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ON THE GENERALISED RIEMANN-HURWITZ FORMULA

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Introduction

Let M be a compact orientable manifold of dimension $2m$, with fundamental class $[M]$ and let E be a complex vector bundle of rank n on M, with Chern classes $\{c_i(E)\}$, $0 < i \leqq n$. If P is a homogeneous polynomial of weight m in n indeterminates, then the associated Chern number is defined to be $\langle P(c_1(E), \cdots, c_n(E)), [M]\rangle = \langle P(E), [M]\rangle$.

Let M_1 be a closed submanifold of M of co-dimension r. We consider another complex vector bundle F of rank n on M, and assume given a morphism ϕ: $E \rightarrow F$ of vector bundles such that $\psi = \phi|_{M-M_1}:E|_{M-M_1} \rightarrow F|_{M-M_1}$ is an isomorphism. Let $S(M_1)$ be the 'double' of a tubular neighbourhood $B(M_1)$ of M_1 in M (see below): $S(M_1)$ is the sphere bundle of $\nu \oplus 1$ where ν is the normal bundle of the embedding $M_1 \subset M$.

We may construct the 'clutched' bundle (E, ψ, F) over $S(M_1)$ as follows: consider the restriction of E to one copy of $B(M_1)$, and F to another copy of $B(M_1)$ and attach them by means of the 'clutching' isomorphism ψ on the boundary of $B(M_1)$.

In [11], Ngô Van Quê proved the *generalised Riemann-Hurwitz formula*:

$$\langle P(E), [M]\rangle - \langle P(F), [M]\rangle = \langle P(E, \psi, F), [S(M_1)]\rangle,$$

explicitly in terms of Chern forms and integration. It is remarked that the techniques work when the Chern forms are replaced by Pontrjagin or Euler forms, in the case of real vector bundles, and the same formula is obtained.

The purpose of this paper is to state and prove a result in the characteristic ring of two G-bundles E and F where G is any group admitting a classifying space BG. Within this context, our main result turns out to be a considerable generalisation of that stated above. We also discuss some applications to the special case of branched covering maps and exemplify our discussion with two examples outside of the smooth category.

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§1. The main result

We assume given a smooth orientable n-manifold M and an r-submanifold M_1 with $n - r \geqq 2$. Then the pair (M, M_1) satisfies

1) $$H_q(M, \mathbb{Z}) = \begin{cases} 0 & q > n \\ \mathbb{Z} & q = n \end{cases},$$

2) $$H_q(M_1, \mathbb{Z}) = 0 \qquad \text{if } q \geqq n-1,$$

3) there exists a neighbourhood $N(M_1)$ of M_1 so that M_1 is a deformation retract of the interior $N^0(M_1)$ of $N(M_1)$ and the inclusion $p: M \to (M, M - N^0)(M_1))$ induces an isomorphism

(1.1) $$p_*: H_n(M) \xrightarrow{\cong} H_n(M, M - N^0(M_1)).$$

We form the subspace $\mathscr{K}$ of $M \times I$ where

(1.2) $$\mathscr{K} = (M \times \partial I) \cup ((M - N^0(M_1)) \times I)$$

and the double $S(M_1) \subset \mathscr{K}$ by

(1.3) $$S(M_1) = (\partial N(M_1) \times I) \cup (N(M_1) \times \partial I)$$

Now let

$$\mathscr{K}_1 = (M \times \{0\}) \cup ((M - N^0(M_1)) \times [0, \tfrac{3}{4}])$$

and

$$\mathscr{K}_2 = (M \times \{1\}) \cup ((M - N^0(M_1)) \times [\tfrac{1}{4}, 1])$$

and let $S(M_1)_i = S(M_1) \cap \mathscr{K}_i \qquad i = 1, 2.$

By this construction, the spaces $\mathscr{K}_i$ are homotopically equivalent to M_1 and the spaces $\mathscr{K}/\mathscr{K}_1$ and $S(M_1)/S(M_1)_1$ are both homotopically equivalent to the Thom space $M/(M - N^0(M_1))$.

