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TRANSFER AND THE ROTHENBERG-STEENROD SPECTRAL SEQUENCES

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1. Introduction

In [Pr_2] Rothenberg-Steenrod spectral sequences were constructed approximating $h(EG \underset{G}{\times} F)$ resp. $h^*(EG \underset{G}{\times} F)$ for h, resp. h^* a generalized homology, resp. cohomology theory, satisfying certain conditions, such that the E_2-term is $\mathrm{Tor}^{h(G)}(h(*), h(F))$, resp. $\mathrm{Ext}_{h(G)}(h(*), h^*(F))$, where EG is the total space of the classifying bundle of the topological group G and F is an arbitrary G–space.

If one has a *triangle*

1.1.
$$\begin{array}{ccc} EG \underset{G}{\times} F \supset V & \xrightarrow{f} & EG \underset{G}{\times} F \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & BG & \end{array}$$

admitting a transfer

$$\tau(f)\colon h(BG) \to h(EG \underset{G}{\times} F),$$

resp.
$$\tau(f)\colon h^*(EG \underset{G}{\times} F) \to h^*(BG),$$

and one considers the filtration of BG

$$\mathscr{B}\colon B_0 \subset B_1 \subset \cdots \subset B_n \subset B_{n+1} \subset \cdots \subset BG$$

induced by the Milnor G-resolution $E_0 \subset E_1 \subset \cdots \subset E_n \subset E_{n+1} \subset \cdots \subset EG$ (see [Pr_2 §1]) one has by [Pr_1 1.4.] the

PROPOSITION (1.2). *$\tau(f)$ induces a transformation of spectral sequences*

$$\tau\colon \{\bar{E}^r, \bar{d}^r\} \to \{E^r, d^r\}$$

resp.
$$\tau\colon \{E_r, d_r\} \to \{\bar{E}_r, \bar{d}_r\}$$

where $\{\bar{E}^r, \bar{d}^r\}$, resp. $\{\bar{E}_r, \bar{d}_r\}$ is the spectral sequence induced by $\mathscr{B}$ for h, resp. h^* and $\{E^r, d^r\}$, resp. $\{E_r, d_r\}$ is the spectral sequence induced by the filtration

$$\mathscr{E}\colon E_0 \underset{G}{\times} F \subset E_1 \underset{G}{\times} F \subset \cdots \subset E_n \underset{G}{\times} F \subset \cdots \subset EG \underset{G}{\times} F. \quad \square$$

Theorem 3.3 in [Pr_2] states:

1.3. *If the homology algebra $h(G)$ is $h(*)$-projective, then*

$$\text{(a)}\quad \begin{cases} \bar{E}^2_{pq} \cong \mathrm{Tor}^{h(G)}_{pq}\,(h(*),\, h(*)) \\ E^2_{pq} \cong \mathrm{Tor}^{h(G)}_{pq}\,(h(*),\, h(F)) \end{cases}$$

and in case that h and h^* are represented by a spectrum satisfying the Adams' condition 2.9 *in* [Pr_2], *then*

$$\text{(b)}\quad \begin{cases} \bar{E}_2^{pq} \cong \mathrm{Ext}^{pq}_{h(G)}\,(h(*),\, h^*(*)) \\ E_2^{pq} \cong \mathrm{Ext}^{pq}_{h(G)}\,(h(*),\, h^*(F)) \end{cases}$$

The purpose of this paper is to show the compatibility of the transfer with these isomorphisms. Namely:

THEOREM (1.4). *Let f be such that $\tau(f_0)$: $h(*) \to h(F)$, resp. $\tau(f_0)$: $h^*(F) \to h^*(*)$ is an $h(G)$-homomorphism (see* 2.7. *below), where*

$$\begin{array}{ccc} F \supset W & \xrightarrow{f_0} & F \\ & \pi_0 \searrow \quad \swarrow \pi_0 & \\ & \{*\} & \end{array}$$

is the restriction of 1.1. *to the base point* (B_0) *of* BG *(by the inclusion* $F \to EG \times_G F$ *given by* $y \mapsto [x_0, y]$*) and* $h(*)$, $h(F)$, $h^*(*)$ *and* $h^*(F)$ *are given the* $h(G)$*-module structures induced by the G-space structures on* $*$ *and F.*

