Boletín de la Sociedad Matemática Mexicana Vol. 30 No. 2, 1985

SPECTRAL ANALYSIS AND MEAN-PERIODIC FUNCTIONS ON RANK ONE SYMMETRIC SPACES

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Introduction

Let X be a symmetric space of the rank one and of noncompact type. The manifold X can be represented as a homogeneous space $X = G/K$, where G is a semisimple noncompact Lie group of finite center and K is a maximal compact subgroup of G. The action of G on the space $E(X)$ of smooth functions with the usual Fréchet topology is defined by the formula:

$$L_g f(x) := f(g^{-1}x)$$

The subject of the present paper is the spectral analysis and synthesis in $E(X)$ with respect to the family $e_{\lambda,b}$ of plane waves on X. (The customary notation of [13], [19], [20] is used).

A number of papers have appeared recently dedicated to the study of the spectral analysis and synthesis on symmetric spaces (see [1], [2], [3], [4], [6], [16], [21]). All of them however, except [16] and [21] are treating the case of the space $E(K\backslash X)$ of spherically symmetric functions on X. Although this is precisely the key to the general theory, only the analysis with respect to the plane waves neglects all deep differences between the classical theory $X = R^n$ (see [5], [7], [12], [17]) and the symmetric space case.

We prove that the spectral analysis holds on X, i.e. any closed and invariant subspace $V \subset E(X)$ contains some plane wave. As known (cf. [21]) the spectral synthesis fails in general. For example the space $E(SL(2, R)/U(1))$ contains a numerable family of invariant subspaces such that in each of them one can distinguish three different subrepresentations of G and only two independent classes of plane waves.

This phenomenon may occur only if the subspace contains a zonal spherical function ϕ_λ for which $c(\lambda)c(-\lambda) = 0$. By $c(\cdot)$ is denoted the Harish-Chandra function of X.

The way of studying the problem is the following. For a given λ one can consider in the space $E(X)$ a maximal invariant subspace which contains the unique zonal spherical function ϕ_λ. It is the space consisting of those $f \in E(X)$ which satisfy for some natural k

(0.1) $$\int_K f(gkx)\, dk = \sum_{j=0}^{k} d_j(g)\phi_\lambda^{(j)}(x),$$

where

$$\phi_\lambda^{(j)} = \frac{d^j}{d\lambda^j}\phi_\lambda, \quad d_j(g) \in C.$$

The space of functions satisfying (0.1) is denoted by $E_\lambda^{(k)}$. Each element of the space $E_\lambda{}^{(k)}$ can be represented as

$$\psi(x) = \sum_{j=0}^{k} \int_B e_{i\lambda+\rho,b}{}^{(j)}(x)\, dT_{\lambda,j}(b) \tag{0.2}$$

where $T_{\lambda,j}$ is an analytic functional (Prop. 4.6). This generalizes a result of Helgason which refers to the case $k = 0$.

Provided that for a given closed, invariant, nontrivial subspace $V \subset E(X)$ we know the set of zonal spherical functions belonging to V and the higher numbers $m(\lambda)$ such that $\phi_\lambda{}^{(j)} \in V$ for all $j \leq m(\lambda)$, we can try to approximate the elements of V by linear combinations of the form

$$\psi(x) = \sum_\lambda \sum_{j=0}^{m(\lambda)} \int_B e_{i\lambda+\rho,b}{}^{(j)}(x)\, dT_{\lambda,j}(b). \tag{0.3}$$

According to [21] (see Them 4.8) it is always possible. In the present paper we prove that a function $f \in E(X)$ can be expanded in a series of the form (0.3) if it is mean-periodic with respect to some slowly decreasing distribution. This type of expansions was announced in [16].

1. Notation and preliminaries

We shall use the standard notation which is established in classical books [14], [15], [19], or in [20].

By $G = KAN$ we denote the Iwasawa decomposition of the semi-simple Lie group G of non-compact type and of finite center.

The constituent A of this decomposition is a vector group and thus the mapping $\exp\colon \mathfrak{a} \to A$ is a diffeomorphism, (here $\mathfrak{a}$ stands for the Lie algebra of the group A). It follows that for every $g \in G$ there exists a unique element $H(g) \in \mathfrak{a}$ such that:

$$g = k \exp H(g) n, \quad \text{where} \quad k \in K,\ n \in N.$$

The subgroup K is a maximal compact subgroup of G so admits a probability Haar measure denoted by dk; the subgroup N is nilpotent. Let M and M' be respectively the centralizer and normalizer of A in K. Then $W := M'/M$ is the Weyl group of G.

The quotient $X := G/K$ is a symmetric space of noncompact type whose boundary is defined as $B := K/M = G/MAN$.

The dimension of $\mathfrak{a}$ is called the rank of X.

We denote by $\mathfrak{a}_c{}^*$ the space of complex valued and R-linear functionals on $\mathfrak{a}$.

By a plane wave on X with frequency $\lambda \in \mathfrak{a}_c{}^*$ and normal $kM \in B$ we mean the following function:

$$e_{\lambda,\cdot kM}(gk) := e^{-\langle \lambda, H(g^{-1}k)\rangle}. \tag{1.1}$$

The plane wave satisfies the multiple identity

$$e_{\lambda,b}(gh) = e_{\lambda,g^{-1}b}(h) e_{\lambda,b}(gk), \quad g \in G,\ h \in X. \tag{1.2}$$

Let db denote the K-invariant probability measure on B.