It follows from the cofibration

(1.4) $$S(M_1)_1 \to S(M_1) \to S(M_1)/S(M_1)_1$$

and 2) above that

$$H_n(S(M_1), \mathbb{Z}) \cong H_n(S(M_1)/S(M_1)_1) \cong \mathbb{Z}.$$

More generally, a pair of CW-complexes (M, M_1) will be called an (n, Λ)-*adapted pair* for any coefficient ring Λ if it satisfies the above conditions 1) – 3) with Λ replacing $\mathbb{Z}$. A choice of generators for $H_n(M, \Lambda)$ and $H_n(S(M_1), \Lambda)$ will be called an *orientation* of M and $S(M_1)$ respectively.

Remark. Examples arise from the following two cases: we could take M to be an orientable manifold and M_1 to be a codimension $\geqq 2$ subcomplex, or even more generally, M could be taken to be an orientable simple n-circuit [3] and M_1 to be an arbitrary subcomplex of codimension $\geqq 2$.

Now let us consider a topological group G with its classifying space BG. If $P \in H^q(BG, \Lambda)$ is a cohomology class and E is a G - bundle over a space X, then we shall denote by $P(E) \in H^q(X, \Lambda)$ the class defined by $P(E) = f_E^*(P)$ where $f_E : X \to BG$ is the classifying map. Also, if $\tau \in H_q(X, \Lambda)$, then we denote the Kronecker pairing by $\langle P(E), \tau \rangle$. We now proceed to our main result:

THEOREM (1.1). *Suppose (M, M_1) is an (n, Λ)-adapted pair and E and F are G-bundles over M such that on $M - M_1$ there exists a homotopy*

$$f_E|_{M-M_1} \underset{\sim}{\theta} f_F|_{M-M_1}. \tag{1.5}$$

Then there exists a bundle ξ_θ over $S(M_1)$ and orientations $[M]$ and $[S(M_1)]$ such that for any class $P \in H^n(BG, \Lambda)$, we have the following equality of Kronecker pairings

$$\langle P(\xi_\theta), [S(M_1)]\rangle = \langle P(E) - P(F), [M]\rangle \tag{1.6}$$

Proof. Henceforth, we assume all cohomology and homology groups to have coefficients in Λ.

For $i = 1, 2$, let us consider $\mathscr{K}$ and $\mathscr{K}_i$ $\quad i = 1, 2$.

We have $\mathscr{K}_1 \cap \mathscr{K}_2 = (M - N^0(M_1)) \times [\frac{1}{4}, \frac{3}{4}]$. The Mayer-Vietoris sequence for $(\mathscr{K}_1, \mathscr{K}_2)$ gives

$$\begin{gathered} \cdots\cdots \to H_n(\mathscr{K}_1 \cap \mathscr{K}_2) \overset{(i_1,i_2)_*}{\to} H_n(\mathscr{K}_1) \oplus H_n(\mathscr{K}_2) \to H_n(\mathscr{K}) \\ \overset{\partial_*}{\to} H_{n-1}(\mathscr{K}_1 \cap \mathscr{K}_2) \overset{(i_1,i_2)_*}{\to} H_{n-1}(\mathscr{K}_1) \oplus H_{n-1}(\mathscr{K}_2) \to \cdots\cdot \end{gathered} \tag{1.7}$$

We claim that the maps $(i_1, i_2)_*$ and ∂_* in dimensions n are the 0-maps and hence we have the isomorphism

$$\mu_* : H_n(\mathscr{K}_1) \oplus H_n(\mathscr{K}_2) \overset{\cong}{\to} H_n(\mathscr{K}). \tag{1.8}$$

This follows from the sequence

$$\begin{gathered} \cdots\cdot \to H_n(M - N^0(M_1)) \overset{i_*}{\to} H_n(M) \overset{p_*}{\to} H_n(M, M - N^0(M_1)) \\ \overset{\partial_*}{\to} H_{n-1}(M - N^0(M_1)) \overset{i_*}{\to} H_{n-1}(M) \to \cdots\cdots . \end{gathered}$$

Since $i_{t^*} = i_*$, for $t = 1, 2$, then from (1.1) i_* is 0 in dimension n and a monomorphism in dimension $n - 1$.

