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BILINEAR FORMS VERSUS GALOIS REPRESENTATIONS

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§1

Throughout this paper all fields will have characteristic different from two. Furthermore a bilinear form $\beta: V \times V \to K$ will mean a non-singular, symmetric bilinear form which is finite dimensional over the field, K. K^* denotes $K - \{0\}$ and K^{**} denotes the subgroup of squares. An element, $a \in K^*$, determines a bilinear form $\langle a \rangle: K \times K \to K$ defined by $\{(x, y) \mapsto axy\}$ and depends up to isomorphism, only on $a \in K^*/K^{**}$.

If $H^i(K, Z/2)$ denotes Galois cohomology [S] there is an isomorphism $l: K^*/K^{**} \xrightarrow{\cong} H^1(K; Z/2)$ where if $g \in G(\bar{K}/K)$, the Galois group of the separable closure, $\bar{K}$, over K then $l(a)(g) = g(\sqrt{a})/\sqrt{a} \in \{\pm 1\}$. A form, (V, β), has Hasse-Witt characteristic classes [D]

$$(1.1) \qquad HW_i(V, \beta) \in H^i(K; Z/2)$$

defined as the i-th symmetric function of $l(a_1), \cdots, l(a_n)$ where $(V, \beta) \cong \langle a_1 \rangle \oplus \cdots \oplus \langle a_n \rangle$.

Now let us turn to orthogonal Galois representations, $\rho: G(N/K) \to O_m(K)$. Here N/K is a finite Galois extension with group $G(N/K)$ and $O_m(K) = \{X \in GL_m(K); XX^t = I\}$. Such a representation determines a continuous 1-cocycle $\rho\pi: G(\bar{K}/K) \xrightarrow{\pi} G(N/K) \xrightarrow{\rho} O_m(K) \to O_m(\bar{K})$ where π is the canonical map. This cocycle gives a class, $(\rho) \in H^1(K; O_m(\bar{K}))$, the Galois cohomology group which classifies bilinear forms of rank m over K [S; S2, pp. 152/3]. The quadratic form (ρ) can be shown to equal Fröhlich's bilinear form defined in [F, §2]. This gives rise to Hasse-Witt classes, $\mathrm{HW}_i(\rho)$. On the other hand $H^*(O(\bar{K}); Z/2)$ [Su; Ka] is a polynomial ring on Stiefel-Whitney classes, w_i. Pulling back these classes gives rise to Stiefel-Whitney class of ρ,

$$(1.2) \qquad \mathrm{SW}_i[\rho] = (\rho\pi)^*(w_i) \in H^i(K; Z/2).$$

In addition ρ possesses a spinor class [F]

$$(1.3) \qquad \mathrm{Sp}[\rho] \in H^2(K; Z/2)$$

defined in the following manner. Let $\theta: O_m(K) \to K^*/K^{**}$ denote the spinor norm [$O'M$, p. 137]. Define

$$\mathrm{Sp}[\rho] \in \mathrm{Hom}(G(N/K), K^*/K^{**}) \cong H^1(G(N/K); Z/2) \otimes H^1(K; Z/2)$$

by $\hat{\mathrm{Sp}}[\rho](g) = \theta(\rho(g))$ and set $\mathrm{Sp}[\rho]$ equal to the image of $\hat{\mathrm{Sp}}[\rho]$ under the cup-product.

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(1.4.) By way of illustration let us consider for a moment the case when K is a number field.

It is well-known that the discriminant, $\mathrm{HW}_1(\rho)$, the Hasse-Witt invariant, $\mathrm{HW}_2(\rho)$, the rank and the signatures at the real places of K, determine the bilinear form, (ρ), up to isomorphism. However, for $i \geq 3$, $H^i(K; Z/2)$ is the sum of copies of $Z/2$, one for each real place. In addition if $(\rho) \otimes K_v \cong q\,\langle -1\rangle \oplus (m-q)\,\langle 1\rangle$ at a real completion K_v then the K_v-component of $\mathrm{HW}_i(\rho)$ is $\binom{q}{i}$ (mod 2). Hence the $\{\mathrm{HW}_i(\rho),\ i \geq 1\}$ determine (ρ) up to isomorphism, which illustrates the usefulness of (1.5) below. Incidentally, any bilinear form is represented by a representation ρ in the above manner, for any K of characteristic different from two.

The following result can be deduced from our Theorem 1.7 but we will give a direct proof.

THEOREM (1.5). *If K is a number field then these classes are related as follows*

$$\mathrm{HW}_i(\rho) = \begin{cases} \mathrm{SW}_i(\rho), & \text{if } i \neq 2 \\ \mathrm{SW}_2(\rho) + \mathrm{Sp}[\rho], & \text{if } i = 2. \end{cases}$$

Proof: It is well-known (c.f. [$S3$]) that $\mathrm{HW}_1(\rho) = \mathrm{SW}_1(\rho)$ while the formula in dimension two is proved in [F, §3.1]. For dimensions $i \geq 3$ the inclusions of K into its real completions, K_v, induces an isomorphism

$$H^i(K; Z/2) \xrightarrow{\cong} \oplus_v\, H^i(K_v; Z/2) \cong \oplus_v\, (Z/2).$$

Consequently, it will suffice to show that $\mathrm{HW}_i(\rho) = \mathrm{SW}_i(\rho)$ for all $i \geq 1$ when $K = \mathbb{R}$ and $G(\overline{K}/K) \cong Z/2$ generated by τ (complex conjugation).

