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NOETHERIAN BASES IN ORDINAL SPACES*

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Summary

A collection $\mathscr{C}$ of subsets of a set X is said to be Noetherian if $\mathscr{C}$ does not contain a strictly increasing infinite chain. In this paper we show that a space of ordinals $[0, \alpha)$, has a Noetherian base if and only if α is smaller than the first strongly inaccessible cardinal.

1. Introduction

W. F. Lindgren and P. J. Nyikos in [2], gave an example due to J. Vaughan of a topological space without a Noetherian base. The space in question is that of the ordinals smaller than the first strong limit cardinal k of uncountable cofinality with the order topology. The proof is based on the following result: *If* $f: [0, k) \rightarrow [0, k)$ is a regressive function, then there exists $b \in [0, k)$ and $A \subset [0, k)$ such that $f(a) \leq b$ for every $a \in A$ and $|A| > 2^{|b|}$. Nevertheless, this proposition is not true in general. (In the case where k is strongly inaccessible it is true. See [1 Theorem A 1.3]), as can be seen from the following example: Let $k = \lim_{a<w_1}\{\exp^a w\}$ where w is the first infinite cardinal and w_1 is the first uncountable one. Let us consider the function

$f: [0, k) \rightarrow [0, k)$ defined by

$$f(b) = \begin{cases} \exp^a w & \text{if } \exp^a w < b < \exp^{a+1} w \\ a & \text{if } b = \exp^a w \\ 0 & \text{if } b < \exp^1 w \end{cases}$$

($\exp^a w$ is defined by induction in the following manner: $\exp^1 w = 2^w$; if $\exp^n w$ is defined by every $n < a$ and a is a limit ordinal, $\exp^a w = \lim_{n<a}\{\exp^n w\}$. If a is not a limit and $a > 1$, $\exp^a w = 2^{\exp^{a-1} w}$).

The main result of this paper completely determines for which ordinals α, $[0, \alpha)$ has a Noetherian base. The theorem is the following: $[0, \alpha)$ and any of its subspaces has a Noetherian base if and only if $[0, \alpha]$ does not contain a strongly inaccessible cardinal.

Recently, J. Vaughan informed me that in 1983 E. van Douwen presented this theorem at a conference and he had a copy of this manuscript sent to me. Nevertheless, the proof appearing in the manuscript only shows that $[0, \alpha)$ has a Noetherian base if α is less than the first weakly inaccessible cardinal and that $[0, \alpha)$ does not have a Noetherian base when α is a strongly inaccessible cardinal.

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In this article inaccessible cardinal will mean uncountable inaccessible cardinal.

2. Spaces of Ordinals and Noetherian Bases

Definition 2.1. A collection $\mathscr{C}$ of subsets of a set X is *Noetherian* if $\mathscr{C}$ does not contain strictly increasing infinite chains. That is, $\mathscr{C}$ is Noetherian if whenever $\{C_n\}_{n\in\mathbb{N}} \subset \mathscr{C}$ is such that $C_1 \subset C_2 \subset \cdots$, then $\{C_n\}_{n\in\mathbb{N}}$ is a finite collection.

LEMMA (2.2). *Let Y be a subset of a T_1 space X and A an open subset of Y. Then there exists a Noetherian collection $\mathscr{B}$ of open sets in X, such that*
(i) *For every $B \in \mathscr{B}$, $B \cap Y = A$.*
(ii) *If C is an open subset of X such that $A \subset C \cap Y$, then exists $B \in \mathscr{B}$ satisfying $B \subset C$.*

Proof. If A is open in X, then we may take $\mathscr{B} = \{A\}$. Let us now suppose that A is not open in X. If C is an open subset of X such that $A \subset Y \cap C$, we can construct by induction a strictly decreasing chain $\mathscr{A}$ of open sets in X that satisfy
(a) For every $B \in \mathscr{A}$, $B \cap Y = A$ and $B \subset C$.
(b) $A \cap (\text{int} \cap \mathscr{A})^c \neq \varnothing$.

