ON MONOIDS AND THEIR DUAL

By Daniel M. Kan

1. Introduction

A monoid (i.e. a multiplicative system which is associative and has a unit) may be considered as a *set* A together with a *multiplication map* $m:A \times A \to A$ satisfying certain axioms. Let $\mathfrak{M}$ denote the category of sets. Then A and $A \times A$ are objects of $\mathfrak{M}$ while the map m is a map in this category. However the axioms are expressed in terms of the elements of the set A and hence make use of the fact that A is a *set* and not merely an object of some category.

In this note we shall formulate the axioms of a monoid in terms of maps of the category $\mathfrak{M}$ without using elements. In fact for an arbitrary category with products $\mathfrak{C}$ we shall define a notion of *object with a monoid structure* which, if $\mathfrak{C}$ is the category of sets (resp. topological spaces), reduces to the usual notion of monoid (resp. topological monoid). Application of this notion to the category of groups yields the following characterization of an abelian group: *a group A is abelian if and only if it admits a monoid structure.*

As the notion of object with a monoid structure is defined entirely in terms of maps of the category $\mathfrak{C}$ it can be dualized. We thus obtain for an arbitrary category with sums $\mathfrak{D}$ a notion of *object with a co-monoid structure.* Application of this notion to the category of groups yields the following characterization of a free group: *a group F is free if and only if it admits a co-monoid structure.* Furthermore a free group admits more than one co-monoid structure; *the co-monoid structures admitted by a free group are in one to one correspondence with its free bases.*

The main lemma of [2] states, roughly speaking, that a functor from countable groups to groups which preserves short exact sequences and finite free products is either trivial or is, in a unique way, naturally equivalent with the inclusion functor. Using the characterizations of abelian groups and free groups mentioned above, the proof of this lemma may be simplified. In particular there is no longer need for the group $Z_2 * Z_2$ to play a special role.

Another application is a characterization of c.s.s. groups which are obtainable by the construction F of J. W. Milnor ([3]).

In an appendix groups and their dual are considered.

Throughout this note we freely use the language of categories and functors of Eilenberg-MacLane ([1]).

2. Objects with a monoid structure

We first specify what we mean by a category with products.

Definition (2.1). A category $\mathfrak{C}$ will be called a *category with products* if for every two objects $A_1, A_2 \in \mathfrak{C}$ there are *given* an object $A_1 \times A_2 \in \mathfrak{C}$ (called *product*) and maps $p_i:A_1 \times A_2 \to A_i \in \mathfrak{C}$ $(i = 1, 2)$ (called *projections*) with the following property. For every two maps $f_i:B \to A_i \in \mathfrak{C}$ $(i = 1, 2)$ there is a unique map

$f:B \to A_1 \times A_2 \in \mathcal{C}$ such that commutativity holds in the diagram

$$\begin{tikzcd} & B \arrow[dl, "f_1"'] \arrow[d, "f"] \arrow[dr, "f_2"] & \\ A_1 & A_1 \times A_2 \arrow[l, "p_1"'] \arrow[r, "p_2"] & A_2 \end{tikzcd}$$

It is readily seen that for every two maps $g_i:C_i \to A_i \in \mathcal{C}$ $(i = 1, 2)$ there is a unique map $g_1 \times g_2:C_1 \times C_2 \to A_1 \times A_2 \in \mathcal{C}$ such that commutativity holds in the diagram

$$\begin{tikzcd} C_1 \arrow[d, "g_1"] & C_1 \times C_2 \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "g_1 \times g_2"] & C_2 \arrow[d, "g_2"] \\ A_1 & A_1 \times A_2 \arrow[l, "p_1"'] \arrow[r, "p_2"] & A_2 \end{tikzcd} \tag{2.1a}$$

Also let $A_i \in \mathcal{C}$ be objects $(i = 1, 2, 3)$ and let

$$s_i:(A_1 \times A_2) \times A_3 \to A_i \qquad \text{and}$$
$$t_i:A_1 \times (A_2 \times A_3) \to A_i$$

be the iterated projections. Then clearly there is a unique equivalence

$$j:(A_1 \times A_2) \times A_3 \to A_1 \times (A_2 \times A_3)$$

such that the following diagram is commutative for $i = 1, 2, 3$.

$$\begin{tikzcd} (A_1 \times A_2) \times A_3 \arrow[d, "j", "\approx"'] \arrow[dr, "s_i"] & \\ A_1 \times (A_2 \times A_3) \arrow[r, "t_i"'] & A_i \end{tikzcd}$$

We may therefore identify the objects $(A_1 \times A_2) \times A_3$ and $A_1 \times (A_2 \times A_3)$ under the map j and denote the resulting object by $A_1 \times A_2 \times A_3$.