Let $[M]$ be an orientation for M; it induces generators $[M]_0$ and $[M]_1$ for $H_n(\mathscr{K})$ by pushing forward $[M]$; we have the sequence

$$H_n(M) \overset{(t_1,t_2)_*}{\to} H_n(\mathscr{K}_1) \oplus H_n(\mathscr{K}_2) \overset{\mu_*}{\to} H_n(\mathscr{K})$$

where for $i = 1, 2$, $t_i : M \to \mathscr{K}_i$ is the inclusion $X \to X \times \{i\}$. Consider now the map $g : \mathscr{K} \to M$ given by $g(x, t) = x$.

Then $g_*[M]_i = [M]$. Now the restriction g_1 of g to $S(M_1)$ sends $S(M_1)$ to M_1 and we have the induced maps

$$\begin{array}{ccc} H_n(S(M_1)) & \xrightarrow{j_*} & H_n(\mathscr{K}) \\ g_{1*}\downarrow & & \downarrow g_* \\ H_n(M_1) & \longrightarrow & H_n(M) \end{array} \tag{1.9}$$

If $[S(M_1)]$ is a generator, then for $a_0,\ a_1 \in \mathbb{R}$, we have

$$j_*[S(M_1)] = a_0[M]_0 + a_1[M]_1$$

and since $g_*j_* = g_{1*}j_* = 0$, it follows that $a_1 = -a_0$. Moreover, consider the diagram with maps j_1, s and h

$$\begin{array}{ccccc} S(M_1)_1 & \longrightarrow & S(M_1) & \xrightarrow{j_1} & S(M_1)/S(M_1)_1 \\ & & g\downarrow & & \downarrow s \\ \mathscr{K}_{1a} & \longrightarrow & \mathscr{K} & \xrightarrow{h} & \mathscr{K}/\mathscr{K}_1 \end{array} \tag{1.10}$$

where $S(M_1)_1 = S(M_1) \cap \mathscr{K}_1$. Then s is a homotopy equivalence and each space has the homotopy type of $N(M_1)/\partial N(M_1)$. On passing to the level of homology we have

$$\begin{array}{ccc} H_n(S(M_1)) & \xrightarrow{j_*} & H_n(\mathscr{K}) \\ \cong\ \downarrow j_{1*} & & \downarrow h_* \\ H_n(SM_1))/S(M_1) & \underset{\cong}{\xrightarrow{s_*}} & H_n(\mathscr{K}/\mathscr{K}_1) \end{array} \tag{1.11}$$

and obtain $h_*j_*[S(M_1)] = a_0h_*[M]_0$. But $h_*j_* = s_*j_{1*}$ is an isomorphism, hence a_0 is a unit in Λ and $h_*[M])_0$ is a generator of $H_n(\mathscr{K}/\mathscr{K}_1)$.

Finally, if we look at

$$H_n(S(M_1)) \xrightarrow{j_{1*}} H_n(S(M_1)/S(M_1)_1) \underset{\cong}{\xrightarrow{s_*}}$$
$$H_n(\mathscr{K}/\mathscr{K}_1) \underset{\cong}{\xrightarrow{i_0}} H_n(M,\ M - N^0(M_1)) \underset{\cong}{\xleftarrow{k_*}} H_n(M),$$

we can choose generators $[S(M_1)]$ and $[M]$ such that $s_*j_{1*}[S(M_1)] = i_{0*}k_*[M]$, and $a_0 = 1$.

