t \geqq 0$), and assume that $x(t, x^0, t_0)$ is defined and continuous on $I^+ \times E^n \times I^+$, where $I^+ = [0, +\infty)$.

The change of variables $t = e^\tau$ replaces (12.1) by

(12.2) $$\frac{dx}{d\tau} = X(x, e^\tau)e^\tau = X(x, t)t, \frac{dt}{d\tau} = t$$

which is a dynamical system relative to the space $E^n \times I^+ = E^n \times [0, +\infty)$, $(-\infty < \tau < +\infty)$. If we assume that $X(0, t) \equiv 0$ then (12.2) has for closed invariant set the t line: $x = 0$.

Thus a non-autonomous system with a critical point may be made into an autonomous one with closed invariant line.

(c) (example due to Robert Bass). Let the autonomous system $\dot{x} = X(x)$ on E^n fail to satisfy the condition of (a). (For example, if $n = 1$ and $X(x) \equiv x^2$, then $x(t, x^0) = x^0/(1 - x^0 t)$, so that no solution $x(t, x^0)$, $0 \leqq t < 1/x^0$, can be extended for $t \geqq 1/x^0$.) It is well known that no solution $x(t, x^0)$, $0 \leqq t < t_\infty$, can have (in the terminology of L. Markus; cf. Rend. del Sem. Mat., Univ. di Torino, vol. 11 (1951–2), pp. 271–7) a finite "escape-time" t_∞, unless $\lim\sup \| x(t, x^0) \| = +\infty$ as $t \rightarrow t_\infty$. But if $\| X(x) \| \leqq \alpha$ for all x, then, by (2.2), $\| x(t, x^0) \| \leqq \| x^0 \| + \alpha t_\infty$. Hence if we are willing to reparameterize the paths of the system, we can use a modified form of (2.2) to define a dynamical system on E^n. In fact, consider

(12.3) $$\frac{dx}{ds} = \frac{X(x)}{1 + \| X(x) \|} \equiv Y(x), \qquad (-\infty < s < +\infty).$$

Because $\| Y(x) \| \leqq 1$, $x(s, x^0)$ is defined on all of $I \times E^n$. Hence (12.3) defines a dynamical system on E^n. Moreover, along any path $x(s, x^0)$, one may define $t(s) = \int_0^s [1 + \| X(x(s, x^0)) \|]^{-1} ds$, and verify easily that $t(s)$ is monotone and has an inverse $s = s(t)$, and that $x(s(t), x^0)$ satisfies (2.2).

III. Stability in Dynamical Systems (Zubov) Generalities

13. In all that follows M will designate a *closed invariant set* of the dynamical system $f(p, t)$. We will also denote by $\mathcal{S}(\varepsilon)$ the spheroid of radius ε and center M, that is, the set of points not as far as ε from M. The boundary of $\mathcal{S}(\varepsilon)$ will be denoted by $H(\varepsilon)$. We now define M as

(a) *Stable* whenever given any $\varepsilon > 0$ there is an $\eta > 0$ such that if $p \in \mathcal{S}(\eta)$ then $\{f(p, t) \mid t \geqq 0\} \subset \mathcal{S}(\varepsilon)$;

(b) *asymptotically stable* whenever it is stable and in addition under the preceding conditions $f(p, t) \to M$ as $t \to +\infty$;

(c) *unstable* whenever for any $\varepsilon > 0$ and whatever $\eta > 0$ (and $< \varepsilon$) there is a point $p \in \mathcal{S}(\eta)$ such that for some $t > 0$, $f(p, t)$ reaches $H(\varepsilon)$ (obvious negation of stability).

When M is asymptotically stable the set A of points whose paths $\to M$ as $t \to +\infty$ is called the *domain of asymptotic stability* of M.