DEFINITION (2.2). Let $\mathcal{C}$ be a category with products. An object $P \in \mathcal{C}$ is called a *point* if for every object $A \in \mathcal{C}$ there is exactly one map $A \to P \in \mathcal{C}$.

This terminology may be justified by the following proposition, the proof of which is straightforward.

PROPOSITION (2.3). Let $\mathcal{C}$ be a category with products and let $P \in \mathcal{C}$ be a point. Then for every object $A \in \mathcal{C}$ the projections $p_1:A \times P \to A$ and $p_2:P \times A \to A$ are equivalences.

EXAMPLES (2.4). Examples of categories with products are:

a) The category $\mathfrak{M}$ of sets. The product (in the above sense) of two sets is what is usually called their direct product. Any set consisting of only one element is a point.

b) The category $\mathfrak{I}$ of topological spaces. The product of two topological spaces is what is usually called their cartesian product. Any space consisting of one point is a point.

c) The category $\mathfrak{G}$ of groups. The product of two groups is their direct product. Any group consisting of only one element is a point.

We shall suppose that in every category with products there is *given* a point P. This is no real restriction, as a category $\mathcal{C}$ which has no points may always be enlarged to a category with a point by adding one object P (the point), its identity map, and for every object $A \in \mathcal{C}$ one map $A \to P$.

We now define the notion of an object with a monoid structure.

DEFINITION (2.5). Let $\mathcal{C}$ be a category with products. An object $A \in \mathcal{C}$ will be called an *object with a monoid structure* if there is given a map $m:A \times A \to A \in \mathcal{C}$ (called *multiplication map*) satisfying the following axioms.

I. *Associativity*: Commutativity holds in the diagram

$$\begin{tikzcd} A \times A \times A \arrow[r, "m \times i_A"] \arrow[d, "i_A \times m"] & A \times A \arrow[d, "m"] \\ A \times A \arrow[r, "m"] & A \end{tikzcd}$$

II L. *Existence of a left unit*: There exists a map $e:P \to A \in \mathcal{C}$ such that commutativity holds in the diagram

$$\begin{tikzcd} A \arrow[r, "(p_2)^{-1}", "\approx"'] \arrow[d, "i_A", "\approx"'] & P \times A \arrow[d, "e \times i_A"] \\ A & A \times A \arrow[l, "m"'] \end{tikzcd}$$

II R. *Existence of a right unit*: There exists a map $e':P \to A \in \mathcal{C}$ such that commutativity holds in the diagram

$$\begin{tikzcd} A \arrow[r, "(p_1)^{-1}", "\approx"'] \arrow[d, "i_A", "\approx"'] & A \times P \arrow[d, "i_A \times e'"] \\ A & A \times A \arrow[l, "m"'] \end{tikzcd}$$

EXAMPLES (2.6)

a) A set with a monoid structure clearly is a monoid in the usual sense.

b) Similarly a topological space with a monoid structure is a topological monoid.

Let A be a group and let $m:A \times A \to A$ be a homomorphism. In order that m satisfies the axioms II L and II R m has to be the multiplication map in the usual sence. But this map is a homomorphism if and only if A is abelian. Hence

THEOREM (2.7). A group A is abelian if and only if it admits a monoid structure. If A is abelian it admits exactly one monoid structure, the multiplication map of which coincides with the multiplication map of A in the usual sense.

Let A be a monoid. Then the multiplication map $m:A \times A \to A$ is an epimorphism. Also, if e is a left unit of A and e' a right unit, then $e = ee' = e'$. Similar results hold for objects with a monoid structure.

PROPOSITION (2.8). Let A be an object with a monoid structure. Then the multiplication map $m:A \times A \to A$ is an epimorphism, i.e. if $f_1, f_2:A \to B$ are maps such that $f_1 \circ m = f_2 \circ m$, then $f_1 = f_2$.

PROOF. In view of axiom II L

$$\begin{aligned} f_1 &= f_1 \circ m \circ (e \times i_A) \circ (p_2)^{-1} \\ &= f_2 \circ m \circ (e \times i_A) \circ (p_2)^{-1} = f_2 . \end{aligned}$$

PROPOSITION (2.9). Let A be an object with a monoid structure and let $e:P \to A$ be a left unit and $e':P \to A$ a right unit. Then $e = e'$.