We now proceed to construct a bundle over $\mathscr{K}$ as follows: we define $f_\theta : \mathscr{K} \to BG$ to be f_E on $M \times \{0\}$ f_F on $M \times \{1\}$, and f_θ on $(M - N^0(M_1)) \times I$.

Let $\overline{\xi}_\theta$ be the corresponding bundle over $\mathscr{K}$ and $\xi_\theta = j_* \overline{\xi}_\theta$ the induced bundle

Now taking account of (1.6), (1.15) and Lemma 1.2, we obtain (see also [11]).

$$(1.17) \qquad \langle \chi(E), [M] \rangle - \langle \chi(F), [M] \rangle = b\langle \chi(L), [M_1] \rangle.$$

§2. Applications to branched covering maps and examples

We shall speak of a (finite) topological branched covering as that defined by Fox in [5]. In fact, such a branched covering can be seen as a special case of a *singular fibration*: let X and Y be topological spaces and $\pi: X \to Y$ a map. Let Y_1 be a closed subset of Y such that

$$(2.1) \qquad \pi^{-1}(Y - Y_1) \to Y - Y_1$$

is a fibration of some specified type, e.g. a Serre or Hurewicz fibration. We call the set $\{\pi^{-1}(y) : y \in Y_1\}$ the *singular fibres.*

Such a singular fibration $\pi: X \to Y$ can be expressed as a composition of singular fibrations

$$X \xrightarrow{\pi_1} Z \xrightarrow{\pi_2} Y$$

where Z is the identification space of X where the singular fibres are identified to points and where for π_1, the non-singular fibres are thus identified with points and for π_2, the singular fibres consist of points.

In the case that a compact Lie group G acts on X, then we obtain a singular fibration $X \to X/G$ where

$$(2.2) \qquad X - \cup_H X_{(H)} \to (X - \cup_H X_{(H)})/G$$

is a principal G-fibration, where $X_{(H)}$ is a principal orbit for H which is conjugate to a subgroup of each isotropy group of G. This may be seen to be a special case of [2, Theorem 3.2, p. 182] (in this case, the group K in Bredon's notation, is taken to be the identity).

For branched coverings we consider a map $\pi: M \to N$ where M and N are equidimensional orientable combinatorial manifolds with branch set N_1 (connected) which is taken to be a pure codimension 2 simplificial complex, tamely embedded in the interior of N. Let $M_1 = \pi^{-1}(N_1)$(M_1 is sometimes called 'the ramification locus' of π); we assume that M_1 and N_1 are combinatorially homeomorphic. By the *degree of* π, we mean the number of pre-images of a point $x \in N - N_1$. We do not assume orientability of N_1.

Considering the special, equi-dimensional case of (2.1), we would set $M = X$ and $N = X/G$ where G is considered to be cyclic of order k and acts freely on $M - M_1$ where M_1 is the fixed point set of the action of G. Here π has degree equal to k. Such (cyclic) branched coverings have been discussed by Hirzebruch in [7].

So let us proceed to set $E = TM, F = \pi^*TN$, assume that M_1 is of codimension 2 in M and take π to have degree equal to k.

Further, let $K_1 = \nu(M_1)$ and $K_2 = \pi^{-1}\nu(N_1)$ (here ν denotes the normal bundle). Thus for K in (1.15), we consider the 'clutched' bundle

$$(2.3) \qquad K = (q_1{}^*\nu(M_1),\ \eta,\ q_2{}^*\pi^*\nu(N_1)).$$

THEOREM (2.1). *Let $\pi:M \to N$ be a smooth branched covering map of degree k. Then with respect to the fibration.*

$$S^2 \xrightarrow{i} S(M_1) \xrightarrow{q} M_1 \tag{2.4}$$

there exists a bundle isomorphism $\xi \cong q^*TM_1 \oplus K$ where K is an orientable 2-dimensional bundle defined by (2.3) *such that*