(13.1) *The domain of asymptotic stability A of M is an open invariant set containing a spheroid $\mathcal{S}(\eta)$; and $\bar{A}$, hence also its boundary, are invariant sets.*

Under the definition $\mathcal{S}(\eta(\varepsilon)) \subset A$ and all that one needs to prove is that A is open. Let $p \in A - M$. By hypothesis the path γ of p crosses $\mathcal{S}(\eta)$ before reaching M, say at a point q. Given then $\alpha > 0$ there is a $\beta > 0$ such that if p' is a point nearer than β to p then its path γ' passes nearer than α to q. Hence if α is small enough the path $\gamma' \to M$ and so q is in A. Thus the points close enough to p are are in A and so A is open.

IV. Stability of Closed Invariant Sets Possessing a Compact Neighborhood

14. We assume then for the present that the invariant set M has a neighborhood U whose closure $\bar{U}$ is compact. We shall solely operate within U; that is, assume that $\mathcal{S}(\varepsilon) \subset U$, so that $\mathcal{S}(\varepsilon)$ is likewise compact. Since $M \subset \bar{U}$, M is thus also compact. We shall prove several n.a.s.c. for stability theorems for M, whose expression is strikingly topological.

(14.1) Theorem. *A necessary and sufficient condition for the stability of M is this: given any $\epsilon > 0$ and p outside $\mathcal{S}(\epsilon)$ there is a $\xi(\epsilon) > 0$ such that γ_p^- remains outside $\mathcal{S}(\xi)$.*

(14.2) Corollary. *The negation of the preceding condition is a necessary and sufficient condition for the instability of M.*

Proof of [14.1]. Necessity. If the condition is violated given ϵ and any $\eta > 0$ for some p outside $\mathcal{S}(\epsilon)\gamma_p^-$ has a point q in $\mathcal{S}(\eta)$. Since γ is also the path of q, γ_q^+ leaves $\mathcal{S}(\epsilon)$ and we have instability, proving necessity.

Sufficiency. If $\eta(\epsilon) = \xi(\epsilon)$ and q is in $\mathcal{S}(\eta)$, $\gamma_q^+ \subset \mathcal{S}(\epsilon)$ and we have stability.

(14.3) Theorem. N.a.s.c. *for asymptotic stability of M is the condition of* (14.1) *and that there exists a neighborhood of M free from complete paths other than those in M itself.*

(a) Proof of Necessity. Since we have stability the condition of (14.1) is fulfilled. Suppose that the other condition does not hold so that every $\mathcal{S}(\varepsilon)$ contains a complete path, say γ for ε. Because of stability there is an $\eta(\varepsilon)$ (of stability) corresponding to ε. And because of compactness and asymptotic stability there is a time T such that if $p \in \mathcal{S}(\varepsilon)$ then $f(p, T) \in \mathcal{S}(\eta)$(proof same as for (19.2)). Now we may take η so small that $\mathcal{S}(\eta)$ does not contain a certain point q of γ. Hence if we trace γ back from q for a time T we arrive at a point $q' = f(q, -T) \in \mathcal{S}(\varepsilon)$ such that $f(q', T) = q \notin \mathcal{S}(\eta)$. This contradiction proves necessity of the conditions.

(b) Proof of Sufficiency. Again if the condition of (14.1) is fulfilled we have stability. Take ε, η as for stability. Thus every path γ issued from a point p of $\mathcal{S}(\eta)$ remains in $\mathcal{S}(\varepsilon)$ (beyond p). Suppose that some such path γ does not tend to M, so that $\Omega(\gamma) \subset \mathcal{S}(\varepsilon)$ does not meet M. Thus γ (beyond p) remains outside a certain $\mathcal{S}(\zeta)$ and so it has an ω-point q outside $\mathcal{S}(\zeta)$. Since the path δ of q is in $\Omega(\gamma)$, it is in $\mathcal{S}(\varepsilon)$; hence the latter contains the complete path δ. Since ε is arbitrarily small this contradicts one of the assumptions and proves sufficiency.

15. *Application.* Take an autonomous system in E^n

$$\dot{x} = X(x), \qquad X(0) = 0, \tag{15.1}$$

behaving as those considered earlier relative to a certain region Φ surrounding the origin. Here M is the origin and it has manifestly a compact neighborhood. We may therefore apply the preceding results and obtain the following conclusions:

(15.2) *A necessary condition for the stability of the origin for* (15.1) *is that no path of the system be issued from the origin* ($\to 0$ as $t \to -\infty$).