In fact, if for some ordinal α, $\{B_\lambda\}_{\lambda<\alpha}$ is a collection of open sets in X such that $B_{\lambda_2} \subsetneq B_{\lambda_1} \Leftrightarrow \lambda_1 < \lambda_2 < \alpha$, satisfying (a) and $A \subset \text{int}(\cap_{\lambda<\alpha} B_\lambda)$, the contention is proper and since X is T_1, there is an open set B_α contained properly in each one of the B_λ and whose intersection with Y is A. This process must finish for some ordinal α_0. $\mathscr{A} = \{B_\lambda\}_{\lambda<\alpha_0}$ is the required chain.

In the same manner we construct $\mathscr{B}$ by induction. Assume that for some ordinal α we have $\{\mathscr{A}_\lambda\}_{\lambda<\alpha}$, where for each $\lambda < \alpha$, $\mathscr{A}_\lambda$ is a strictly decreasing chain of open sets in X which satisfies (a) and (b) for some open $C \subset X$ such that $A \subset C \cap Y$ and if $\lambda_1 < \lambda_2 < \alpha$ and $B \in \mathscr{A}_{\lambda_2}$, then B does not contain any element belonging to $\mathscr{A}_{\lambda_1}$. If $\cup_{\lambda<\alpha}\mathscr{A}_\lambda$ does not satisfy (ii), then there is $C \subset X$ an open set such that $A \subset C \cap Y$ and C does not contain any element in $\cup_{\lambda<\alpha}\mathscr{A}_\lambda$. Let $\mathscr{A}_\alpha$ be a strictly decreasing chain of open sets in X that satisfies (a) and (b) with respect to C. This process must finish for some α_0. So $\mathscr{B} = \cup_{\lambda<\alpha_0}\mathscr{A}_\lambda$ satisfies (i) and (ii) and is a Noetherian collection since $\mathscr{A}_\lambda$ is, and if $B_1 \in \mathscr{A}_{\lambda_1} y B_2 \in \mathscr{A}_{\lambda_2}$ with $B_1 \subsetneq B_2$ then $\lambda_1 \geq \lambda_2$.

COROLLARY (2.3). *Let Y be a subspace of a T_1 space X and let $\mathscr{B}'$ be a Noetherian base for Y. Then there exists a Noetherian collection of open sets $\mathscr{B}$ in X such that $\mathscr{B}' = \{B \cap Y : B \in \mathscr{B}\}$ and for each $y \in Y$, $\mathscr{B}$ contains a local base of y in X.*

Proof. For each $B' \in \mathscr{B}'$ let $\mathscr{B}(B')$ be the collection whose existence is guaranteed by Lemma 2.2. $\mathscr{B} = \cup\{\mathscr{B}(B') : B' \in \mathscr{B}'\}$ is the required Noetherian collection.

COROLLARY (2.4). *Let X be a T_1 space. For each $x \in X$, x has a Noetherian local base of neighborhoods.*

LEMMA (2.5). *Any open subspace of a space with a Noetherian base, has a Noetherian base and the disjoint topological union of spaces with a Noetherian base, also possesses a Noetherian base.*

Definition 2.6. A collection $\mathscr{A}$ of subsets of a set X is an *antichain* if for each pair of different elements in $\mathscr{A}$, A_1 and A_2, we have $A_1 \not\subset A_2$ and $A_2 \not\subset A_2$.

LEMMA (2.7).[i] *Let X be an infinite set of cardinality α. Then there exists an antichain $\mathscr{A} \subset \mathscr{P}(X)$ such that $|\mathscr{A}| = 2^\alpha$ and for each $A \in \mathscr{A}$, $|A| = \alpha$.*

Proof. Let $\mathscr{P} \subset \mathscr{P}(X)$ be a partition of X consisting of subsets of X, each one with exactly two elements. The collection of choice functions defined on $\mathscr{P}$, determines an antichain with the desired properties.

THEOREM (2.8). *Let α be an ordinal, $[0, \alpha)$ has a Noetherian base if and only if $[0, \alpha]$ does not contain a strongly inaccessible cardinal.*

Proof. Assume that $[0, \alpha]$ does not contain a strongly inaccessible cardinal and that for every $\beta < \alpha$, $[0, \beta)$ has a Noetherian base. If α is a successor ordinal, Corollary 2.4 and the hypothesis of induction show that $[0, \alpha)$ has a Noetherian base.