$$\langle \chi(K), i_*[S^2] \rangle = \pm (k-1). \tag{2.5}$$

Proof. This is a special case of Theorem 1.1 where we take ξ to be ξ_θ and the result follows from (1.15) and from an explicit description of the clutching map

$$q_1{}^*K_1 \mid_{\partial B(M_1)} \xrightarrow{\eta} q_2{}^*K_2 \mid_{\partial B(M_1)}$$

where q_i, $i = 1, 2$, are the restrictions in (2.4) to $B_i(M_1)$.

Let $x \in M_1$ and let U be a neighbourhood of x and let V be a neighbourhood of $\eta(x)$ in M_1.

Now for $y \in S^1 \subset D^2$ and $z \in q^*TM_1$, we define the clutching map η by

$$(x, y, z) = (x, y, ky^{k-1}z) \tag{2.6}$$

where the first two components constitute the identity and the third component represents the differential of the map. But the explicit description in (2.6) actually reduces to describing a map $S^1 \to SO(2)$ which, in this case, is none other than $y \to y^{k-1}$, whence we deduce that with respect to (2.4).

$$\langle \chi(K), i_*[S^2] \rangle \text{ is } (k-1) \text{ to within a sign.} \qquad \square$$

We shall choose (2.5) to have a minus sign. Note that generally, K may not be unique but the evaluation of $\chi(K)$ on $i_*[S^2]$ is. Thus using (1.17), we recover the formula of [11] for a smooth branched covering map $\pi:M \to N$ of degree k with branch set N_1:

$$\chi(M) = k\chi(N) - (k-1)\chi(N_1). \tag{2.7}$$

However, the restriction that π should be smooth can, in fact, be weakened and (2.7) is true in the combinatorial category. To see this, let us recall that for a fibration $F \to X \to B$, we have $\chi(X) = \chi(F)\chi(B)$. Thus commencing from (2.1) and using the fact that

$$\chi(M - M_1) = \chi(M) + (-1)^{c+1}\chi(M_1)$$

where c denotes the codimension of M_1 in M, we can deduce (2.7) in the combinatorial category. But even more generally, we assert the following:

THEOREM (2.2). *Let M and N be topological manifolds and let $\phi:M - M_1 \to N - N_1$ a k-fold covering where M_1 and N_1 are subspaces of codimension c_1 and c_2 respectively such that*

$$c = c_1 \equiv \text{mod}(2) \text{ and } \chi(M_1) = \chi(N_1).$$

Then we have

$$\chi(M) = k\chi(N) + (-1)^{c+1}(k-1)\chi(N_1). \tag{2.8}$$

Proof. By Lefschetz duality and the exactness of the triangle

$$\begin{array}{ccc} H_*(M) & \longleftarrow & H_*(M_1) \\ & \searrow \quad \nearrow & \\ & H_*(M, M_1) & \end{array}$$

we deduce that $\chi(M - M_1) \cong \chi(M, M_1) \cong \chi(M) + (-1)^{c_1+1}\chi(M_1)$.

But also, we have

$$\begin{aligned} \chi(M - M_1) &= k\chi(N - N_1) \\ &= k\chi(N) + (-1)^{c_2}k\chi(N_1), \end{aligned}$$

whence we deduce (2.8). □

We now proceed to discuss two examples of topological branched coverings from outside of the smooth category in order to exemplify how the topology is regulated by (2.8) for the case $c = 2$.

Example (2.3). In [8] Kuiper considers the quotient space Y of $\mathbb{C}P^2$ under the identification of complex conjugation. The quotient map $\pi:\mathbb{C}P^2 \to Y$ is a piecewise-smooth branched covering of degree 2 with branch set $\mathbb{R}P^2$. The possibility of global smoothness is lost on the branch set.