The Radon-Nikodym derivative (dgb/db) is a plane wave which is denoted by $e_{2\rho,b}(gk)$. The function

$$\phi_\lambda(x) := \int_B e_{i\lambda+\rho,b}(x)\, db$$

is called the zonal spherical function on X of the frequency λ. Applying the identity (1.2) we state the relations called the spherical equations:

$$\int_K \phi_\lambda(gkx)\, dk = \phi_\lambda(gK)\phi_\lambda(x) \tag{1.3}$$

and

$$\int_K e_{i\lambda+\rho,b}(gkx)\, dk = e_{i\lambda+\rho,b}(gK)\phi_\lambda(x). \tag{1.4}$$

The zonal spherical functions satisfy the following symmetry relations, which can be found in [14], [15], [18]

$$\phi_\lambda(gk) = \phi_{-\lambda}(g^{-1}) \tag{1.5}$$

$$\phi_{w\lambda} = \phi_\lambda, \quad w \in W. \tag{1.6}$$

By $E(X)$ is denoted the space of infinitely differentiable functions on X with its customary topology. The group action on $E(X)$ is defined as:

$$L_g f(x) := f(g^{-1}x).$$

$E(K\backslash X)$ stands for the space of those elements $f \in E(X)$ which satisfy

$$L_k f = f, \quad k \in K.$$

In particular the zonal spherical functions are contained in $E(K\backslash X)$, as well as the functions

$$\phi_\lambda{}^{(n)} := \frac{d^n}{d\lambda^n}\phi_\lambda.$$

Let us denote also

$$e_{\lambda,b}{}^{(n)} := \frac{d^n}{d\lambda^n}e_{\lambda,b}.$$

The spaces $E(X)$ and $E(K\backslash X)$ are locally convex Fréchet reflexive complete spaces whose duals $E'(X)$ and $E'(K\backslash X)$ can be identified with the space of compactly supported distributions and K-invariant compactly supported distributions, respectively.

One can consider the space $E'(X)$ as a subspace of $E'(G)$ composed of those elements which are right K-invariant i.e. satisfy

$$T_x(f(xk)) = T(f) \quad \text{for all} \quad f \in E(G),\, k \in K.$$

The space $E'(G)$ forms a convolution algebra with the convolution defined as

$$T * S(f) := (T_x \otimes S_y)f(xy).$$

If S is right K-invariant then for any $T \in E'(G)$ also $T * S$ is invariant. Thus both $E'(X)$ and $E'(K\backslash X)$ form convolution subalgebras of $E'(G)$. The convolution of $T \in E'(G)$ and $S = fdg$ (dg denotes a Haar measure on G) is given by the formula

(1.7) $$T * f(g) = T(L_g \check{f}), \quad \text{where} \quad \check{f}(g) := f(g^{-1}).$$

A linear subspace $V \subset E(X)$ is called G-invariant if $L_g V \subset V$ for all $g \in G$. A closed subspace is G-invariant iff $T * V \subset V$ for all $T \in E'(G)$.

Having given a representation (τ, U) of the compact subgroup K in a Fréchet space U we define the operator

$$Pu := \int_K \tau(k)u \, dk.$$

Obviously

$$PE(X) = E(K\backslash X) \quad \text{and} \quad PE'(X) = E'(K\backslash X).$$

A subspace $V \subset E(K\backslash X)$ will be called invariant if $T * V \subset V$ for all $T \in E'(K\backslash X)$. If $V \subset E(X)$ is G-invariant then PV is invariant in $E(K\backslash X)$.

2. Spectral analysis

We are interested in the spectral analysis in the spaces $E(K\backslash X)$ and $E(X)$. As the elementary functions in the first space we consider the zonal spherical functions ϕ_λ, which according to (1.6) are parametrized by the set $W/\mathfrak{a}_c^*$. In the space $E(X)$ we take the plane waves $e_{i\lambda+\rho,b}$, $(\lambda, b) \in \mathfrak{a}_c^* \times B$ as the family of elementary functions.

We will say that the spectral analysis holds in the space of $E(K\backslash X)$ if any closed and invariant subspace $V \subset E(K\backslash X)$ contains some zonal spherical function. The spectrum of V (denoted Sp V) is the set of those $\lambda \in \mathfrak{a}_c^*$ for which $\phi_\lambda \in V$. If $\{\phi_\lambda, \phi_\lambda^{(1)}, \cdots, \phi_\lambda^{(m)}\} \subset V$ and $\phi_\lambda^{(m+1)} \notin V$ then the number m is called the multiplicity of λ in Sp V and is denoted by $m(\lambda)$.

The spectral synthesis problem in the space $E(K\backslash X)$ consists in the question if the spectrum Sp V and the multiplicities $m(\lambda)$, $\lambda \in$ Sp V determine V uniquely.

The main purpose of this section is to prove that the spectral analysis holds in the space $E(X)$ that is, every closed and invariant subspace $V \subset E(X)$ contains some plane wave.