PROOF. Axiom II L yields the commutative diagram

$$\begin{tikzcd} P \arrow[r, "d", "\approx"'] \arrow[d, "e'"] & P \times P \arrow[d, "i_P \times e'"] \arrow[dd, bend left=70, "e \times e'"] \\ A \arrow[r, "(p_2)^{-1}", "\approx"'] \arrow[d, "i_A", "\approx"'] & P \times A \arrow[d, "e \times i_A"] \\ A & A \times A \arrow[l, "m"'] \end{tikzcd}$$

where $d:P \to P \times P$ is the unique such map (proposition (2.3)). Hence

$$m \circ (e \times e') \circ d = i_A \circ e' = e'.$$

Similarly axiom II R yields

$$m \circ (e \times e') \circ d = e.$$

Hence $e = e'$.

Remark (2.10). In the proof of proposition (2.8) we used only the fact that A has a left (or right) unit; in the proof of proposition (2.9) also associativity was not needed.

3. Objects with a co-monoid structure

The definitions of the preceding section may be dualized as follows.

Definition (3.1). A category $\mathfrak{D}$ will be called a *category with sums* if for every two objects $A_1, A_2 \in \mathfrak{D}$ there are *given* an object $A_1 + A_2 \in \mathfrak{D}$ (called *sum*) and maps $j_i : A_i \to A_1 + A_2 \in \mathfrak{D}$ $(i = 1, 2)$ (called *injections*) with the following property. For every two maps $f_i : A_i \to B \in \mathfrak{D}$ $(i = 1, 2)$ there is a unique map $f : A_1 + A_2 \to B \in \mathfrak{D}$ such that commutativity holds in the diagram

$$\begin{tikzcd} & B & \\ A \arrow[ur, "f_1"] \arrow[r, "j_1"] & A_1 + A_2 \arrow[u, "f"] & A_2 \arrow[l, "j_2"'] \arrow[ul, "f_2"'] \end{tikzcd}$$

Again it is readily seen that for every two maps $g_i : A_i \to C_i \in \mathfrak{D}$ $(i = 1, 2)$ there is a unique map $g_1 + g_2 : A_1 + A_2 \to C_1 + C_2 \in \mathfrak{D}$ such that the dual of diagram (2.1a) is commutative. Also $(A_1 + A_2) + A_3$ and $A_1 + (A_2 + A_3)$ will be identified under the canonical isomorphism and the resulting object will be denoted by $A_1 + A_2 + A_3$.

Definition (3.2). Let $\mathfrak{D}$ be a category with sums. An object $Q \in \mathfrak{D}$ is called *empty* if for every object $A \in \mathfrak{D}$ there is exactly one map $Q \to A \in \mathfrak{D}$.

This terminology may be justified by the following dual of proposition (2.3).

Proposition (3.3). Let $\mathfrak{D}$ be a category with sums and let $Q \in \mathfrak{D}$ be an empty object. Then for every object $A \in \mathfrak{D}$ the injections $j_1 : A \to A + Q$ and $j_2 : A \to Q + A$ are equivalences.

Examples (3.4). Examples of categories with sums are

a) The category $\mathfrak{M}$ of sets. The sum of two sets $A_1, A_2 \in \mathfrak{M}$ is what is usually called their union $A_1 \cup A_2$. The empty set is an empty object.

b) The category $\mathcal{G}$ of groups. The sum of two groups A_1, A_2 in the sense of definition (3.1) is what is usually called their *free product* $A_1 * A_2$. Any group consisting of only one element is an empty object.

We shall suppose that in every category with sums $\mathfrak{D}$ there is given an empty object Q. Again this is no real restriction on $\mathfrak{D}$. We then may define

Definition (3.5). Let $\mathfrak{D}$ be a category with sums. An object $A \in \mathfrak{D}$ will be called an *object with a co-monoid structure* if there is given a map $n : A \to A + A \in \mathfrak{D}$ (called *co-multiplication map*) satisfying the following axioms.