We deduce from (2.8) that $\chi(Y) = 2$. Once we can show that Y is simply-connected, we can then deduce that Y is a homotopy 4-sphere and then apply the results of Freedman [6] to deduce that Y is actually S^4. Thus we need only to establish:

LEMMA (2.4). *The space* $Y \cong \mathbb{C}P^2/(\text{complex conjugation})$ *is simply-connected.*

Proof. Consider the isomorphic presentation

$$\mathbb{C}P^1 \times \mathbb{C}P^1/(a, b) \sim (b, a) \overset{T}{\cong} \mathbb{C}P^2$$

given by $T([z_1, z_2], [w_1, w_2]) = [z_1w_2, z_1w_2 + z_2w_1, z_2w_2]$.

The following diagram commutes

$$\begin{array}{ccc} \mathbb{C}P^1 \times \mathbb{C}P^1/\sim & \xrightarrow{T} & \mathbb{C}P^2 \\ J \downarrow & & \downarrow I \\ \mathbb{C}P^1 \times \mathbb{C}P^1/\sim & \longrightarrow & \mathbb{C}P^2 \end{array}$$

where $I[z_0, z_1, z_2] = [\bar{z}_0, \bar{z}_1, \bar{z}_2]$ and $J[a, b] = [\bar{a}, \bar{b}]$. Then we have $\mathbb{C}P^2/I = \{\{a, b\}, \{\bar{a}, \bar{b}\}: a, b, \in \mathbb{C}\}$ $(\cong (\mathbb{C}P^1 \times \mathbb{C}P^1/\sim)/J)$

Now consider the map $\gamma: S^1 \to \mathbb{C}P^2/I$ given by

$$\gamma(t) = \{\{\gamma_1(t), \gamma_2(t)\}, \quad \{\bar{\gamma}_1(t), \bar{\gamma}_2(t)\}\}.$$

By a codimension argument, we may assume that $\gamma_1(t), \gamma_2(t) \neq \infty\ \forall t \in S^1$. Consider then the homotopy, for $s \in [0, 1]$, given by

$$\Gamma_s(t) = \{\{s\gamma_1(t), s\gamma_2(t)\}\{s\bar{\gamma}_1(t), s\bar{\gamma}_2(t)\}\}$$

where $\Gamma_1 = \gamma$ and $\Gamma_0 =$ constant. Exhibiting such a homotopy suffices to show that Y is simply-connected. □

Example (2.5). This is an example of an n-circuit pair (M, M_1) where M_1 is not locally flat in M but nevertheless (2.8) applies.

Let $n_1, \cdots, n_m$ be integers $\geqq 1$. Consider the complex hypersurfaces $p^{-1}(0) \subset \mathbb{C}^m$ defined by the weighted homogeneous polynomial $p(z_1, \cdots, z_n) = z_1{}^{n_1} + \cdots + z_m{}^{n_m}$. Then following [9], we see that $V = V(n_1, \cdots, n_m) = p^{-1}(0)$ has the origin as its unique singularity. Furthermore, V intersects each sphere $S(r)^{2m-1} = \{(z_1, \cdots, z_m) \in \mathbb{C}^m: \sum_{i=1}^m |z_i|^2 = r^2\}$ transversally. To see this, assume that $(z_1, \cdots, z_n) \in V$ is a point different from the origin and consider the map $\tau: \mathbb{R}^+ \to \mathbb{C}^m$ defined by $\tau(t) = (t^{\alpha_1}z_1, t^{\alpha_2}z_2, \cdots, t^{\alpha_m}z_m)$ where $\alpha_i = \Pi_{j\neq i} n_j$.

Then $\tau(t)$ lies in V for all $t > 0$. The function $t \to \|\tau(t)\|^2$ is strictly increasing since $(d/dt)\|\tau(t)\|^2 > 0$.