The problem of spectral analysis and synthesis in the space $E(K\backslash X)$ was solved in [1] in case of the symmetric space of rank one (see also [21]). On the other hand the examples constructed in [3] show that in spaces of rank higher than one even the spectral analysis is not possible.

THEOREM (2.1). [21]. *Let X be a rank one symmetric space of the noncompact type. If $0 \neq V \neq E(K\backslash X)$ is closed and invariant, then it is equal to the closed linear span of the set of functions $\phi_\lambda^{(n)}$, $\lambda \in$ Sp V, $n \leq m(\lambda)$.*

The proof is based on the theorem of Schwartz [17] about the spectral analysis and synthesis in the space $E(R)$ applied in the same way as in [11], where the case $G = SL(2, R)$ is considered.

The problem of spectral analysis in $E(X)$ for rank one symmetric space X will be solved by reducing it to Thm 2.1. Throughout what follows, all symmetric spaces considered are assumed to be of rank one.

Let $V \subset E(X)$ be a closed, invariant and nontrivial subspace. The space PV is nontrivial closed and invariant in $E(K \backslash X)$. There exists, according to Thm 2.1, some $\lambda \in \operatorname{Sp} PV$.

Our aim is to prove that at least one of the plane waves $e_{i\lambda+\rho,b}$ or $e_{-i\lambda+\rho,b}$ belongs to V.

Denote by V_λ the closed linear span of the set $\{L_g\phi_\lambda \mid g \in G\}$. The space V_λ is contained in another G-invariant subspace of $E(X)$ defined as follows.

$$E_\lambda = \{f \in E(X) \mid PL_g f = c(g)\phi_\lambda \text{ for all } g \in G, \text{ and some } c(g) \in \mathbb{C}\}$$

On putting the argument $x = eK$ in the defining equation of E_λ we observe

$$E_\lambda = \{f \in E(X) \mid PL_g f = f(g^{-1})\phi_\lambda,\ g \in G\}.$$

In virtue of the equations (1.3), (1.4) all plane waves $e_{iw\lambda+\rho,b}$, $b \in B$, $w \in W$ belong to E_λ as well as the zonal spherical function ϕ_λ and then the whole space V_λ. Because of (1.6) we have $E_{w\lambda} = E_\lambda$ for all $w \in W$, what in our rank one case means that $E_{-\lambda} = E_\lambda$. In general the space E_λ is not equal to V_λ, although it is true for "almost all" values of λ. This fact was proved by Helgason [14] or [15] in a slightly different form.

Let $D(X)$ be the algebra of all differential operators on X which commute with the G-action.

THEOREM (2.2). [15]. $E_\lambda = \{f \in E(X) \mid Df = (D\phi_\lambda)(eK) f,\ D \in D(X)\}$.

Let us define the continuous operator F_λ: $L^2(B) \to E(X)$ by the formula:

$$F_\lambda \varphi(x) := \int_B \varphi(b) e_{i\lambda+\rho,b}(x)\, db. \tag{2.1}$$

It is easily seen in virtue of (1.2) that the operator F_λ intertwines the representation U^λ defined in $L^2(B)$ by the formula:

$$U_g{}^\lambda \psi(x) := e_{i\lambda+\rho,b}(gK)\psi(g^{-1}b) \tag{2.2}$$

with the regular representation in $E(X)$:

$$F_\lambda U_g{}^\lambda = L_g F_\lambda. \tag{2.3}$$

The closure of the image $F_\lambda L^2(B)$ is an invariant subspace of E_λ containing ϕ_λ and hence V_λ.

Definition. The functional $\lambda \in \mathfrak{a}_c{}^*$ is called simple if the mapping F_λ is injective.

The following result is used in the proof of Thm 4.1 in [14].

THEOREM (2.3). *If $-\lambda$ is simple then the image $F_\lambda L^2(B)$ belongs to V_λ.*

Proof. Applying the multiplier identity (1.2) and the fact that $e_{2\rho,b}(x) = (dgb/db)$ we obtain

$$\begin{aligned}\phi_\lambda(g^{-1}x) &= \textstyle\int_B e_{i\lambda+\rho,b}(x)\, e_{-i\lambda+\rho,b}(gK)\, db \\ &= F_\lambda e_{-i\lambda+\rho,b}(gK), \quad g \in G,\end{aligned} \tag{2.4}$$

what means that the translations of ϕ_λ span exactly the space $F_\lambda(\Omega)$, where Ω is the linear span of the family of functions.

$$B \ni b \longrightarrow e_{-i\lambda+\rho,b}(gK), \quad g \in G.$$

On the other hand the condition $-\lambda$ to be simple can be interpreted as the density of Ω in $L^2(B)$. Hence the operator F_λ is uniquely determined by its values on Ω. Taking into account that the space V_λ is complete and passing to the extension of F_λ we obtain $F_\lambda L^2(B) = F_\lambda(\overline{\Omega}) \subset V_\lambda$, what was to be proved.