Any homomorphism from a finite 2-group into $O_{2n}(\mathbb{R})$ may be conjugated, by a result of Borel-Serre (see [Ta]) to land in the wreath product $\Sigma_n \int O_2(\mathbb{R})^n$ generated by the diagonal 2×2 blocks, $O_2(\mathbb{R})^n$, and the permutation matrices which, by conjugation, permute these blocks. Since a homomorphism, $Z/2 \to \Sigma_n$ is conjugate to a homomorphism of the form $Z/2 \xrightarrow{\text{diagonal}} (Z/2)^r = (\Sigma_2)^r \subset \Sigma_{2r} \to \Sigma_n$ we may conjugate $\rho(\tau) \in O_{2n}(\mathbb{R})$ into $\Sigma_n \int O_2(\mathbb{R})$ and then by a permutation to have the form

$$\rho(\tau) = \begin{pmatrix} A_1 & & & & \\ & A_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \end{pmatrix}$$

where $A_i \in O_2(\mathbb{R})$ or $A_i = \begin{pmatrix} 0 & X \\ X^{-1} & 0 \end{pmatrix} \in \Sigma_2 \int O_2(\mathbb{R})$. In the latter case $\begin{pmatrix} 0 & X \\ I & 0 \end{pmatrix} A_i \begin{pmatrix} 0 & I \\ X^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ which is in turn conjugate to a diagonal matrix of (± 1)'s. Therefore, by conjugations in $O_{2n}(\mathbb{R})$ which leaves $\mathrm{HW}_i(\rho)$ and $\mathrm{SW}_i(\rho)$ unaltered, we may suppose ρ is the sum of representations

$$\rho_\alpha : Z/2 = G(\mathbb{C}/\mathbb{R}) \to O_\epsilon(\mathbb{R})(\epsilon = 1 \quad \text{or} \quad 2),$$

$$\rho = \oplus_\alpha \, \rho_\alpha$$

Since $\mathrm{HW}_i(\rho' \oplus \rho'') = \sum_m \mathrm{HW}_m(\rho')\mathrm{HW}_{i-m}(\rho'')$, and similarly for SW_i, it suffices to show that $\mathrm{HW}_i(\rho_\alpha) = \mathrm{SW}_i(\rho_\alpha)$ for $i = 1$ and 2. As remarked earlier, this is well-known when $i = 1$ and from [F, §3.1)] $\mathrm{HW}_2(\rho_\alpha) = \mathrm{SW}_2(\rho_\alpha) + \mathrm{Sp}[\rho_\alpha]$, so we must show that $\mathrm{Sp}[\rho_\alpha] = 0 \in H^2(\mathbb{R}); Z/2)$, when $\dim \rho_\alpha = 2$.

If $\det \rho_\alpha(\tau) = 1$ then

$$\rho_\alpha(\tau) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad \text{with} \quad a^2 + b^2 = 1, \qquad ab \neq 0$$

and $\mathrm{Sp}[\rho_\alpha] = 0$ since the spinor norm of $\rho_\alpha(\tau) = \dfrac{a+1}{2}$ [$O'M$, p. 137], which is a square in $\mathbb{R}$. Similarly, if $\det \rho_\alpha(\tau) = -1$ then

$$\rho_\alpha(\tau) = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \quad \text{with} \quad a^2 + b^2 = 1, \qquad ab \neq 0$$

so that the spinor norm of $\rho_\alpha(\tau)$ is the same (since 2 is a square in $\mathbb{R}$) as that of

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ b & -a \end{pmatrix} = \begin{pmatrix} b & -a \\ a & b \end{pmatrix}$$

which is $\left(\dfrac{b+1}{2}\right)$ by [$O'M$, p. 137] which is again a square in $\mathbb{R}$.

1.6.

Write $\mathrm{HW} = 1 + \mathrm{HW}_1 + \mathrm{HW}_2 + \cdots$ and $\mathrm{SW} = 1 + \mathrm{SW}_1 + \mathrm{SW}_2 + \cdots$ for the total Hasse-Witt and Stiefel-Whitney classes, respectively. In many cases ρ, as an element of the Grothendieck group of orthogonal K-representations, is the formal sum of representations induced up from one- and two-dimensional representations (see, for example [S4; *De*; *Ta*]). This justifies interest in the following formula, which generalizes those of [S2; F, §3.1; K]. Let $\mathrm{Ind}_{E/K}(\rho)$ denote the representation of $G(N/K)$ induced from ρ on $G(N/E)$.

Inside the special orthogonal group, $SO_2(K)$, let C_n denote the cyclic group of order n. The matrix $\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in O_2(K)$ acts on $x \in C_n$ by the formula $\tau x \tau = x^{-1}$ and we have the wreath product $Z/2 \ltimes C_n \subset O_2(K)$. This is the dihedral group of order $2n$.

We say that $\rho : G \to O_\epsilon(K)$ ($\epsilon = 1$ or 2) is special dihedral if either $\epsilon = 1$ or if $\epsilon = 2$ and $\mathrm{im}\,(\rho) \subset Z/2 \ltimes C_n$.

THEOREM (1.7). *Let $N \supset E \supset K$ be a chain of finite extensions of fields in* *which N/K is Galois and* char $(K) \neq 2$. *Let $\rho: G(N/E) \to O_\epsilon(K)$ be a special dihedral representation of dimension $\epsilon = 1$ or 2.*

Then, in $H^*(K; Z/2)$,

$$\mathrm{HW}(\mathrm{Ind}_{E/K}(\rho)) = \{\mathrm{SW}[\mathrm{Ind}_{E/K}(\rho)]\}(1 + \mathrm{Sp}[\mathrm{Ind}_{E/K}(\rho)]).$$

1.8.
Remarks.
(i) If $\mathrm{Tr}_{E/K}$ denotes the Galois cohomology transfer then

$$\mathrm{Sp}[\mathrm{Ind}_{E/K}(\rho)] = \mathrm{Tr}_{E/K}(\mathrm{Sp}[\rho]) + (\mathrm{rank}\ \rho) l(2) d_{E/K} \tag{1.9}$$

where $d_{E/K} \in H^1(K; Z/2)$ is the discriminant of E/K. This formula is derived in [*F*, §4, Theorem 6(ii)]. Also if rank$(\rho) = 1$, then $0 = \mathrm{Sp}[\rho]$.

(ii) The representation $\mathrm{Ind}_{E/K}(\rho)$ represents the bilinear form which is the Scharlau transfer of the form represented by ρ.