I* *Associativity*: Commutativity holds in the diagram

$$\begin{tikzcd} A + A + A & A + A \arrow[l, "n + i_A"'] \\ A + A \arrow[u, "i_A + n"] & A \arrow[l, "n"] \arrow[u, "n"'] \end{tikzcd}$$

II*L *Existence of a left co-unit*: There exists a map $a: A \to Q \in \mathfrak{D}$ such that commutativity holds in the diagram

$$\begin{tikzcd} A & Q + A \arrow[l, "(j_2)^{-1}"', "\approx"] \\ A \arrow[u, "i_A"', "\approx"] \arrow[r, "n"] & A + A \arrow[u, "a + i_A"'] \end{tikzcd}$$

II*R *Existence of a right co-unit*: There exists a map $a': A \to Q \in \mathfrak{D}$ such that commutativity holds in the diagram

$$\begin{tikzcd} A & A + Q \arrow[l, "(j_1)^{-1}"', "\approx"] \\ A \arrow[u, "i_A"', "\approx"] \arrow[r, "n"] & A + A \arrow[u, "i_A + a'"'] \end{tikzcd}$$

Dualizing proposition (2.8) and (2.9) we get

PROPOSITION (3.6). Let A be an object with a co-monoid structure. Then the co-multiplication map $n: A \to A + A$ is a monomorphism.

PROPOSITION (3.7). Let A be an object with a co-monoid structure with left co-unit $a: A \to Q$ and right co-unit $a': A \to Q$. Then $a = a'$.

We now consider the possibilities of converting the objects of the categories $\mathfrak{M}$ (sets) and $\mathcal{G}$ (groups) into objects with a co-monoid structure.

THEOREM (3.8). Let $A \in \mathfrak{M}$ be a non-empty set. Then A does not admit a co-monoid structure.

PROOF. This is an immediate consequence of axiom II*L and the non-existence of maps $A \to Q \in \mathfrak{M}$.

THEOREM (3.9). Let $F \in \mathcal{G}$ be a free group and let the subset $B \subset F$ be a free basis of F. Then the map $n: F \to F * F$ given by $n(b) = j_1(b) \cdot j_2(b)$ for all $b \in B$, defines a co-monoid structure on F.

The proof of theorem (3.9) is straightforward.

THEOREM (3.10). Let $F \in \mathcal{G}$ be an object with a co-monoid structure and let $n:F \to F * F$ be the co-multiplication map. Then F is a free group and the set $B = \{b \mid b \in F, b \neq 1, n(b) = j_1(b) \cdot j_2(b)\}$ is a free basis of F.

COROLLARY (3.11). A group $F \in \mathcal{G}$ is free if and only if it admits a co-monoid structure.

COROLLARY (3.12). Let $F \in \mathcal{G}$ be a free group. Then there exists a one to one correspondence between the co-monoid structures admitted by F and its free bases.

PROOF OF THEOREM (3.10). Suppose B is a basis of F and let w be a non-empty reduced word in the elements of B and their inverses. Then it follows from the definition of B that $n(w) \neq 1$. Hence $w \neq 1$, i.e. B is a free basis of F. It thus remains to show that B generates F.

Let $p \in F$, $p \neq 1$. Then there exists an integer k and elements $b_1, q_2, \cdots, q_k, r_1, \cdots, r_k \in F$ such that

$$n(p) = j_1(b_1) \cdot j_2(r_1) \cdot j_1(q_2) \cdots j_1(q_k) \cdot j_2(r_k)$$

and such that $q_2, \cdots, q_k, r_1, \cdots, r_{k-1}$ are $\neq 1$. If $k = 1$ then it follows immediately from axiom II*L and II*R that $b_1 = r_1 = p$, i.e. $p \in B$. Now suppose $k > 1$. Let

$$n(b_1) = j_1(s_1) \cdot j_2(t_1) \cdots j_1(s_m) \cdot j_2(t_m)$$

$$n(r_1) = j_1(u_1) \cdot j_2(v_1) \cdots j_1(u_n) \cdot j_2(v_n).$$

By hypothesis $r_1 \neq 1$ and in view of axiom II*L we may therefore suppose that $v_1 \neq 1$. Application of axiom I* yields

$$j_1(b_1) \cdot j_2(u_1) \cdot j_3(v_1) \cdots j_2(u_n) \cdot j_3(v_n) \cdots$$
$$= j_1(s_1) \cdot j_2(t_1) \cdots j_1(s_m) \cdot j_2(t_m) \cdot j_3(r_1) \cdots .$$