COROLLARY (2.4). *If $-\lambda$ is simple then the space V_λ contains the plane waves $e_{i\lambda+\rho,b}$, $b \in B$.*

Proof. It suffices to prove that the plane wave $e_{i\lambda+\rho,eM}$ is contained in V_λ, because

$$e_{i\lambda+\rho,kM} = L_k e_{i\lambda+\rho,eM} \tag{2.5}$$

by the very definition of the plane wave.

Let δ_n be an approximative unit in $L^2(B)$ i.e. such a sequence of functions that

$$\delta_n \geq 0, \quad spt\delta_n \searrow eM \quad and \quad \textstyle\int_B \delta_n\, db = 1.$$

Then $F_\lambda \delta_n \longrightarrow e_{i\lambda+\rho,eM}$ and the corollary is proved.

THEOREM (2.5). *Let X be a symmetric space of rank one and of noncompact type and let V be a nontrivial closed invariant subspace of $E(X)$. Assume $\lambda \in$ Sp PV. Then the functions $e_{i\lambda+\rho,b}$ or $e_{-i\lambda+\rho,b}$ are contained in V.*

Proof. The decisive step in the proof is based on another theorem belonging to Helgason.

Denote by $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ the Iwasawa decomposition of the Lie algebra of the group G, corresponding to the global decomposition $G = KAN$. The positive restricted roots of $\mathfrak{g}$ corresponding to the above decomposition are denoted by α and 2α if the latter root also appears. By $m_{2\alpha}$ and m_α we denote the dimensions of the root spaces of α and 2α respectively. Let us define:

$$e(\lambda) := \Gamma^{-1}(\tfrac{1}{2}(\tfrac{1}{2}m_\alpha + 1 + i\lambda))\Gamma^{-1}(\tfrac{1}{2}(\tfrac{1}{2}m_\alpha + m_{2\alpha} + i\lambda)).$$

The function e^{-1} is just the denominator of the Harish-Chandra function $c(\cdot)$ appearing in the harmonic analysis on symmetric spaces.

THEOREM (2.6). [15]. *First, the functional λ is simple iff $e(\lambda) \neq 0$. Second, $E_\lambda = V_\lambda$ and both spaces are irreducible under the action of G iff $e(\lambda)e(-\lambda) \neq 0$.*

Coming back to the proof of Thm 2.5, if $\lambda \in \mathrm{Sp}\, PV$ then $V_\lambda \subset V$. Because of the form of the function $e(\cdot)$ at least one of the functionals λ and $-\lambda$ is simple. In virtue of Corollary 2.4 the proof follows.

The results of this section can be summarized in the following way:

THEOREM (2.7). *If X is a symmetric space of rank one and of noncompact type then the spectral analysis holds in the space $E(X)$.*

3. Mean periodic functions and spectral synthesis in $E(K\backslash X)$

Definition. A function $f \in E(X)$ is called mean-periodic if there exists a non-zero distribution $T \in E'(X)$ such that

(3.1) $$T(L_g f) = 0 \quad \text{for all } g \in G.$$

In terms of convolution the above condition can be written as

(3.2) $$T * \check{f} = 0 \quad \text{or} \quad f * \check{T} = 0.$$

For a fixed T the functions satisfying (3.1) will be called mean periodic with respect to T or T-mean periodic.

The space of all T-mean periodic functions is denoted by $V(T)$. A function f is mean-periodic iff the space V_f being the closed linear span of all translations of f is a propert subspace of $E(X)$.

According to Thm 2.7 the space V_f contains some plane wave. Our aim now is to establish sufficient conditions for V_f to be uniquely determined by the functions $e_{i\lambda+\rho,b}{}^{(n)}$ contained in it. Next we are going to find some approximation formulas of f by the plane waves.

Such formulas were proved in [1] for K-invariant mean periodic functions and announced in [16] for K-finite functions mean periodic with respect to a slowly decreasing distribution.

Assume first T to be K-invariant. The spherical Fourier transform of T is defined as

(3.3) $$\hat{T}(\lambda) := T(\phi_\lambda).$$

The Fourier transform as a map

$$E'(K\backslash X) \ni T \to \hat{T}(\cdot)$$

is a homomorphism of the commutative convolution algebra $E'(K\backslash X)$ into the algebra $A(\mathfrak{a}_c{}^*)$ of entire functions on $\mathfrak{a}_c{}^*$. The image of $E'(K\backslash X)$ is determined by the symmetric space version of the Paley-Wiener-Schwartz theorem [8]. The image of the space $\mathscr{D}(K\backslash X)$ under the spherical Fourier transform is equal in rank one case to the set of even functions on $\mathfrak{a}_c{}^* \cong \mathbb{C}$ which satisfy: there exists such a $C > 0$ that for any $N > 0$ there exists $C_N >$

0 for which

$$|X(\lambda)| \leq C_N |1 + |\lambda||^{-N} \exp(-C|\operatorname{Im} \lambda|), \quad \lambda \in \mathfrak{a}_c^*.$$

On reasoning exactly as in the classical case one can prove that the image of the space $E'(K\backslash X)$ is precisely the set $A_0(\mathfrak{a}_c^*)$ of entire and even functions which satisfy

$$|X(\lambda)| \leq A|1 + |\lambda||^B \exp C|\operatorname{Im} \lambda|$$

for appropriate $A, B, C > 0$.