(iii) By the induction theorems of [*De*; S4; *Ta*] Theorem 1.7 is sufficient to determine $\mathrm{HW}(\rho)$ for any $\rho: G(N/K) \to O_m(\mathbb{R})$, where $K \supset \mathbb{R}$, the real field, in terms of $\mathrm{SW}(\rho)$ and suitable spinor classes.

(iv) In fact, the proof of Theorem 1.7 actually yields the following stronger form.

THEOREM. *If ρ is a virtual, orthogonal representation of $G(N/K)$ of the form $\rho = \sum_i n_i \rho_i$ with $\rho_i = \mathrm{Ind}_{E_i/K}(\rho_i{}')$ and $\rho_i{}'$ is a* 1- *or* 2-*dimensional special dihedral representation, as in Theorem* 1.7, *then*

$$\mathrm{HW}(\rho) = \mathrm{SW}[\rho](1 + \mathrm{Sp}[\rho]) \in H^*(K; Z/2).$$

1.10.
In [S3] Serre discovered the first formulae of the type of §1.7 when dim $\rho = 1$ in dimensions less than or equal to three. Fröhlich generalized the two dimensional formula to arbitrary representations in [*F*] and in [*K*] B. Kahn found the formula in arbitrary dimensions when dim $\rho = 1$. I am very grateful to these authors for their correspondence and their preprints. I am particularly grateful to Bruno Kahn for providing me with the counterexample given in §5 and for numerous other helpful comments.

I will prove Theorem 1.7 by means of Koslowski's transfer [*Kos*] when E/K is Galois and $[E:K] = 2^j$. This is done in §3. The general case is deduced in §4 using a double coset formula argument.

§2. The Spinor class of a 2-dimensional representation

Through this section N/K will be a finite Galois extension for which $G(N/K)$ is a 2-group. Let $\rho: G(N/K) \to O_2(K)$ be an orthogonal representation. If $\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then $O_2(K) \cong Z/2 \ltimes SO_2K$ where

$$SO_2(K) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix};\ a^2 + b^2 = 1 \right\}$$

and τ generates $Z/2$. Suppose that $\operatorname{im}(\rho) \subseteq Z/2 \ltimes C_{2^j}$ where $C_{2^j} \subset SO_2(K)$ is the cyclic group of order 2^j.

We have homomorphisms ($i = 1, 2$) $x_i : Z/2 \ltimes C_{2^j} \to Z/2$ given by composing the i-th projection with the canonical map

$$1 \ltimes r : Z/2 \ltimes C_{2^j} \to Z/2 \ltimes C_2 \cong Z/2 \times C_2.$$

Since

$$H^1(Z/2 \ltimes C_{2^j}; Z/2) \cong \operatorname{Hom}(Z/2 \ltimes C_{2^j}; Z/2)$$

we have

$$H^1(Z/2 \ltimes C_{2^j}; Z/2) \cong \begin{cases} Z/2 \oplus Z/2\ \langle x_1, x_2\rangle & \text{if } j \geq 1 \\ Z/2\langle x_1\rangle & \text{if } j = 0. \end{cases} \tag{2.1}$$

Suppose that $\operatorname{im}(\rho) \cap SO_2(K) = C_{2^j}$ with $j \geq 2$ with generator $g_j = \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix}$ then, since $\theta(\tau^\epsilon g^\nu) = 2^\epsilon \left(\frac{1 + \alpha_j}{2}\right)^\nu$ [$O'M$, p. 137] we see that

$$\operatorname{Sp}[\rho] = \begin{cases} (x_1 + x_2)l(2) + x_2 l(1 + \alpha_j) & \text{if } j \geq 2, \\ x_1 l(2) \quad \text{if } j = 1 \quad \text{or } j = 0. \end{cases} \tag{2.2}$$

The equation $b^2 + (1 + a)^2 = 2(1 + a)$, if $a^2 + b^2 = 1$, shows that $l(2)l(-1) = 0 = l(\alpha_j + 1)l(-1)$ so that

$$l(-1)\operatorname{Sp}[\rho] = 0 \quad \text{in} \quad H^3(K; Z/2). \tag{2.3}$$

LEMMA (2.4). *For $\rho : G(N/K) \to O_2(K)$ as in* §2.1

$$\operatorname{HW}(\rho) = \operatorname{SW}[\rho](1 + \operatorname{Sp}[\rho]) \text{ in } H^*(K; Z/2).$$

Proof. It is well-known that $\operatorname{HW}_1(\rho) = \operatorname{SW}_1[\rho]$ and by [F; §3.1] $\operatorname{HW}_2(\rho) = \operatorname{SW}_2[\rho] + \operatorname{Sp}[\rho]$. Since $\operatorname{Sp}[\rho]$ and $\operatorname{HW}_2(\rho)$ are sums of products of one dimensional classes and $l(Z)^2 = l(Z)l(-1)$, $0 = \operatorname{Sq}^1(\operatorname{HW}_2(\rho)) = \operatorname{Sq}^1\operatorname{SW}_2[\rho] = \operatorname{SW}_1[\rho]\operatorname{SW}_2[\rho]$ by the *Wu* formulae. Also, $\operatorname{Sq}^1(\operatorname{HW}_2(\rho)) = \operatorname{HW}_1(\rho)\operatorname{HW}_2(\rho)$ so the formula $\operatorname{HW}_2(\rho) = \operatorname{SW}_2[\rho] + \operatorname{Sp}[\rho]$ implies that $0 = \operatorname{SW}_1[\rho]\operatorname{Sp}[\rho]$. It remains to show that $\operatorname{Sp}[\rho]\operatorname{HW}_2[\rho] = 0$.

There exist $a, b \in K^*$ so that

$$\begin{cases} \operatorname{SW}_1[\rho] = l(a) + l(b) \quad \text{and} \\ l(a)l(b) = \operatorname{HW}_2(\rho). \end{cases} \tag{2.5}$$

Since $\operatorname{SW}_1[\rho]\operatorname{Sp}[\rho] = 0$ we may multiply by $l(a)$ to obtain

$$0 = Sp[\rho]l(a)l(b) + \operatorname{Sp}[\rho]l(a)l(-1) = \operatorname{Sp}[\rho]\operatorname{HW}_2(\rho)$$

by (2.3).