Hence V meets $S^{2m-1}(r)$ transversally. Let $\tilde{V} = V \cap D^{2m} = \{(z_1, \cdots, z_m) \in V: \sum_{i=1}^m |z_i|^2 \leqq 1\}$.

Then it follows from [9] that V is homeomorphic to the cone $\mathscr{C}V_1(n_1, \cdots, n_m)$ where $V_1(n_1, \cdots, n_m)$ is the Brieskorn manifold $V_1(n_1, \cdots, n_m) = V \cap S^{2m-1}$.

In fact, the pair $(\mathscr{C}S^{2m-1}(1), \mathscr{C}V(n_1, \cdots, n_m))$ is homeomorphic to the pair $(D^{2m}, \tilde{V})$. Consider now the map $\pi_0: \tilde{V} \subset \mathbb{C}^m \to D^{2m-2} \subset \mathbb{C}^{m-1}$ given by

$$\pi_0(z_1, \cdots, z_m) = (z_2, \cdots, z_m).$$

Then π_0 is a branched covering of degree n_1, with ramification locus

$$W = \{(z_2, \cdots, z_m) \in \mathbb{C}^{m-1}: z_2{}^{n_2} + \cdots z_m{}^{n_m} = 0, \sum_{i=2}^m |z_i|^2 \leqq 1\}.$$

Consider the double $S(\tilde{V})$ of V obtained by taking two disjoint copies $\tilde{V}^+ = \tilde{V}\times\{0\}$, $\tilde{V}^- = \tilde{V}\times\{1\}$ of $\tilde{V}$, by identifying $(x, 0)$ with $(x, 1)$, for all $x \in V_1(n_1, \cdots, n_m)$. Then by a standard construction [10], we can provide $S(W)$, with a differentiable structure in the complement of the two points that correspond to the origin (i.e. (0, 0) (0, 1)). Consider $\pi: S(\tilde{V}) \to D^{2m-2}$ where $2D^{2m-2} \simeq S^{2m-2}$ is the double of D^{2m-2} constructed in the obvious fashion. This describes $S(\tilde{V})$ as a branched covering of order m over the sphere S^{2m-2} with ramification locus the double of W, $S(W)$. We see that W is a topological manifold if and only if $V_1(n_1, \cdots, n_m)$ is a homotopy sphere (which may, in many cases, be exotic).

If we consider the particular case $V_1(5, 3, 2, 2, 2)$. Then we obtain a map $\pi: S^8 \to S^8$ which is a 5-fold PL-branched covering for some PL-triangulation of S^8.

The ramification locus of such a covering is a PL triangulated 6-sphere embedded in S^8 in a knotted fashion (since its complement is a manifold that fibres over S^1 with fibre a bouquet of eight 4-spheres).

To exemplify the fact that the degree of the map and the topological type of the branch set N_1 does not in fact impose topological conditions on N, we consider the following interesting observation. Let $\pi: \mathbb{C}P^2 \to N$ be a topological branched covering of degree $k \geqq 0$, where N is a simply connected combinatorial 4-manifold and the branch set of π, N_1, is a compact orientable surface (possibly with singularities). Then applying (2.8), we have

$$3 = k(2 + b_2(N_1)) + (2g - 2)(k - 1) \tag{2.9}$$

where $b_2(N)$ is the second betti number of N and g here denotes the genus of N_1. Now (2.9) only makes sense for $k = b_2(N) = 1$. Thus we deduce that π is, in fact, forced to be a homeomorphism.

We conclude with a further observation. Assume given a topological branched covering $\pi: \mathbb{H}P^n \to \mathbb{H}P^n$ (quaternionic projective space of n-dimensions) that is actually the restriction of a map $\mathbb{H}H^\infty \to \mathbb{H}P^\infty$. Then following [4], such a map has degree $k = (2p + 1)^2$, where $p \in \mathbb{Z}$. The formula (2.8) thus tells us that the branch set N_1 must have $\chi(N_1) = n + 1$.

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