(15.3) *A sufficient condition for the instability of the orgin for* (15.1) *is that some path be issued from the origin.*

(15.4) *Necessary conditions for the asymptotic stability of the origin are the condition of* (15.2) *plus this: there exists a neighborhood of the origin free from complete paths of the system.*

V. The Generalized Liapunov Function and Stability

16. Proceeding with the dynamical system $f(p, t)$ in the space R and closed invariant set M, we remove the restriction that M possess a compact neighborhood. By a *generalized Liapunov function* $V(p)$ attached to M is meant a function defined in a certain set $\mathcal{S}(\alpha)$ and with the following properties:

(a) Given any $0 < \varepsilon < \alpha$ there exists a $\lambda > 0$ such that $V > \lambda$ for p outside of $\mathcal{S}(\varepsilon)$.

(b) Given $\lambda > 0$ there exists an $\eta(\lambda) > 0$ such that $V < \lambda$ for $p \in \mathcal{S}(\eta)$.

(c) $V[f(p, t)]$ is, in $\mathcal{S}(\alpha)$, a non-increasing function of t.

In a certain sense V generalizes the concept of a positive definite function with $\dot{V}$ of negative sign.

We have now at once:

(16.1) THEOREM. *A* n.a.s.c. *for the stability of M is the existence of a generalized Liapunov function V.*

PROOF OF SUFFICIENCY. Given ε, take $\eta = \eta(\lambda(\varepsilon))$. Let the path γ start at $p \in \mathcal{S}(\eta)$. Since in $\mathcal{S}(\eta)$, $V < \lambda$ and along the path γ^+ from p the function V is non-increasing, it will never take the value λ and so γ will remain within $\mathcal{S}(\varepsilon)$, or M is stable.

PROOF OF NECESSITY. If $p \in \mathcal{S}(\alpha)$, define $V(p)$ by:

$$V(p) = \sup_{t \geqq 0} d(f(p, t), M).$$

Thus $V(p)$ is the largest distance from M of the path γ_p^+ through p beyond p. It is easily seen that $V(p)$ has all the required properties. In fact, given ε we may take $\lambda(\varepsilon) = \varepsilon$. Then given $\lambda(\leqq \alpha)$ we take $\eta(\lambda)$ as in the definition of stability. And finally V is evidently non-increasing along a path $f(p, t)$ as $t \to +\infty$. This construction of Yoshizawa (after ideas of Okamura) proves necessity. (See Yoshizawa, Memoirs of the College of Sciences, Kyoto, 29, Math., 27-33, (1955).)

(16.2) THEOREM. N.a.s.c. *for the asymptotic stability of M is that there exist a $V(p)$, defined for p in a certain $\mathcal{S}(\alpha)$ and such that $V(f(p, t)) \to 0$ as $t \to +\infty$.*

PROOF OF NECESSITY. At all events since M is stable, for some ε we have a $V(p)$ defined for $p \in \mathcal{S}(\eta(\varepsilon))$. Take the path γ through p and suppose that $V(f(p, t))$, which is non-increasing, does not $\to 0$. It will tend from above to a lower limit λ. There is then a $\zeta(\lambda)$ such that in $\mathcal{S}(\zeta)$ we have $V < \lambda$, while for all $t \geqq 0$, $V(f(p, t)) \geqq \lambda$. Since $\gamma \to M$ it will penetrate and remain in $\mathcal{S}(\zeta)$ for t sufficiently large, and we will then have $V(f(p, t)) < \lambda$. This contradiction proves necessity.

PROOF OF SUFFICIENCY. The existence of $V(p)$ implies stability. If it is not asymptotic there is an ε, η pair and a point p of $\mathcal{S}(\eta)$ whose path γ_p^+ beyond p does not $\to M$. Let it remain outside $\mathcal{S}(\zeta)$. There is then a $\lambda(\zeta) > 0$ such that outside $\mathcal{S}(\zeta)$ we have $V > \lambda$. Hence $V(f(p, t))$ does not $\to 0$ as $t \to +\infty$, and this contradiction proves sufficiency.