§3. Koslowski's transfer and the spinor class

3.1.

Let X be a topological space, in applications it will be the classifying space of a finite Galois group.

Let $G(X)$ denote $\prod_{i\geq 1} H^i(X; Z/2)$ considered as the subset of $\prod_{i\geq 0} H^i(X; Z/2)$ of elements having one as their zero component. $G(X)$ is made into a group by means of the cup-product. We will use only *connected* spaces, X, in which case, following [*Kos*], we set

$$\hat{G}(X) = Z \times G(X) \tag{3.2}$$

with product given by $(m, a)(n, b) = (m + n, ab)$.

If $\pi: X \to Y$ is a double covering of connected spaces there is a transfer [*Kos*, §2]

$$\hat{N}_{X/Y}: \hat{G}(X) \to \hat{G}(Y) \tag{3.3}$$

which is a homomorphism, natural for double covers and satisfying for each vector bundle, $V \to X$,

$$\hat{N}_{X/Y}(\dim V, \mathrm{SW}(V) = (2 \dim V, \mathrm{SW}(\mathrm{Ind}_{X/Y}(V))), \tag{3.4}$$

where SW is the total Stiefel-Whitney class and $\mathrm{Ind}_{X/Y}(V)$ is the induced bundle, whose fibre at $y \in Y$ is $V_{x_1} \oplus V_{x_2}$ if $\{x_1, x_2\} = \pi^{-1}(y)$.

3.5.

In this section we will study the following problem. Suppose that $K = K_n \subset K_{n-1} \subset \cdots, \subset K_0 = E$ is a chain of quadratic extensions and that $\rho: G(N/E) \to O_2(K)$ is, as in §2.1, a special Galois representation of a 2-group, $G(N/E)$. We will establish the following formula. Write $\hat{N}_{E/K}$ for $\hat{N}_{K_{n-1}K} \cdot \hat{N}_{K_{n-2}/K_{n-1}} \cdot \cdots \hat{N}_{E/K_1}$.

LEMMA (3.6). *In the situation of* §3.5

$$\hat{N}_{E/K}(0, 1 + \mathrm{Sp}[\rho]) = (0, 1 + \mathrm{Sp}[\mathrm{Ind}_{E/K}(\rho)]),$$

in $\hat{G}(K)$, the Galois cohomology analogue of $\hat{G}(X)$.

3.7.

For finite groups $i: A \subset D$ of index two $\hat{N}_{BA/BD}$ is defined as the composite of the pretransfer of $[K - P]$, ϕ^*, with a map $\hat{D}$ as follows

$$\hat{N}_{BA/BD}: \hat{G}(BA) \xrightarrow{\hat{D}} \hat{G}(B(\Sigma_2 \ltimes (A^2)) \xrightarrow{\phi^*} \hat{G}(BD).$$

From [*Kos*, §2.2/2.4], if $0 \neq w \in H^1(B\Sigma_2; Z/2)$ and if $\dim(x) = 2$,

$$\hat{D}(0, 1 + x) = (0, 1 + \mathrm{Tr}(1 \otimes x) + P(x)(1 + w)^{-2}) \tag{3.8}$$

where $\mathrm{Tr}: H^*(BA; Z/2) \to H^*(BD; Z/2)$ is the transfer and $P: H^i(BA; Z/2) \to H^{2i}(B(\Sigma_2 \ltimes (A)^2); Z/2)$ is the power map used to define Steenrod operations [$S - E$]. P satisfies (see [*K*, Prop I.2.5(*d*)])

$$\begin{cases} P(u + v) = P(u) + P(v) + \mathrm{Tr}(u \otimes v) \\ \phi^*(P(u))(z^2 + zd) = \phi^* P(ui^*(z)) \end{cases} \tag{3.9}$$

$u, v \in H^*(BA; Z/2)$, $z \in H^1(BD; Z/2)$, where d is the image of $o \neq w \in H^1(B\Sigma_n; Z/2)$ in $H^1(BD; Z/2)$.

3.10.

Proof of §3.6. We will use the formulae of (2.2) for $\mathrm{Sp}[\rho]$. However, firstly we observe that if $j \geq 3$ then $\sqrt{2} \in K$ so that $l(2) = 0$ while if $j = 2$, $\alpha_2 = 0$ and $l(1 + \alpha_j) = 0$. Hence we will suppose, by induction on the length of the chain of quadratic extensions, that we have shown

$$\hat{N}_{E/K_m}(0, 1 + \mathrm{Sp}[\rho]) = (0, 1 + \mathrm{Sp}[\mathrm{Ind}_{E/K_m}(\rho)]) \tag{3.11}$$

and that

$$\mathrm{Sp}[\mathrm{Ind}_{E/K_m}(\rho)] = \begin{cases} Z_2 l(1 + \alpha_j) & \text{if } j \geq 3 \\ (Z_1 + Z_2) l(2) & \text{if } j = 2 \\ Z_1 l(2) & \text{if } j = 0, 1 \end{cases} \tag{3.12}$$

for some $Z_1, Z_2 \in H^1(K_m; Z/2)$. Therefore by the second equation in (3.9)

$$\phi^* P(\mathrm{Sp}[\mathrm{Ind}_{E/K_m}(\rho)]) = \begin{cases} \phi^*P(Z_2)(l(1 + \alpha_j)^2 + l(1 + \alpha_j)d) & \text{if } j \geq 3 \\ \phi^*P(Z_1 + Z_2)(l(2)^2 + l(2)d) & \text{if } j \geq 2 \\ \phi^*P(Z_1)(l(2)^2 + l(2)d) & \text{if } j = 0, 1 \end{cases}$$

where d is the discriminant of K_m/K_{m-1}. However these expressions are zero since $\phi^*P(Z)d = 0$ by [K, Prop II.1.4] and the squares $l(1, \alpha_j)^2$ and $l(2)^2$ are zero, as remarked in §2. Hence, by (3.8) and (3.11)

$$\begin{aligned} \hat{N}_{E/K_{m+1}}(0, 1 + \mathrm{Sp}[\rho]) &= (0, 1 + \mathrm{Tr}_{K_m/K_{m+1}}(\mathrm{Sp}[\mathrm{Ind}_{E/K_m}(\rho)])) \\ &= (0, 1 + \mathrm{Sp}[\mathrm{Ind}_{E/K_{m+1}}(\rho)]), \end{aligned}$$

by (1.9) as $\dim(\mathrm{Ind}_{E/K_m}(\rho))$ is even. This completes the induction step.