The space $A_0(\mathfrak{a}_c^*)$ can be topologized in such a way that the Fourier transform becomes a homeomorphism [10].

Definition. The distribution $T \in E'(K\backslash X)$ will be called slowly decreasing if for any $S \in E'(K\backslash X)$ and $f \in A(\mathfrak{a}_c^*)$ the condition $\hat{T}f = \hat{S}$ implies $f \in A_0(\mathfrak{a}_c^*)$.

This concept was introduced and studied by Ehrenpreis in case of classical Fourier transform [9].

Since the image of the spherical Fourier transform is just the "even part" of the image of the classical Fourier transform one can extend to easily Ehrenpreis results about slowly decreasing distributions onto the symmetric space case.

In the sequel we describe roughly, following the exposition [5], the way of deducing the spectral synthesis theorem and approximation formulas under the assumption of T being slowly decreasing. The results obtained are also contained in [1].

One of the most important results of this assumption is that the principal ideal generated in $A_0(\mathfrak{a}_c^*)$ by $\hat{T}$ is closed and consequently also the ideal $J = \{S * T \mid S \in E'(K\backslash X)\}$ is closed in $E'(K\backslash X)$.

The dual of the quotient space $E'(K\backslash X)/J$ is naturally identified with the space $V(T) = \{f \in E(K\backslash X) \mid T * \check{f} = 0\}$.

Hence

$$V(T)' = E'(K\backslash X)/J = A_0(\mathfrak{a}_c^*)/\hat{J}.$$

Let $Z(T)$ denote the set of zeros of the function T. If $\lambda \in Z(T)$ then $m(\lambda)$ will stand for the multiplicity of λ. Let us observe that the condition $\lambda \in Z(T)$ means that $\lambda \in \operatorname{Sp} PV(T)$. In fact, the condition $T(\phi_\lambda) = 0$ implies $T(L_g\phi_\lambda) = T(PL_g\phi_\lambda) = \phi_\lambda(g^{-1})T(\phi_\lambda) = 0$.

In the same way one can check that $m(\lambda) = k$ means that $\phi_\lambda^{(i)} \in V(T)$ for all $i \leq k$.

The equivalence class of an element $\psi \in A_0(\mathfrak{a}_c^*)$ modulo $\hat{J}$ is uniquely determined by the values $\psi^{(j)}(\lambda)$ for $\lambda \in Z(T), j \leq m(\lambda)$.

In order to describe those double sequences $\psi_{\lambda,j}$ which correspond to elements of the quotient space $A_0(\mathfrak{a}_c^*)/\hat{J}$ one introduces a grouping in the parameter space. Let

$$\Omega := \{(\lambda, j) \mid \lambda \in Z(T), j \leq m(\lambda)\}.$$

By a grouping of Ω we mean a decomposition

$$\Omega = \bigcup_{k=1}^{\infty} \Omega_k \quad \text{where} \quad \Omega_k \cap \Omega_\iota = \phi \quad \text{for} \quad k \neq \iota \quad \text{and} \quad \# \Omega_k < \infty.$$

Denote by E_k the vector space $\mathbb{C}^{\# \Omega_k}$.

As proved in [9] for a given slowly decreasing distribution T one can find such a grouping $\{\Omega_k\}$ and such a system of norms $\| \cdot \|_k$ in E_k that the space $V(T)' = A_0(\mathfrak{a}_c^*)/\hat{J}$ becomes isomorphic to the space of all double sequences $\psi_{\lambda,j}$, $(\lambda, j) \in \Omega$ for which the following condition is satisfied:

(3.4) for some $C > 0$ $\sup_k \| \psi_k \|_k e^{-Cp_k} < \infty$, where ψ_k stands for the vector $(\psi_{\lambda,j})$, $(\lambda, j) \in \Omega_k$ ordered lexicographically;

$$p_k = \min_{\Gamma_k} (\operatorname{Im} |\lambda| + \log(1 + |\lambda|^2)).$$

The dual space of $V(T)$ is represented by the space of sequences $F_k := (F_{\lambda,j})$, $(\lambda, j) \in \Omega_k$ such that

$$\sum_k \| F_k \|_k^* e^{Cp_k} < \infty \tag{3.5}$$

for every $C > 0$; here $\| \cdot \|_k^*$ is the normal dual to $\| \cdot \|_k$ in E_k^*.

We put

$$F(\psi) := \sum_k F_k(\psi_k). \tag{3.6}$$

For fixed F the series converges absolutely, uniformly over the bounded subsets of ψ's and F's.

In particular, putting $\psi = \hat{\delta}_x$ and $f \in V(T)$ we have $\psi^{(j)}(\lambda) = \phi_\lambda^{(j)}(x)$ and the formula (3.6) gives the Fourier expansion:

$$f(x) = f(\delta_x) = \sum_{(\lambda,j) \in \Omega_k} F_{\lambda,j} \phi_\lambda^{(j)}(x). \tag{3.7}$$

The series is uniformly convergent on compact sets in X and for compact families of f's.

By taking in place of δ_x its partial derivatives with respect to x we obtain the uniform convergence on compact sets of the series of derivatives. Hence the series (3.7) converges in the space $E(K\backslash X)$.