(16.3) THEOREM. N.a.s.c. *for instability of M are the existence of a function $V(p)$ defined in a certain $\mathcal{S}(\alpha)$ and with the following properties:*

(a) *$|V(p)|$ is bounded in $\mathcal{S}(\alpha)$.*

(b) *Every $\mathcal{S}(\varepsilon)$ contains at least one point p where $V(p) > 0$.*

(c) *At every point $p \in \mathcal{S}(\alpha)$ along its path the derivative $\dot{V}$ is defined and*

$$\dot{V} = \lambda V + W(p)$$

where $\lambda > 0$ and $W(p)$ is non-negative in $\mathcal{S}(\alpha)$.

PROOF OF NECESSITY. When M is unstable there is an $\varepsilon > 0$ such that no matter how small δ, $\mathcal{S}(\delta)$ contains a point p whose path γ reaches $H(\varepsilon)$. Given then

$p \in \mathcal{S}(\varepsilon)$ there are two possibilities: (i) its path γ remains in $\mathcal{S}(\varepsilon)$; (ii) there is a first time $t(p)$ at which γ reaches $H(\varepsilon)$: $f(p, t(p)) \in H(\varepsilon), f(p, t') \in \mathcal{S}(\varepsilon)$ for $t' \in [0, t(p)]$. In case (i) define $V(p) = 0$; in case (ii) define $V(p) = e^{-t(p)}$. Thus $V(p)$ satisfies both (a) and (b). Now along γ we have $V = e^{t-t(p)}$ and so $\dot{V} = V$, so that (c) holds with $\lambda = 1$, $W = 0$. This proves necessity.

Proof of Sufficiency. Let $V(p)$ exist as stated and suppose that we have stability. Let $\varepsilon \leqq \alpha$ and η correspond as in stability. Thus if $p \in \mathcal{S}(\eta)$ then on its path γ: $| V(f(p, t)) |$ is bounded. Take p such that $V(p) > 0$. Since on γ

$$\dot{V} = \lambda V + W, \qquad V(f(p, 0)) = V(p),$$

integration yields on γ

$$V = V(p)e^{\lambda t} \to +\infty$$

with t, which contradicts the assumption that $| V |$ is bounded on $\mathcal{S}(\alpha) \supset \mathcal{S}(\varepsilon)$. Hence M is unstable.

17. *Some consequences.* If M has a compact neighborhood in R then the n.a.s.c. of (14.2) and (16.1), likewise of (14.3) and (16.2) must be equivalent. Therefore we have:

(17.1) *If M has a compact neighborhood a necessary condition for the existence of a generalized Liapunov function is that no path from outside M has an α-point in M.*

(17.2) *Under the same conditions this Liapunov function V will tend to zero along every path $[V(f(p, t)) \to 0$ as $t \to +\infty]$ if and only if M has a neighborhood free from complete paths of the system* (Krasovskii).

18. An observation of an entirly different nature is the following. Suppose that the space $R - M$ has several components and let U be one of them. Then all the preceding results continue to hold if one replaces R by U and M by $N = M \cap U$. We have then stability asymptotic or otherwise, likewise instability, of M *relative to U alone.* If there are many components $\{U_\mu\}$ then the behaviors in the various components need not be related. In fact the U_μ need *not be* the components of $R - M$ but merely those of some set $\overline{\mathcal{S}(\alpha)} - M$, and everything goes on in the same way.

There are noteworthy and well known applications of the above remarks. Consider a planar system

$$\dot{x} = X(x, y), \qquad \dot{y} = Y(x, y) \tag{18.1}$$

where x, y are now coordinates, and to simplify matters assume that X and Y are analytic wherever considered. Suppose that the system has a limit-cycle γ (isolated periodic solution). Thus there is a neighborhood U of γ free from limit-cycles. Let $M = \gamma$. Then $U - \gamma$ has two components U_1 and U_2. If U_i is stable a path issued from $U_i - \gamma$, for U_i narrow enough, will spiral towards γ;

while if U_i is unstable it will spiral away from γ. Now γ may be

stable on both sides
unstable on both sides
stable on one side, unstable on the other,

and simple examples of each of these cases can be readily given.