Then

$$1.5.\qquad \begin{array}{ccc} \bar{E}^2_{pq} & \cong & \mathrm{Tor}^{h(G)}_{pq}\,(h(*),\, h(*)) \\ \tau\downarrow & & \downarrow \mathrm{Tor}(1,\, \tau(f_0)) \\ E^2_{pq} & \cong & \mathrm{Tor}^{h(G)}_{pq}\,(h(*),\, h(F)) \end{array}$$

and

$$1.6.\qquad \begin{array}{ccc} E_2^{pq} & \cong & \mathrm{Ext}^{pq}_{h(G)}\,(h(*),\, h^*(F)) \\ \tau\downarrow & & \downarrow \mathrm{Ext}(1,\, \tau(f_0)) \\ \bar{E}_2^{pq} & \cong & \mathrm{Ext}^{pq}_{h(G)}\,(h(*),\, h^*(*)) \end{array}$$

are commutative.

1.7. *Remark.* Theorem 1.4 can be useful for proving splitting results for maps $BH \to BG$, $H \subset G$ compact Lie groups, in adequate homology or cohomology theories, by looking for maps f (1.1.), such that $\tau(f_0)$ splits the map $G/H \to *$.

2. Proof of the Main Theorem

For the proof we need a series of lemmas. First recall that the *complete* G-resolution

$$G = E_0 \subset D_0 \subset E_1 \subset D_1 \subset \cdots \subset E_{n-1} \subset D_n \subset \cdots \subset EG$$

such that the G-action $\varphi\colon EG \times G \to EG$ induces relative homeomorphisms

$$\varphi_n\colon (D_n, E_{n-1}) \times G \to (E_n, E_{n-1})$$

(in particular $\varphi_0\colon D_0 \times G \approx E_0$).

LEMMA (2.1). *The diagram*

$$\begin{array}{ccc} D_n \times F & \xrightarrow{\xi_n} & E_n \underset{G}{\times} F \\ \downarrow & & \downarrow \pi \\ D_n & \xrightarrow{\xi_n} & B_n \end{array}$$

is a pullback–diagram, where ξ_n *is the restriction of the canonical identification and the* π*'s denote the corresponding projections.*

Proof. Decompose the diagram as follows

$$\begin{array}{ccccc} D_n \times F & \xrightarrow{\mu} & \pi_n^*(E_n \underset{G}{\times} F) & \xrightarrow{\pi_n} & E_n \underset{G}{\times} F \\ & \searrow & \swarrow & & \downarrow \pi \\ & & D_n & \xrightarrow{\quad \pi_n \quad} & B_n \end{array}$$

where $\pi_n^*(E_n \underset{G}{\times} F)$ denotes the canonical pullback and $\mu(x, y) = (x, [x, y])$ ($[x, y] = \xi_n(x, y)$).

We need only show that μ is a homeomorphism (over D_n). Being EG principal, there is a continuous *translation map*

$$\alpha\colon E^* = \{(x, x') \in EG \times EG \mid \pi x = \pi x' \in BG\} \to G$$

such that $x' = x\alpha(x, x')$ for all $(x, x') \in E^*$ (cf. [Hm, 4.2.2.]). Let $\nu\colon \pi^*(E_n \underset{G}{\times} F) \to D_n \times F$ be such that $\nu(x', [x, y]) = (x', \alpha(x', x)y)$; ν is well-defined, continuous and inverse to μ, as one sees easily. □

Let now $\bar{f}\colon \xi_n^{-1}(V \cap (E_n \underset{G}{\times} F)) \to D_n \times F$ be the pullback of the restriction of f to $V \cap (E_n \underset{G}{\times} F) \to E_n \underset{G}{\times} F$. This map has the form

$$\bar{f}(x, y) = (x, \varphi(x, y)).$$

On the other hand one has the map $id \times f_0\colon D_n \times F \to D_n \times F$ (f_0 is defined only on $W = \{y \in F \mid [x_0, y] \in V\}$). Both maps admit transfers between the same groups. Namely one has

LEMMA (2.2). *$\tau(\bar{f}) = \tau(id \times f_0)$: $h(D_n, E_{n-1}) \to h((D_n, E_{n-1}) \times F)$ and correspondingly for h^*.*

Proof. Recall that $D_n = C(E_{n-1})$ (the cone over E_{n-1}), whence it is contractible. Applying Lemma 2.10. in [Pr₁] we get the result. □