Suppose that α is a limit ordinal.

First Case. $\operatorname{cof} \alpha < \alpha$.

By the induction hypothesis $[0, \operatorname{cof} \alpha)$ possesses a Noetherian base. Let $A \subset [0, \alpha)$ be a closed cofinal subset in $[0, \alpha)$ homeomorphic to $[0, \operatorname{cof} \alpha)$. $[0, \alpha) - A$ has a Noetherian base $\mathscr{B}_1$ (by Lemma 2.5 and the induction hypothesis). Let $\mathscr{B}_2$ be a Noetherian collection of open sets that contains a local base for each $a \in A$ (Corollary 2.3). $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$ is then a Noetherian base of $[0, \alpha)$.

Second Case. α is a non-limit cardinal.

Consider the set $A = \{\gamma + \alpha^-: \gamma \in [0, \alpha)\}$ where α^- is the immediate predecessor cardinal of α (α^- exists because α is a nonlimit cardinal).

Thus we have that:

(**a**) A is a cofinal set in $[0, \alpha)$ and the closure of A in $[0, \alpha)$, $\bar{A}$, is homeomorphic to $[0, \alpha)$ and it follows that $Y = [0, \alpha) - \bar{A}$ is a disjoint topological union of spaces with Noetherian bases. Let $\mathscr{B}_1$ be a Noetherian base of Y.

(**b**) For each $\lambda = \gamma + \alpha^- \in A$, let $L = \{\xi \in (\gamma, \gamma + \alpha^-): \xi \text{ is isolated}\}$. L

[i] I wish to express my sincere thanks to Professor Victor Neumann for having called my attention to this result.

satisfies:

(i) $|L| = \alpha^-$.

(ii) If $L' \subset L$ is such that $|L'| = \alpha^-$, then L' has order type α^- and is a cofinal set in $[0, \lambda)$.

(**c**) For each $\lambda \in A$, there exists an antichain $\mathscr{L} \subset \mathscr{P}(L)$ such that $|\mathscr{L}| = 2^{\alpha^-}$, and if $L' \in \mathscr{L}$, $|L'| = \alpha^-$ (Lemma 2.7).

For each $\lambda \in A$, let φ_λ: $(\lambda, \alpha) \to \mathscr{L}$ be an injective function. Each $\varphi_\lambda(\beta)(\lambda < \beta < \alpha)$ is a cofinal set in $[0, \lambda)$ and it has order type α^-, and so it can be indicated as $\varphi_\lambda(\beta) = \{x_\delta\}_{\delta<\alpha^-}$ such that $\delta_1 < \delta_2 \Leftrightarrow x_{\delta_1} < x_{\delta_2}$. If $\delta_0 < \alpha^-$, let $\varphi_\lambda{}^{\delta_0}(\beta) = \{x_\delta\}_{\delta_0<\delta<\alpha^-}$ and for each triple $(\lambda, \beta, \delta) \in A \times (\lambda, \alpha) \times [0, \alpha^-)$, let $S(\lambda, \beta, \delta) = \varphi_\lambda{}^\delta(\beta) \cup (\lambda, \beta]$. Each $S(\lambda, \beta, \delta)$ is an open set in $[0, \alpha)$ and the collection $\mathscr{B}_2 = \{S(\lambda, \beta, \delta):(\lambda, \beta, \delta) \in A \times (\lambda, \alpha) \times [0, \alpha^-)\}$ is Noetherian. In fact, if $S(\lambda_1, \beta_1, \delta_1) \subset S(\lambda_2, \beta_2, \delta_2)$ and if $\lambda_1 < \lambda_2$, then $\varphi_{\lambda_1}{}^{\delta_1}(\beta_1) \subset \varphi_{\lambda_2}{}^{\delta_2}(\beta_2)$ and this means, because of (b), that $\varphi_{\lambda_1}{}^{\delta_1}(\beta_1)$ is a cofinal set in $[0, \lambda_2)$, which contradicts the inclusion $\varphi_{\lambda_1}{}^{\delta_1}(\beta_1) \subset [0, \lambda_1)$. Then $\lambda_1 \geq \lambda_2$. So if $S(\lambda_1, \beta_1, \delta_1) \subset S(\lambda_2, \beta_2, \delta_2) \subset \cdots \subset S(\lambda_n, \beta_n, \delta_n) \subset \cdots$, there exists n_0 such that $\lambda_s = \lambda_{n_0}$ for all $s \geq n_0$ and, by construction, $\beta_s = \beta_{n_0}$ for all $s \geq n_0$ and $\delta_{n_0} \geq \delta_{n_0+1} \geq \delta_{n_0+2} \geq \cdots$, which means that the sequence $\{S(\lambda_n, \beta_n, \delta_n)\}_{n\in\mathbb{N}}$ is finite.