Consider again the system (18.1) and suppose that $X(0, 0) = Y(0, 0) = 0$, while X, Y are holomorphic at the origin. Then the curves $X = 0$, $Y = 0$ subdivide a neighborhood of the origin into a finite (arbitrarily large) number of sectors, whose stability behaviors may well be entirely independent.

VI. Uniform Asymptotic Stability

19. We continue to consider the same invariant closed set M but without a necessary neighborhood with compact closure. We operate again within a certain $\mathcal{S}(\alpha)$. Let $\eta(\varepsilon)$ be the stability function. Asymptotic stability means that given any ξ and if ε is small enough then any path γ_p from $p \in \mathcal{S}(\varepsilon)$ will after a certain time T_p remain in $\mathcal{S}(\xi)$. Asymptotic stability is uniform if one may choose a $T_p = T$, the same for all $p \in \mathcal{S}(\varepsilon)$.

(19.1) Theorem. N.a.s.c. *for uniform asymptotic stability of M is the existence of a function $V(p)$ such as in Theorem* 16.2, *which in addition is such that $V(f(p, t)) \to 0$ as $t \to +\infty$, uniformly in p for $p \in \mathcal{S}(\varepsilon)$.*

Proof of Necessity. Suppose that M is uniformly asymptotically stable. There exists then ε (fixed) such that given any small ξ there is a $T(\xi)$ such that for $t \geqq T(\xi)$ we have $f(p, t) \in \mathcal{S}(\xi)$, with T the same for all $p \in \mathcal{S}(\varepsilon)$. The $V(p)$ defined in (16.1) is such that

$$V(f(p, t)) = \sup_{\tau>0} d(f(p, t + \tau), M).$$

Thus for $p \in \mathcal{S}(\varepsilon)$,

$$V(f(p, t)) = \sup_{\theta>t} d(f(p, \theta), M)$$

for $t \geqq T(\xi)$. Thus $V(f(p, t)) \to 0$ as $t \to +\infty$, uniformly in p for $p \in \mathcal{S}(\varepsilon)$.

Proof of Sufficiency. At all events if the conditions are satisfied we have asymptotic stability. Let ε, ξ be as above. If the asymptotic stability is non-uniform, then whatever T, there exists a $p \in \mathcal{S}(\varepsilon)$ such that $f(p, T)$ is outside of $\mathcal{S}(\xi)$. Hence there is a $\lambda(\xi)$ such that $V(f(p, T)) > \lambda(\xi)$, which contradicts the assumption that $V(f(p, t)) \to 0$ as $t \to +\infty$ uniformly in p for $p \in \mathcal{S}(\varepsilon)$. This proves sufficiency.

(19.2) *If M has a neighborhood $\mathcal{S}(\alpha)$ with compact closure then asymptotic stability is always uniform.*

Take $\varepsilon < \alpha$ and ξ arbitrarily small and set $\overline{\mathcal{S}(\varepsilon)} = B$, $\mathcal{S}(\xi) = C$, $\mathcal{S}(\eta(\xi)) = D$. We suppose ε such that given $p \in B$ there is a time T such that $f(p, T) \in D$ and so

$f(p, t) \in C$ for $t \geqq T$. Since B is compact and $f(p, T)$ is continuous in p, if μ is such that $\mathcal{S}(f(p, T), \mu) \subset D$ there is a ν such that whenever $q \in \mathcal{S}(p, \nu) = U$ then $f(q, T) \in D$ and hence $f(q, t) \in C$ for $t \geqq T$. Since B is compact its open covering $\{U\}$ has a finite subcovering $\{U_k\}$ with $T = T_k$ for U_k. Hence if T denotes now the largest T_k then $f(B, t) \in C$ for $t \geqq T$; that is, asymptotic stability is uniform.

As a special case take the system (2.1) in E^n with the origin O asymptotically stable. Since the origin always has a neighborhood with compact closure within the domain Φ of operation of the differential equation, asymptotic stability of O is always uniform and this is generally tacitly assumed.

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