This implies $n(b_1) = j_1(b_1) \cdot j_2(u_1)$. Application of axiom II*L now yields $u_1 = b_1$, i.e. $b_1 \in B$ or $b_1 = 1$. Consequently

$$n(b_1^{-1} \cdot p) = j_2(c_1) \cdot j_1(q_2) \cdots j_1(q_k) \cdot j_2(r_k)$$

where $c_1 = b_1^{-1} \cdot r_1$. By the same method, using axiom II*R instead of axiom II*L we obtain that $n(c_1) = j_2(c_1) \cdot j_1(c_1)$, i.e. $c_1^{-1} \in B$ or $c_1 = 1$ and hence

$$n(c_1^{-1} \cdot b_1^{-1} \cdot p) = j_1(b_2) \cdot j_2(r_2) \cdots j_1(q_k) \cdot j_2(r_k)$$

where $b_2 = c_1^{-1} \cdot q_2$. It now follows by induction that there exist elements $b_1, \cdots, b_{k-1}, c_1^{-1}, \cdots, c_{k-1}^{-1} \in F$ which are in B or $= 1$ such that

$$n(p') = j_1(b_k) \cdot j_2(r_k)$$

where

$$p' = c_{k-1}^{-1} \cdot b_{k-1}^{-1} \cdots c_1^{-1} \cdot b_1^{-1} \cdot p.$$

By axiom II*L and II*R we have that $p' = b_k = r_k$, i.e. $p' \in B$ or $p' = 1$. Hence

p is in the subgroup of F generated by B. As p was an arbitrary element of F it follows that F is generated by B.

4. On the proof of lemma 1 of [2]

We recall lemma 1 of [2].

Let $\mathcal{G}$ be the category of groups, $\mathcal{G}_c$ the subcategory of countable groups and let $I:\mathcal{G}_c \to \mathcal{G}$ denote the inclusion functor. Let $M:\mathcal{G}_c \to \mathcal{G}$ be a functor which preserves

a) *short exact sequences (i.e. extensions)*

b) *(two-fold) free products. Then either there exists a unique neutral equivalence $n:I \to M$ or $M(A) = 1$ for all $A \in \mathcal{G}_c$.*

The proof of this lemma may be divided as follows.

(i) *If $A \in \mathcal{G}_c$ is abelian, then so is $M(A)$.* This is proved by first showing that M preserves direct products ([2], proposition 10). It follows that M preserves monoid structures and hence the result follows by applying the characterization of abelian groups of theorem (2.7). In this part of the proof no use is made of the fact that M preserves free products.

(ii) *If $A \in \mathcal{G}_c$ is free, then so is $M(A)$.* As A is free it admits a co-monoid structure (theorem (3.9)) which clearly is preserved by the functor M. Hence $M(A)$ is free (theorem (3.10)). This part of the proof uses only the fact that M preserves free products.

(iii) *If Z is infinite cyclic, then either $M(Z)$ is infinite cyclic or $M(Z) = 1$.* This follows immediately from (i) and (ii).

(iv) *If $M(Z)$ is infinite cyclic, then there exists a unique natural equivalence $m:I \to M$.* The problem is to choose for the isomorphism $m(Z)$ one of the two possible isomorphisms $Z \to M(Z)$. Let $z \in Z$ be a generator, then z induces on Z a co-monoid structure (theorem (3.9)). Application of the functor M yields a co-monoid structure on $M(Z)$. Denote by z' the corresponding generator of $M(Z)$ (theorem (3.10)). It is readily seen that the isomorphism $m(Z):Z \to M(Z)$ should be compatible with the co-monoid structures of Z and $M(Z)$. Hence this must be the isomorphism given by $(m(Z))z = z'$. This is the isomorphism h of [2], proposition 14. The uniqueness and existence of the natural equivalence $m:I \to M$ then can be shown as in [2], propositions 15 and 16.

(v) *If $M(Z) = 1$, then $M(A) = 1$ for all $A \in \mathcal{G}_c$.* This is proposition 17 of [2] and need not be changed.

5. A topological application

For a c.s.s. complex K with base point φ let FK be the c.s.s. group of J. W. Milnor ([3]) which has the homotopy type of the loops on the suspension of K. We recall its definition. For every integer $n \geqslant 0$ F_nK is the group which has

(i) one generator $F\sigma$ for every simplex $\sigma \in K_n$

(ii) one relation $F(\varphi\eta^0 \cdots \eta^{n-1}) = 1$.