We are now prepared to prove Theorem 1.7 in the special circumstances of §3.5.

PROPOSITION (3.13). *Let $N \supset E = K_0 \supset K_1 \supset \cdots \supset K_n = K$ be a chain of extensions as in* §3.5 *and let $\rho: G(N/E) \to O_\epsilon(K)$, $\epsilon = 1$ or 2, be a special dihedral Galois representation, as in* §2.1. *Then*

$$\mathrm{HW}(\mathrm{Ind}_{E/K}(\rho)) = \mathrm{SW}[\mathrm{Ind}_{E/K}(\rho)]\{1 + \mathrm{Sp}[\mathrm{Ind}_{E/K}(\rho)]\} \quad \text{in} \quad H^*(K; Z/2).$$

3.14.

Remark. When $\dim \rho = 1$ we easily find, from (1.9), that $\mathrm{Sp}[\mathrm{Ind}_{E/K}(\rho)] = l(2)d_{E/K}$ where $d_{E/K} \in H^1(K; Z/2)$.

3.15.

Proof of 3.13. If $n = 0$ there is nothing to prove and if $n = 1$, $\epsilon = 1$ the formula follows from (1.9) and §2.4 applied to $\mathrm{Ind}_{E/K_1}(\rho)$. Hence we may assume $\epsilon = 2$. In this case

$$\hat{N}_{E/K}(2, \mathrm{HW}(\rho)) = (2^{n+1}, \mathrm{HW}(\mathrm{Ind}_{E/K}(\rho))),$$

by (Sn, §3.9(ii)]. On the other hand, by §2.4,

$$\begin{aligned}\hat{N}_{E/K}(2,\ \mathrm{HW}(\rho)) &= \hat{N}_{E/K}((2,\ \mathrm{SW}[\rho])(0,\ 1+\mathrm{Sp}[\rho]))\\ &= (2^{n+1},\ \mathrm{SW}[\mathrm{Ind}_{E/K}(\rho)](0,\ 1+\mathrm{Sp}[\mathrm{Ind}_{E/K}(\rho)])\end{aligned}$$

by (3.4) and §3.6. Equating second coordinates of these two expressions yields the required formula.

§4

4.1.

Let $N \supset E \supset K$ be a chain of field extensions as in the statement of Theorem 1.7. Let $\rho: G(N/E) \to O_\epsilon(K)$ ($\epsilon = 1$ or 2) be an orthogonal representation. Since, when proving Theorem 1.7, we will be calculating in $H^*(K; Z/2)$ we may assume that the Galois extension, N/E, contains all over-fields of E that appear in the course of the argument to follow.

Following [K, §3], for example, the finite, separable extension E/K sits inside a finite Galois extension L/K and we denote by S a Sylow 2-subgroup of $G(L/K)$. Set $F = L^S$ then

$$F \otimes_K E \cong \oplus_{i=1}^{t} E_i \tag{4.2}$$

$$[F:K] \quad \text{is odd} \tag{4.3}$$

and for each $i = 1, \cdots, t$ there exists a chain of quadratic extensions

$$F_i = F_{i,0} \supset F_{i,1} \supset F_{i,2} \supset \cdots \supset F_{i,n_i} = F. \tag{4.4}$$

In addition the double coset formula for the transfer yields

$$\mathrm{res}_{F/K}(\mathrm{Ind}_{E/K}(\rho)) = \textstyle\sum_{i=1}^{t} \mathrm{Ind}_{E_i/F}(\mathrm{res}_{E_i/E}(\rho)) \tag{4.5}$$

where 'res' denotes the restriction, for example $\mathrm{res}_{F/K}$ is induced by $G(N/F) \hookrightarrow G(N/K)$.

4.6.

Proof of Theorem 1.7. From (4.3) $H^*(K; Z/2) \to H^*(F; Z/2)$ is injective so that it suffices to evaluate $\mathrm{res}_{F/K}{}^*\mathrm{HW}(\mathrm{Ind}_{E/K}(\rho)) = \prod_{i=1}^{t} \mathrm{HW}(\mathrm{Ind}_{E_i/F}(\mathrm{res}_{E_i/E}(\rho)))$, by (4.5). By §3.13, this equals

$$\begin{cases}[\prod_{i=1}^{t} \mathrm{SW}(\mathrm{Ind}_{E_i/F}(\mathrm{res}_{E_i/E}(\rho)))] \prod_{i=1}^{t} (1+\mathrm{Sp}[\mathrm{Ind}_{E_i/F}(\mathrm{res}_{E_i/E}(\rho))])\\ = \mathrm{res}_{F/K}{}^*(\mathrm{SW}[\mathrm{Ind}_{E/K}(\rho)]) \prod_{i=1}^{t} (1+\mathrm{Sp}[\mathrm{Ind}_{E_i/F}(\mathrm{res}_{E_i/E}(\rho))]).\end{cases} \tag{4.7}$$

Since $\mathrm{Sp}[-]$ is additive and natural

$$\textstyle\sum_{i=1}^{t} \mathrm{Sp}[\mathrm{Ind}_{E_i/F}(\mathrm{res}_{E_i/E}(\rho))] = \mathrm{res}_{F/K}{}^*\mathrm{Sp}[\mathrm{Ind}_{E/K}(\rho)], \qquad \text{by (4.5)}.$$

It remains to show that the products

$$\mathrm{Sp}[\mathrm{Ind}_{E_i/F}(\mathrm{res}_{E_i/E}(\rho))]\mathrm{Sp}[\mathrm{Ind}_{E_j/F}(\mathrm{res}_{E_j/E}(\rho))] \tag{4.8}$$

all vanish when one expands out (4.7). However this is easy, by means of (3.12). Suppose that $\rho G(N/E_i) = Z/2 \ltimes C_{2}j_i$, where we interpret $j_i = -1$ as the case of trivial image. Let $j = \max_i(j_i)$. Since $(1+\alpha_s) = 2\alpha_{s+1}{}^2$ and $\alpha_j \in K$ this means that $l(1+\alpha_s) = l(2)$ if $s < j$ and if $j \geq 3$ then $l(2) = 0$. Hence if $j \geq 3$ the only non-zero terms in the products (4.8) involve $l(1+\alpha_j)^2 = l(-1)l(1+\alpha_j) = 0$ while if $j = 0, 1, 2$ or -1 these terms all have a factor $l(2)^2 = 0$.