In virtue of the above studies one obtains

THEOREM (3.1). *For any slowly decreasing $T \in E'(K\backslash X)$ every solution f of the equation $T * \check{f} = 0$ admits the Fourier series expansion* (3.7).

The series is convergent on $E(K\backslash X)$ after applying an appropriate grouping procedure subordinated to T.

Let V_f be the smallest invariant subspace of $E(K\backslash X)$ containing the function

f. If f is given by (3.7) then

(3.8) $\quad S * f(x) = \sum_{(\lambda,j)\in\Omega_k} F_{\lambda,j}\hat{S}(\lambda)\phi_\lambda^{(j)}(x) \quad$ for every $S \in E'(K\backslash X)$.

Since the transforms $\hat{S}(\lambda)$, $S \in E'(K\backslash X)$, separate the points of $Z(T)$ we have

PROPOSITION (3.2). *The functions $\phi_\lambda, \phi_\lambda^{(1)}, \cdots, \phi_\lambda^{(j)}$ belong to the space V_f generated in $E(K\backslash X)$ by the function f iff $F_{\lambda,0}, F_{\lambda,1}, \cdots, F_{\lambda,j} \neq 0$.*

Thus the sum $\sum_k$ in (3.7) ranges in fact over the spectrum of the space V_f. Let us conclude with

THEOREM (3.3). *Every invariant subspace $V \subset V(T)$ is uniquely determined by its spectrum: the set of functions ϕ_λ contained in V and their corresponding multiplicities.*

4. On the spectral synthesis in the space $E(X)$

The former results will be used for proving the Fourier expansion formulas for mean periodic functions in the space $E(X)$.

We still assume $T \in E'(K\backslash X)$ but now we consider the solutions of the equation $T * \check{f} = 0$ in the space $E(X)$.

As a preparatory step we introduce a family of invariant subspaces in $E(X)$ extending the family E_λ defined in Sec 2. Denote

(4.1) $$E_\lambda^{(j)} := \{f \in E(X) \mid \textstyle\int_K f(gkx)\,dk = \sum_{m=0}^{j} c_m(g)\phi_\lambda^{(m)}(x)\}.$$

The elements of the space $E_\lambda^{(j)}$ are then projected by the operator P into the space spanned by the functions $\phi_\lambda, \phi_\lambda^{(1)}, \cdots, \phi_\lambda^{(j)}$.

By derivation of the eq. (1.4) with respect to the variable λ we find

(4.2) $$\int_K e_{i\lambda+\rho,b}^{(n)}(gkx)\,dk = \sum_{j=0}^{n} \binom{n}{j} e_{i\lambda+\rho,b}^{(j)}(g)\phi_\lambda^{(n-j)}(x).$$

Thus the functions $e_{i\lambda+\rho,b}^{(n)}$, $\phi_\lambda^{(n)}$ belong to $E_\lambda^{(n)}$.

Let us observe the following filtration:

(4.3) $$E_\lambda^{(n)} \supset E_\lambda^{(n-1)} \supset \cdots \supset E_\lambda^{(0)} = E_\lambda.$$

Let again T denote a slowly decreasing distribution on X. If $f \in V(T)$ then the function

$$\psi_g: X \ni x \to \int_K f(gkx)\,dk = PL_{g^{-1}}f(x)$$

belongs to $V(T)$. When g ranges over a compact set $C \subset G$ then the functions ψ_g form a bounded subset in $V(T)$.

By applying the formula (3.7) to ψ_g we obtain

(4.4) $$\int_K f(gkx)\,dk = \sum_{(\lambda,j)\in\Omega_k} F_{\lambda,j}(g)\phi_\lambda^{(j)}(x)$$

The functions $F_{\lambda,j}$ are differentiable, right K-invariant and the convergence is uniform on compact sets in $G \times X$ with all derivatives.

Now apply the formula (4.4) to the function

$$x \to \int_K \int_K f(gkh\bar{k}x)\, dk\, d\bar{k} =: F(g, h, x).$$

Since

$$F(g, h, x) = \int_K F(gkh, e, x)\, dk = \int_K F(g, e, hkx)\, dx$$

we obtain the relations

$$\sum_{(\lambda,j)\in\Omega_k} \int_K F_{\lambda,j}(gkh)\, dk\, \phi_\lambda^{(j)}(x) = \sum_{(\lambda,j)\in\Omega_k} F_{\lambda,j}(g) \int_K \phi_\lambda^{(j)}(hkx)\, dk$$

$$= \sum_{(\lambda,j)\in\Omega_k} F_{\lambda,j}(g) \sum_{m=0}^{j} \binom{j}{m} \phi_\lambda^{(m)}(h)\phi_\lambda^{(j-m)}(x).$$

The uniqueness of the coefficients implies that the functions $F_{\lambda,j}$ satisfy

$$\int_K F_{\lambda,m(\lambda)-j}(gkh)\, dk = \sum_{m=0}^{j} F_{\lambda,m(\lambda)-j+m}(g)\phi_\lambda^{(m)}(h) \cdot \binom{m(\lambda)-j+m}{m}. \tag{4.5}$$

In particular we observe that all the functions $F_{\lambda,j}$ belong to the space $E_\lambda^{m(\lambda)-j}$. Putting $x = eK$ in (4.4) we conclude after consulting Prop. 3.2.