Combining 2.1. and 2.2. with the naturality of the transfer, we obtain

COROLLARY (2.3). *The diagrams*

$$\begin{array}{ccc}
h(D_n, E_{n-1}) & \xrightarrow{(\pi_h)_*} & h(B_n, B_{n-1}) \\
\tau(id \times f_0) \downarrow & & \downarrow \tau(f) \\
h((D_n, E_{n-1}) \times F) & \xrightarrow{(\xi_n)_*} & h((E_n, E_{n-1}) \underset{G}{\times} F)
\end{array}$$

and

$$\begin{array}{ccc}
h^*((D_n, E_{n-1}) \times F) & \xleftarrow{(\xi_n)^*} & h^*((E_n, E_{n-1}) \underset{G}{\times} F) \\
\tau(id \times f_0) \downarrow & & \downarrow \tau(f) \\
h^*(D_n, E_{n-1}) & \xleftarrow{(\pi_n)^*} & h^*(B_n, B_{n-1})
\end{array}$$

are commutative.

For the proof of Theorem 1.4. we must show the commutativity of the diagrams

2.4.
$$\begin{array}{ccc}
h(E_n, E_{n-1}) \otimes_{h(g)} h(*) & \xrightarrow{\beta_{[*]}} & h(B_n, B_{n-1}) \\
1 \otimes \tau(f_0) \downarrow & & \downarrow \tau(f) \\
h(E_n, E_{n-1}) \otimes_{h(G)} h(F) & \xrightarrow{\beta_F} & h((E_n, E_{n-1}) \underset{G}{\times} F)
\end{array}$$

and

2.5.
$$\begin{array}{ccc}
\mathrm{Hom}_{h(G)}(h(E_n, E_{n-1}), h^*(F)) & \xleftarrow{\beta_F} & h^*((E_n, E_{n-1}) \underset{G}{\times} F) \\
\mathrm{Hom}(1, \tau(f_0)) \downarrow & & \downarrow \tau(f) \\
\mathrm{Hom}_{h(G)}(h(E_n, E_{n-1}), h^*(*)) & \xleftarrow{\beta_{[*]}} & h^*(B_n, B_{n-1})
\end{array}$$

To that purpose we put these diagrams on the right of the diagram

2.6.
$$\begin{array}{ccccccc}
h(D_n, E_{n-1}) & \underset{\cong}{\overset{\alpha}{\longleftarrow}} & h(D_n, E_{n-1}) \otimes h(*) & \overset{i_*\otimes 1}{\longrightarrow} & h(E_n, E_{n-1}) \otimes h(*) & \overset{\gamma_1}{\longrightarrow} & h(E_n, E_{n-1})_h \otimes_{(G)} h(*) \\
\downarrow{\scriptstyle \tau(id \times f_0)} & & \downarrow{\scriptstyle 1\otimes\tau(f_0)} & & \downarrow{\scriptstyle 1\otimes\tau(f_0)} & & \downarrow{\scriptstyle 1\otimes\tau(f_0)} \\
h((D_n, E_{n-1}) \times F) & \underset{\cong}{\overset{\alpha}{\longleftarrow}} & h(D_n, E_{n-1}) \otimes h(F) & \overset{i_*\otimes 1}{\longrightarrow} & h(E_n, E_{n-1}) \otimes h(F) & \overset{\gamma_1}{\longrightarrow} & h(E_n, E_{n-1})_h \otimes_{(G)} h(F)
\end{array}$$

and of its corresponding dual respectively. Diagram 2.6. and its dual are commutative (those on the extreme left, since α -and its dual $\bar{\alpha}$- are natural transformations between homology -and cohomology- theories).

The rows of 2.6. composed with 2.4. coincide with the isomorphisms $(\pi_n)_*$ and $(\xi_n)_*$ -and those of its dual composed with 2.5. coincide with $(\xi_n)^*$ and $(\pi_n)^*$- as seen in the proofs 3.15 and 3.15* in Pr_2. Hence the composed diagrams are nothing else but the commutative diagrams of 2.3. Since the rows of 2.6. and of its dual are isomorphisms, it follows that the diagrams in question are also commutative, as we wanted. □

2.7. N.B. The assumption that $\tau(f_0)$: $h(*) \longrightarrow h(F)$, resp. $\tau(f_0)$: $h^*(F) \longrightarrow h^*(*)$, has to be an $h(G)$-homomorphism is needed to guarantee that the statements of Theorem 1.4. make sense. One may easily prove that this will be the case if f_0 is a G-map. This condition is satisfied, for example, if $f = id \times g_0$, g_0 a G-map, (then $f_0 = g_0$), or if $f = f' \underset{G}{\times} f''$ for f' and f'' G-maps; in this case G being abelian.

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