Therefore (4.7) equals

$$\mathrm{res}_{F/K}{}^*(\mathrm{SW}[\mathrm{Ind}_{E/K}(\rho)](1 + \mathrm{Sp}[\mathrm{Ind}_{E/K}(\rho)]))$$

as required.

§5. A counterexample

The example which follows is due to Bruno Kahn.

Let $\mathbb{Q}$ denote the rationals and let $E = \mathbb{Q}(x_1, x_2, x_3)$. The cyclic group, $Z/3$, acts on E by permuting the x_i cyclically and we set $K = E^{Z/3}$, the fixed field of this action. Let $L = E((x_1 - 1)^{1/2}, (x_1)^{1/2}, (t)^{1/2})$, where $t = (1 + (x_1)^{-1/2})(x_1{}^{-1})$.

LEMMA (5.1). *L/E is Galois with group, $G(L/E)$, isomorphic to the dihedral group of order eight.*

Proof. Define subfields as follows

$$L_1 = E((x_1 - 1)^{1/2}),\ L_2 = L_1((x_1)^{1/2}),\ L_3 = E((x_1)^{1/2}) \quad \text{and} \quad L_4 = L_3((t)^{1/2}).$$

Hence we have the following picture

(5.2)
$$\begin{tikzcd}[column sep=small, row sep=tiny]
& L \arrow[dd, dash] \\
L_4 \arrow[ur, dash] \arrow[dd, dash] & \\
& L_2 \arrow[dd, dash] \\
L_3 \arrow[ur, dash] \arrow[dd, dash] & \\
& L_1 \\
E \arrow[ur, dash] &
\end{tikzcd}$$

In (5.2) L/L_1 is cyclic of order four. This is because

(5.3)
$$\begin{aligned} N_{L_2/L_1}(t) &= (1 + (x_1)^{-1/2})x_1{}^{-1}(1 - (x_1)^{-1/2})x_1{}^{-1} \\ &= (x_1 - 1)x_1{}^{-3} \in d(L_1{}^*)^2 \end{aligned}$$

where $x_1 = d = \mathrm{disc}(L_2/L_1)$, so that $N_{L/L_1}((t)^{1/2})$ is not a square. The generator of $G(L_2/L_3)$ fixes t so that the generator of $G(L_1/E)$ leaves L globally invariant, which proves L/E is Galois.

Let $\tau \in G(L/L_4)$ and $\sigma \in G(L/L_1)$ be generators so that $\tau^2 = 1 = \sigma^4$. From (5.3) one finds that $\tau\sigma\tau = \sigma^3$, by checking directly on $(x_1 - 1)^{1/2}$ and $(t)^{1/2}$.

5.4.
Now define a (non-special) dihedral representation

(5.5)
$$\rho: G(L/E) \to O_2(\mathbb{Q})$$
$$\text{by} \quad \rho(\tau) = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \qquad \rho(\sigma) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$\text{where} \quad a^2 + b^2 = 1,\ ab \neq 0.$$

Since the spinor norms of $\rho(\tau)$ and $\rho(\sigma)$ are respectively $2(a+1)$ and 2 (modulo squares) we have, in $H^2(E; Z/2)$,

(5.6) $$\mathrm{Sp}[\rho] = l(2(a+1))l(x_1 - 1) + l(2)l(x_1).$$

Set $\Lambda(\rho) = \mathrm{HW}(\rho)/\mathrm{SW}(\rho)$ then we will show that

(5.7) $$\Lambda(\mathrm{Ind}_{E/K}(\rho)) \neq 1 + \mathrm{Sp}[\mathrm{Ind}_{E/K}(\rho)]$$

for ρ, E, K as above. For suppose (5.7) were not true then we would have

(5.8) $$\mathrm{Res}_{E/K}(\Lambda(\mathrm{Ind}_{E/K}(\rho)) = 1 + \mathrm{Res}(\mathrm{Sp}[\mathrm{Ind}_{E/K}(\rho)]) \quad \text{in} \quad H^*(E_j Z/2).$$

However, by the proof of Lemma 2.4, $\Lambda(\rho) = I + \mathrm{Sp}[\rho]$ and if $g \in G(E/K) \cong Z/3$ is a generator (5.8) implies

(5.9) $$\prod_{j=0}^{2} (1 + (g^j)^*(\mathrm{Sp}([\rho])) = 1 + \sum_{j=0} (g^j)^*(\mathrm{Sp}[\rho]).$$

To show that (5.9) fails, it suffices to show that the four dimensional component of the left side

(5.10) $$c = l(2(a+1))l(2) \sum_{1 \le i \neq j \le 3} l(x_1 - 1)l(x_j) \in H^4(E; Z/2)$$

can be non-zero.