PROPOSITION (4.1). *For every T-mean periodic function f there exists a sequence* $(F_{\lambda,j}) \in E^{m(\lambda)}$, $\lambda \in Z(T), j \leq m(\lambda)$, *such that*

$$f(x) = \sum_k \sum_{(\lambda,j)\in\Omega_k} F_{\lambda,j}(x) \tag{4.6}$$

and the series is convergent in $E(X)$.

The nonzero coefficients $F_{\lambda,j}$ correspond to the values $\lambda \in \mathrm{Sp}\, PV_f$.

In order to utilize the result obtained above we need some information about the structure of the spaces $E_\lambda^{(j)}$ and the position of the functions $e_{i\lambda+\rho,b}^{(j)}$ in them.

As proved by Helgason [14] in case $e(\lambda)e(-\lambda) \neq 0$, the space E_λ is irreducible under the action of G. One can prove also that under this condition the quotient space $E_\lambda^{(j+1)}/E_\lambda^{(j)}$ is also irreducible (cf. [21]). In virtue of these results we have the following statement about the spectral synthesis in the space $V(T)$:

THEOREM (4.2). *Let us assume that the distribution* $T \in E'(K\backslash X)$ *is slowly decreasing and* $e(\lambda)e(-\lambda) \neq 0$ *for all* $\lambda \in Z(T)$. *Then every closed and invariant subspace* $V \subset V(T)$ *is completely determined by the spectrum of the space PV. The functions* $e_{i\lambda+\rho,b}^{(j)}$ *contained in V generate the space.*

Proof. Consider the space PV which is nontrivial iff V is nontrivial. According to Thm 3.3 the space PV is determined uniquely by its spectrum. Proposition 4.1 says that V is contained in the closure of the sum $\lambda \in \oplus$ Sp $PV^{E_\lambda^{n(\lambda)}}$, $n(\lambda)$ denoting the multiplicity of λ in Sp PV. By the irreducibility of the spaces E_λ and $E_\lambda^{(j+1)}/E_\lambda^{(j)}$ the functions $\phi_\lambda, \cdots, \phi_\lambda^{n(\lambda)}$ generate by translations the space $E^{n(\lambda)}$. The same is true for the corresponding functions $e_{i\lambda+\rho,b}{}^{(j)}$.

If the functional $\lambda \in \mathfrak{a}_c{}^*$ satisfies $e(\lambda)e(-\lambda) = 0$ the spectral synthesis fails. In the case $G = SL(2, R)$ one can observe for all singular values λ that E_λ contains 3 invariant subspaces and only 2 classes of plane waves $e_{i\lambda+\rho,b}$ and $e_{-i\lambda+\rho,b}$ (cf. [21]).

The structure of the space E_λ was studied in consecutive papers by Helgason [14], I, II. For the rank one case the problem was completely solved in the latter paper.

Definition. Let (τ, V) be a representation of the group G. A vector $v \in V$ is called K-finite if the vectors $\tau(k)v$, $k \in K$ span a finite dimensional subspace in V.

Definition. Analytic functionals on B are linear continuous functionals on the space $A(B)$ of real analytic functions of B with its usual topology (see [14] or [15]).

THEOREM (4.3). [14; II]. *The K-finite elements of the spaces E_λ are precisely*

(4.7) $$f(x) = \int_B e_{i\lambda+\rho,b}(x)F(b)\, db,$$

where F is a K-finite function on B.

THEOREM (4.4) [14; I]. *If rank $X = 1$ then each element $f \in E_\lambda$ has the form*

(4.8) $$f(x) = \int_B e_{i\lambda+\rho,b}(x)\, dT,$$

where T is an analytic functional on B.

THEOREM (4.5). *Under the assumptions of Prop.* 4.1 *for every T-mean periodic function f and every $\lambda \in Z(T)$ there exists such a family of analytic functionals $T_{\lambda,j}$, $j = 1, 2, \cdots, m(\lambda)$ that*

(4.9) $$f(x) = \sum_k \sum_{\Omega_k} \int_B e_{i\lambda+\rho,b}{}^{(j)}(x)\, dT_{\lambda,j}.$$

Proof. We already know that

$$f(x) = \sum_k \sum_{\Omega_k} F_{\lambda,j}(x),$$

where $F_{\lambda,j} \in E_\lambda{}^{(j)}$. The spaces $E_\lambda{}^{(j)}$ can be described in the following way:

PROPOSITION (4.6). *All elements of the space $E_\lambda{}^{(j)}$ are given by the formula*

(4.10) $$f(x) = \sum_{n=0}^{j} \int_B e_{i\lambda+\rho,b}{}^{(n)}(x)\, dT_n$$

where T_n are analytic functionals on B.

Proof of the Proposition. We apply induction with respect to j. For $j = 0$ the conclusion is just the theorem of Helgason.