The face and degeneracy homomorphisms are given by

$$(F\sigma)\varepsilon^i = F(\sigma\varepsilon^i) \qquad (F\sigma)\eta^i = F(\sigma\eta^i) \qquad 0 \leqslant i \leqslant n.$$

The following theorems then are immediate consequences of theorem (3.9) and (3.10).

THEOREM (5.1). The c.s.s. homomorphism $n:FK \to FK * FK$ given by $n(F\sigma) = j_1(F\sigma) \cdot j_2(F\sigma)$ for every $\sigma \in K$ defines on FK a co-monoid structure.

THEOREM (5.2). Let G be a c.s.s. group with a co-monoid structure and let $n:G \to G * G$ be the co-multiplication map. Then

(i) the simplices $\sigma \in G$ for which $n(\sigma) = j_1(\sigma) \cdot j_2(\sigma)$ form a c.s.s. complex L

(ii) the c.s.s. homomorphism $h:FL \to G$ given by $h(F\sigma) = \sigma$ is an isomorphism which is compatible with the co-monoid structures.

COROLLARY (5.3). Let $\mathcal{S}$ be the category of c.s.s. complexes with base point and let $\mathfrak{F}$ be the category of c.s.s. groups with a co-monoid structure. Then the functor $F:\mathcal{S} \to \mathfrak{F}$ is an isomorphism of categories.

APPENDIX

6. Groups and their dual

Analogously to definition (2.5) the notion of an object with a group structure may be defined.

DEFINITION (6.1). Let $\mathcal{C}$ be a category with products. An object $A \in \mathcal{C}$ will be called an *object with a group structure* if there is *given* a map $m:A \times A \to A$ (called *multiplication map*) satisfying the following axioms.

I *Associativity* (see definition (2.5)).

II L *Existence of a left unit* (see definition (2.5)).

III L *Existence of a left inverse*: There exists a map $t:A \to A \in \mathcal{C}$ such that commutativity holds in the diagram

(6.1a)
$$\begin{tikzcd} A \arrow[r] \arrow[d, "d"] & P \arrow[r, "e"] & A \\ A \times A \arrow[rr, "t \times i_A"] & & A \times A \arrow[u, "m"'] \end{tikzcd}$$

where $d:A \to A \times A \in \mathcal{C}$ is the unique map such that commutativity holds in the diagram

(6.1b)
$$\begin{tikzcd} & A \arrow[dl, "i_A"'] \arrow[d, "d"] \arrow[dr, "i_A"] & \\ A & A \times A \arrow[l, "p_1"'] \arrow[r, "p_2"] & A \end{tikzcd}$$

EXAMPLES (6.2)

a) A set with a group structure clearly is a group.

b) A topological space with a group structure is a topological group.

c) The monoid structure of theorem (2.7) is actually a group structure.

Many of the elementary properties of groups also hold for objects with a group structure, for instance, the uniqueness and two-sidedness of unit and inverse.

PROPOSITION (6.3). Let A be an object with a group structure. Then

a) there exists a right unit $e':P \to A$ (see definition (2.5))

b) $e = e'$

c) there exists a right inverse, i.e. a map $t':A \to A \in \mathfrak{C}$ such that commutativity holds in diagram (6.1a) with e' instead of e and $i_A \times t'$ instead of $t \times i_A$.

d) $t = t'$.

The proof is similar to that of proposition (2.9) although more complicated. Dually we get

DEFINITION (6.4). Let $\mathfrak{D}$ be a category with sums. An object $A \in \mathfrak{D}$ will be called an *object with a co-group structure* if there is *given* a map $n:A \to A + A \in \mathfrak{D}$ (called *co-multiplication map*) satisfying the following axioms

I* *Associativity* (see definition (3.5)).

II*L *Existence of a left co-unit* (see definition (3.5))

III*L *Existence of a left co-inverse*: There exists a map $s:A \to A \in \mathfrak{D}$ such that commutativity holds in the diagram

(6.4a)
$$\begin{tikzcd}
A & Q \arrow[l] & A \arrow[l, "a"'] \arrow[d, "n"] \\
A + A \arrow[u, "b"] & & A + A \arrow[ll, "s + i_A"']
\end{tikzcd}$$

where $b:A + A \to A \in \mathfrak{D}$ is the unique map such that commutativity holds in the dual of diagram (6.1b).

EXAMPLES (6.5). The co-monoid structures of theorem (3.9) and (5.1) are actually co-group structures.

Dualization of proposition (6.3) yields that co-unit and co-inverse are also two-sided and unique.

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