Let $F = \mathbb{Q}(x_2, x_3)$ which is the residue field of the principal ideal ring $\mathrm{A} = \mathbb{Q}(\mathrm{x}_2, \mathrm{x}_3)[\mathrm{x}_1]$ at $P = \langle x_1 \rangle$. Hence, we have coboundary maps (from the localisation sequence for A and P).

$$\partial : H^i(E; Z/2) \to H^{i-1}(F; Z/2)$$

such that

$$\partial(c) = l(2(a+1))l(2)\partial(\textstyle\sum_{1 \le i \neq j \le 3} l(x_i - 1)l(x_j)).$$

However, ∂ is a derivation which satisfies $\partial(l(z)) = 0$ if $z \in A$ is prime to x_1, and $\partial(l(x_1)) = 0$ so that

$$\partial(c) = l(2(a+1))l(2)(l(x_2 - 1) + l(x_3 - 1)) \in H^3(F; Z/2).$$

Repeating this process with $A = \mathbb{Q}(x_3)[x_2]$ and $P = \langle x_2 - 1 \rangle$ we obtain

$$\partial' : H^i(F; Z/2) \to H^{i-1}(\mathbb{Q}(x_3); Z/2)$$

such that

$$\partial'\partial(c) = l(2(a+1))l(2) \in H^2(\mathbb{Q}(x_3); Z/2).$$

However, $l(2(a+1))l(2)$ is in the image of the injection $H^2(\mathbb{Q}; Z/2) \to H^2(\mathbb{Q}(x_3); Z/2)$ so that $\partial'\partial(c) = 0$ if and only if

$$l(2(a+1)l(2) = 0 \in H^2(\mathbb{Q}; Z/2)$$

This in turn is true if and only if there exists $u, v, w \in \mathbb{Q}$ such that

$$2(a+1)u^2 + 2v^2 = w^2.$$

In particular, if $a = 3/5$, $b = 4/5$ then $2(a+1) = 5 \in \mathbb{Q}^*/\mathbb{Q}^{**}$ but $5u^2 + 2v^2 = w^2$ is impossible (with $u, v, w \in Z$) since 2 is not a square mod 5.

§6. Characteristic classes and $RO_K(G)$

Let G be a finite group and let K be a field of characteristic different from two. Following [Q, §5] one may define the Grothendieck group of orthogonal K-representations of G, $RO_K(G)$. We will recall the definition in §6.1 below.

Henceforth, let N/K be a finite Galois extension and let $G(N/K)$ be its Galois group. In §§1–5 we studied various characteristic classes—Hasse-Witt invariants, the spinor class, Stiefel-Whitney classes—of orthogonal Galois representations. However, these classes do not factor through $RO_K(G(N/K))$. For example, if $G(N/K$ is a 2-group and K is a number field, then $RO_{\mathbf{Q}}(G(N/K))$ is generated by permutation representations [Seg, p. 379 et seq.] so that if the spinor class extended to a map

$$\mathrm{Sp}: RO_{\mathbf{Q}}(G(N/K)) \to H^2(K; Z/2)$$

we would have $\mathrm{Sp}[\rho] = l(2)\mathrm{SW}_1[\rho]$, since this is true for a permutation representation. However, such a formula contradicts the example given in §5. For this reason it seems appropriate to record here the manner in which these characteristic classes transform under the equivalence relations which define $RO_K(G)$.

6.1.

$RO_K(G)$ is defined as a quotient of the free abelian group on the isomorphism classes of finite dimensional, orthogonal representations (ρ, V, b). Here $b: V \times V \to K$ is a non-singular, symmetric bilinear form and $\rho: G \to O(V, b)$ is a homomorphism to the orthogonal group of b. The relations imposed are of three types.

a) *Scaling*

If $\alpha \in K^*$ and $(\alpha b)(x, y) = \alpha b(x, y)$ then $O(V, b) = O(V, \alpha b)$ and we set

$$(\rho, V, b) \backsim (\rho, V, \alpha b).$$

b) *Sum*

If (ρ, V, b) is the orthogonal sum of (ρ', V', b') and (ρ'', V'', b'') we set

$$(\rho, V, b) \backsim (\rho', V', b') + (\rho'', V'', b'').$$

c) *Hyperbolic*

Let (ρ, V, b) be an orthogonal representation. Suppose that $W \subset \{v \in V \mid b(v, W) = 0\} = W^0$ and that W (and hence W^0) is a G-invariant subspace. Define forms

$$b_1: (W^0/W)^2 \to K$$

and

$$b_2: (W \oplus V/W^0)^2 \to K$$

by

$$b_1(v + W, v' + W) = b(v, v')$$

$$b_2(w \oplus (v + W^0), w' \oplus (v' + W^0)) = b(w, v') + b(w', v).$$

Also ρ induces $\rho_1: G \to O(W^0/W, b_1)$ and $\rho_2: G \to O(W + (V/W^0), b_2)$. With these conventions set

$$(\rho, V, b) \backsim (\rho_1, W^0/W, b_1) + (\rho_2, W \oplus (V/W^0), b_2).$$

6.2.

The orthogonal representation $\rho: G(N/K) \to O(V, b)$ of §6.1, considered as a 1-cocycle in $H^1(K; O(V \oplus_K \bar{K}), b \otimes_K \bar{K})$, corresponds to the symmetric bilinear form of [*F*, §2] which we will also denote by (ρ, V, b), as more briefly by (ρ). Hence, as in §1, (ρ, V, b) has Hasse-Witt classes, $\mathrm{HW}_i(\rho)$, while the representation

(6.3) $\qquad [\rho]: G(\bar{K}/K) \to G(N/K) \to O(V, b) \to O(V \otimes_K \bar{K}, b \otimes_K \bar{K})$

has Stiefel-Whitney classes, $\mathrm{SW}_i[\rho]$. Also (ρ, V, b) has a spinor class corresponding to

$$\mathrm{Sp}[\rho]: G(\bar{K}/K) \to G(N/K) \to O(V, b) \xrightarrow{\theta} K^*/K^{**}$$

where θ is the spinor norm. From [*F*, §3] or [SN, §2.10] we have the following result.

THEOREM (6.4). *Let* $(\rho) = (\rho, V, b)$ *as above and let* $[\rho]$ *be as in* (6.3). *Then*

(i) $\qquad \mathrm{SW}_1[\rho] + \mathrm{HW}_1(b) = \mathrm{HW}_1(\rho)$ *in* $H^1(K; Z/2)$

and

(ii) $\quad \mathrm{HW}_2(\rho) + \mathrm{HW}_2(b) = \mathrm{Sp}[\rho] + \mathrm{SW}_2[\rho] + \mathrm{HW}_1(b)\mathrm{SW}_1[\rho]$ *in* $H^2(K; Z/2)$.