Assume that for $j - 1$ the statement has been proved and let $f \in E_\lambda{}^{(j)}$. Then we have

$$\int_K f(gkx)\, dk = \sum_{n=0}^{j} c_n(g)\phi_\lambda{}^{(n)}(x).$$

The functions c_n are all right K-invariant, then can be treated as functions on X. We observe that c_j belongs to E_λ. To see this let us calculate

$$I = \int_K \int_K f(gkh\bar{k}x)\, dk\, d\bar{k} = \sum_{n=0}^{j} \int_K c_n(gkh)\, dk\, \phi_\lambda{}^{(n)}(x).$$

On the other hand

$$I = \sum_{n=0}^{j} c_n(g) \int_K \phi_\lambda{}^{(n)}(hkx)\, dk$$
$$= \sum_{n=0}^{j} c_n(g) \sum_{m=0}^{n} \binom{n}{m} \phi_\lambda{}^{(m)}(h)\phi_\lambda{}^{(n-m)}(x).$$

Comparing the coefficients of $\phi^{(j)}(x)$ in both expressions we find

$$\int_K c_j(gkh)\, dk = c_j(g)\phi_\lambda(h),$$

what means that $c_j \in E_\lambda$.

Note. By comparing other terms we can see that $c_n \in E_\lambda{}^{(j-n)}$.

According to Thm 4.3 there exists such an analytic functional T_j that

$$c_j(gk) = \int_B e_{i\lambda+\rho,b}(x)\, dT_j.$$

Let us define the function

$$\psi(x) = f(x) - \int_B e_{i\lambda+\rho,b}{}^{(j)}(x)\, dT_j.$$

We obtain

$$\int_K \psi(gkx)\, dk = \sum_{n=0}^{j} c_n(g)\phi_\lambda{}^{(n)}(x) - \int_B \int_K e_{i\lambda+\rho,b}{}^{(j)}(gkx)\, dT_j$$
$$= \sum_{n=0}^{j} (c_n(g) - \int_B e_{i\lambda+\rho,b}{}^{(j-n)}(g)\, dT_j)\phi_\lambda{}^{(n)}(x).$$

By the definition of T_j the coefficient of $\phi_\lambda{}^{(j)}$ is zero, hence $\psi \in E_\lambda{}^{(j-1)}$.

Applying the induction hypothesis we end the proof of Prop. 4.5 and of the theorem.

By using Thm 4.4 in place of 4.3 we obtain the same way:

THEOREM (4.7). *Assume that a T-mean periodic function f is K-finite. Then there exists a sequence of K-finite functions $t_{\lambda,j}$ on B such that*

$$f(x) = \sum_k \sum_{\Omega_k} \int_B e_{i\lambda+\rho,b}{}^{(j)}(x) t_{\lambda,j}(b)\, db. \tag{4.11}$$

The last formula may be compared with the Ehrenpreis Fundamental Principle Theorem which states that in $\mathbb{C}^n$ a solution of the differential equation

$$P(D)f = 0 \left(P \text{ is a polynomial of } n \text{ variables, } D_k = \frac{1}{i}\frac{d}{dx_k}\right)$$

is represented by

$$f(x) = \sum_{k=0}^{r} \int_{V_k} \partial_k\, e^{-i\langle x,z\rangle}\, d_{\nu_k}(z),$$

where V_k are algebraic varieties contained in the variety of zeros of P, ∂_k are differential operators of constant coefficients and d_{ν_k} are Radon measures on V_k.

Finally, let us discuss the case of $T \in E'(X)$ without the assumption of K-invariance. As before, let $V(T)$ denote the set of solutions of the equation

$$T(L_g f) = 0 \quad \text{for all} \quad g \in G.$$

If $f \in V(T)$ then, in particular, it satisfies also the equation $T(PL_g f) = 0$ for all $g \in G$, that is $f \in V(PT)$.

The latter space is the greatest one which contains exactly the same zonal spherical functions and their derivatives $\phi_\lambda{}^{(i)}$ which are contained in $V(T)$. If the set $Z(PT)$ does not contain the zeroes of the function $e(\lambda)e(-\lambda)$ then $V(PT) = V(T)$. This suggests the following

Definition. A distribution $T \in E'(x)$ is called slowly decreasing if its projection PT is slowly decreasing.

All results of this section are valid for general slowly decreasing distributions on rank one symmetric spaces of noncompact type.

It was proved in [21] (Thm 5.1) that for an arbitrary closed and invariant subspace $V \subset E(X)$ all elements of V can be approximated by means of linear combinations $\sum_{\lambda_j,k} F_{\lambda_j,k}$ with $\lambda_j \in \operatorname{Sp} PV$, $k \leq m(\lambda_j)$, $F_{\lambda_j,k} \in E_\lambda{}^{(k)}$. By applying proposition 4.6 we obtain

THEOREM (4.8). *Let V be a closed invariant nontrivial subspace of $E(X)$. Then each element of V can be approximated in $E(X)$ by functions of the form*

$$f(x) = \sum_{j=1}^{n} \sum_{m=0}^{m(\lambda_j)} \int_B e_{i\lambda_j+\rho,b}{}^{(m)}(x)\varphi_{j,m}(b)\, db$$

with $\lambda_j \in \operatorname{Sp} PV$ and $\varphi_{j,m}$ being K-finite functions on B.

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