(**d**) Let $A' = \{a \in A: a$ is an isolated element in $A\}$. For each $a \in A'$ we can take a Noetherian local base $\mathscr{B}(a)$ of a in $[0, \alpha)$ such that if $B \in \mathscr{B}(a)$, $B \cap A = \{a\}$. Let $\mathscr{B}_3 = \bigcup_{a\in A'}\mathscr{B}(a)$.

(**e**) $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2 \cup \mathscr{B}_3$ is a Noetherian collection of open sets in $[0, \alpha)$ and it is clear that it contains a local base for each point in $[0, \alpha) - (\bar{A} - A')$. Let $\beta \in \bar{A} - A'$ and $\gamma < \beta$. Since $\beta \in \bar{A} - A'$, there exists $\lambda \in A$ such that $\gamma < \lambda < \beta$ and $\delta < \alpha^-$ satisfying $\varphi_\lambda{}^\delta(\beta) \cup (\lambda, \beta] \subset (\gamma, \beta]$. Thus $\mathscr{B}$ is a Noetherian base of $[0, \alpha)$.

Third Case. α is a limit regular cardinal which is not strongly inaccessible.

Let α_0 be a regular cardinal smaller than α such that $2^{\alpha_0} \geq \alpha$.

The proof for this case is analogous to the one given in the preceding case; the only difference being the substitution of α^- for α_0.

As mentioned in the Introduction, the argument which appears in Example 5.5 in [2] is valid for k strongly inaccessible. That is, $[0, k)$ does not have a Noetherian base in this situation. Because of Lemma 2.5, if $\alpha \geq k$, $[0, \alpha)$ does not have a Noetherian base. This ends the proof of our theorem.

LEMMA (2.9). *Let $Y \subset [0, \alpha)$ and $y \in Y$. Let $\mathscr{B}(y)$ be a local base of y in (Y, τ_0), where τ_0 is the order topology in Y. Then $\mathscr{B}(y)$ is a local base of y in (Y, τ_R) or $\{y\} \in \tau_R$, where τ_R is the relative topology of Y.*

Proof. If y is an isolated point in Y, then $\{y\} \in \tau_R$. Assume that y is not isolated and that A is an open set in $[0, \alpha)$ that contains y. Let $x < y$ be such that $(x, y] \subset A$. Then there exists $z \in Y$ such that $x < z < y$ and so $(z, y] \cap Y \in \tau_0$. Thus there exists $B \in \mathscr{B}(y)$ such that $y \in B \subset (z, y] \cap Y \subset A \cap Y$.

THEOREM (2.10). *Let $Y \subset [0, \alpha)$, where α is an ordinal such that $[0, \alpha]$ does not contain a strongly inaccessible cardinal. Then (Y, τ_R) has a Noetherian base.*

Proof. (Y, τ_0) is homeomorphic to some $[0, \beta)$, with $\beta \leq \alpha$. Let $\mathscr{B}_1$ be a Noetherian base of (Y, τ_0) and let $\mathscr{B}_2 = \{\{y\}:y$ is an isolated element in $Y\}$. From the previous Lemma, $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$ is a Noetherian base for (Y, τ_R).

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