Scaling b to αb changes (ρ, V, b) to $\alpha(\rho, V, b)$, as one sees from Fröhlich's explicit description of the bilinear form (ρ, V, b) [*F*, §2.Sn]. This leaves $[\rho]$ unchanged but changes the spinor norm of $\rho(g)$ (only when $\deg \rho(g) = -1$) by a factor, α[*O'M*, p. 138]. These facts have the following easy consequences.

PROPOSITION (6.5). *Let* $\alpha \in K^*$ *and set* $(\rho) = (\rho, V, b)$ *and* $(\rho_\alpha) = (\rho, V, \alpha b)$, *as above. Then*

a) $\mathrm{SW}_i[\rho] = \mathrm{SW}_i[\rho_\alpha]$.

b) $\mathrm{HW}_1(\alpha b) + \mathrm{HW}_1(b) = \mathrm{rank}(b) l(\alpha) = \mathrm{HW}_1(\rho) + \mathrm{HW}_1(\rho_\alpha)$.

c) $\mathrm{Sp}[\rho_\alpha] = \mathrm{Sp}[\rho] + l(\alpha)\mathrm{SW}_1[\rho]$.

d) $\mathrm{HW}_2(\alpha b) + \mathrm{HW}_2(b) + (\mathrm{rank}(b) - 1) l(\alpha)\mathrm{HW}_1(b) = \mathrm{HW}_2(\rho_\alpha) + \mathrm{HW}_2(\rho) + (\mathrm{rank}(b) - 1) l(\alpha)\mathrm{HW}_1(\rho) = \frac{1}{2}\,\mathrm{rank}(b)(\mathrm{rank}(b) - 1) l(\alpha)^2$.

The following behaviour of SW_i and HW_i under orthogonal direct sum is well-known. Additivity of the spinor class is clear from the definition and [O'M, p. 139].

PROPOSITION (6.6). *Let* $(\rho) = (\rho, V, b)$ *and* $(\rho') = (\rho', V', b')$ *be orthogonal representations and set* $(\rho'') = (\rho') \oplus (\rho')$, *the orthogonal sum of* (ρ) *and* (ρ'). *Then*

(a) $\mathrm{HW}_1(\rho) + \mathrm{HW}_1(\rho') = \mathrm{HW}_1(\rho'')$.

(b) $\mathrm{SW}_1[\rho] + \mathrm{SW}_1[\rho'] = \mathrm{SW}_1[\rho'']$.

(c) $\mathrm{Sp}[\rho] + \mathrm{Sp}[\rho'] = \mathrm{Sp}[\rho'']$.

(d) $\mathrm{HW}_2(\rho) + \mathrm{HW}_2(\rho') + \mathrm{HW}_1(\rho)\mathrm{HW}_1(\rho') = \mathrm{HW}_2(\rho'')$.

(e) $\mathrm{SW}_2[\rho] + \mathrm{SW}_2[\rho'] + \mathrm{SW}_1[\rho]\mathrm{SW}_1[\rho'] = \mathrm{SW}_2[\rho'']$.

6.7. Now let us sketch the effect of the hyperbolic relation, §6.1(c). By [M-H, p. 13, §6.3] and the argument of [M-H, p. 56, §1.1] V has a basis $v_1, v_2, \cdots, v_a, v_{a+1}, \cdots, v_{a+t}, \cdots, v_{2a+t}$ with respect to which the matrix of b is

$$\beta = \begin{pmatrix} 0 & 0 & I_a \\ 0 & \Lambda & 0 \\ I_a & 0 & 0 \end{pmatrix}$$

and $\langle v_1, \cdots, v_a\rangle = W$, $\langle v_1, \cdots, v_{a+t}\rangle = W^0$. The matrix of $\rho(g)$ $(g \in G(N/K))$ has the form (X^T = transpose of X)

$$\rho(G) = \begin{pmatrix} A(g) & D(g) & E(g) \\ 0 & B(g) & F(g) \\ 0 & 0 & (A(g)^T)^{-1} \end{pmatrix} \tag{6.8}$$

where $A(-)$ and $B(-)$ are homomorphisms. In addition, $B(g) \in O(K^t, \Lambda)$,

$$\begin{cases} 0 = E(g)A(g)^T + A(g)E(g)^T + D(g) \wedge D(g)^T \\ F(g) = ((A(g)^T)^{-1})B(g) \wedge D(g)^T \end{cases} \tag{6.9}$$

Hence, in §6.1(c), $(\rho_1 \oplus \rho_2)$ is given in matrix form by

$$(\rho_1 \oplus \rho_2) = \left(\begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & (A^T)^{-1} \end{pmatrix}, \quad K^{2a+t}, \begin{pmatrix} 0 & 0 & I_a \\ 0 & \Lambda & 0 \\ I_a & 0 & 0 \end{pmatrix}\right)$$

That is, $(\rho_1 + \rho_2)$ is represented by replacing D, E and F in (6.8) by zero. We have $\pi:\mathrm{Im}(\rho) \to O(K^t, \Lambda) \times O\left(K^{2a}, \begin{pmatrix} 0 & I_a \\ I_a & 0 \end{pmatrix}\right)$ given by $\pi(\rho(g)) = \left(B(g), \begin{pmatrix} A(g) & 0 \\ 0 & A(g)^T)^{-1} \end{pmatrix}\right)$. The kernel of π has a decomposition series with two-divisible quotients. Hence, all 2-primary cohomology characteristic classes of ρ depend only on the homomorphisms A and B. This shows the following result.

Proposition (6.10). *Let* (ρ), (ρ_1) *and* (ρ_2) *denote the orthogonal representations of* §6.1(c) *when* $G = G(N/K)$. *Then each of the characteristic classes* SW_i, HW_i *and* Sp *take the same value on* (ρ) *as on* $(\rho_1 \oplus p_2)$.

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