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CLASSIFICATION OF LOW DIMENSIONAL ORTHOGONAL PAIRINGS

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Introduction

Let (F^n, q) be a quadratic space over a field F of characteristic $\neq 2$, where the quadratic map q is given by $q(x) = x_1^2 + \cdots + x_n^2$ for column vectors $x = (x_1, \cdots, x_n)^t \in F^n$. Since all our quadratic spaces will be of this form, we briefly write F^n for (F^n, q). If $\Phi: F^r \times F^s \to F^n$ is a bilinear map then, for $x \in F^r$ and $y \in F^s$ we have $\Phi(x, y) = (z_1, \cdots, z_n)^t$, where each z_i is a bilinear form in x and y with coefficients in F. Here, x and y are viewed as variables. A bilinear map Φ is called an *orthogonal pairing* of type $[r, s, n]$ over F, if

$$(x_1^2 + \cdots + x_r^2)(y_1^2 + \cdots + y_s^2) = z_1^2 + \cdots + z_n^2,$$

for all $x \in F^r$ and $y \in F^s$. Conversely, if there exists such a formula then, there exists an orthogonal pairing Φ. To use a short form to express this, we will say that the triple $[r, s, n]$ is *admissible* over F [6; p 236].

Following S. Yuzvinsky [7; p 139], we say that two orthogonal pairings Φ, Ψ or type $[r, s, n]$ over F are *equivalent* if there exist orthogonal matrices S, Q and P of orders r, s and n, respectively, such that $P\Phi(Sx, Qy) = \Psi(x, y)$, for all $x \in F^r$, $y \in F^s$. This is a well defined equivalence relation and it provides a classification. Let $EO_F(r, s, n)$ denote the set of equivalence classes of orthogonal pairings of type $[r, s, n]$ over F.

Almost all that is known about these sets may be summarized as follows.

If $F = \mathbb{R}$ is the field of real numbers, Yuzvinsky's paper [7] gives several partial results of various cases and also a complete description of $EO_{\mathbb{R}}(2, s, n)$. For instance, if s is even and $n = s + 2$, the $EO_{\mathbb{R}}(2, s, s + 2)$ can be identified with the points of the closed interval $[0, 1]$.

For other fields, first notice that any orthogonal pairing over an arbitrary F can always be viewed as a pairing over an algebraically closed extension of F. Therefore, we may suppose from the beginning that F is an algebraically closed field. Using canonical forms for matrices, it was proved by the author [2; (2.5)], that each set $EO_F(2, s, n)$ contains only one equivalence class, for any s and $n = s + 1$ and for s odd and $n = s + 2$ (see (4.3)).

Avoiding the use of matrices, Yuzvinsky studied in [8] the structure of orthogonal pairings of type $[2, s, n]$, again for s and n as above.

Some of these results were developed and used in order to establish that certain triples are not admissible over any field. For an account of these applications the reader is referred to Shapiro's expository paper [6] and in particular to [8], [3] and [4].

The main aim of this paper is to determine the set $EO_F(2, s, s + 2)$, for s an even integer. This set is described in theorem (5.25) and, as we may notice there, besides a part of $\mathrm{EO}_F(2, s, s + 2)$, that generalizes for arbitrary F the closed interval $\mathrm{EO}_{\mathbb{R}}(2, s, s + 2) = \{a \mid 0 \le a \le 1\}$, we have *five* more classes. This seems significant regarding orthogonal pairings, since it starts a different pattern between the field of real numbers and any algebraically closed field.

The method of proof uses similar techniques to those developed in [1] and [2].

1. Equivalence of orthogonal pairings

Assume that F is a field of characteristic different from two and let (F^n, q) denote a quadratic space, where F^n is the usual n-dimensional vector space over F, whose elements are column vectors $x = (x_1, \cdots, x_n)^t$, where t is the transpose operation and $q: F^n \to F$ is the quadratic map given by $q(x) = x_1{}^2 + \cdots + x_n{}^2$.

Let (F^r, q_1), (F^s, q_2) and (F^n, q) be quadratic spaces as above, respectively for the dimensions r, s and n. Let us recall that a bilinear map $\Phi: F^r \times F^s \to F^n$ is called an *orthogonal pairing* of type $[r, s, n]$ over F, if

$$q(\Phi(x, y)) = q_1(x)q_2(y) \tag{1.1}$$

for $x \in F^r$ and $y \in F^s$. Indistinctively, Φ is also called a *normed map* and the triple $[r, s, n]$ is said to be *admissible* over F if there exists such a map Φ.

Let us also recall that the existence of an orthogonal pairing Φ of size $[r, s, n]$ over F, is equivalent to the existence of an r-tuple $\Delta = (N_1, \cdots, N_r)$ of $n \times s$ matrices N_i over F, fulfilling the Hurwitz equations (see [1; p 32], [6; p 238]):

$$N_i{}^t N_i = I_s \quad \text{if} \quad 1 \le i \le r, \tag{1.2}$$

$$N_i{}^t N_j + N_j{}^t N_i = 0 \quad \text{if} \quad i \neq j,\ 1 \le i, j \le r. \tag{1.3}$$

The matrices $\Delta = (N_1, \cdots, N_r)$ are determined by Φ as follows. Consider orthogonal standard bases for F^r, F^s, F^n, and let $(e_1, \cdots, e_r)$ be such a basis for F^r. Then let the matrix $N_i: F^s \to F^n$ be defined by

$$N_i y = \Phi(e_i, y) \quad \text{for} \quad y \in F^s\ (i = 1, \cdots, r). \tag{1.4}$$

Using (1.1) it follows that these matrices satisfy the Hurwitz equations (see [loc.cit.]).

Conversely, let $\Delta = (N_1, \cdots, N_r)$ be an r-tuple of $n \times s$ matrices satisfying the Hurwitz equations. For $x \in F^r$, with $x = (x_1, \cdots, x_r)^t$ consider

$$N = \Delta \circ x = N_1 x_1 + \cdots + N_r x_r, \tag{1.5}$$

where $\circ$ is used to denote a well defined hybrid product of an r-row Δ, whose elements are matrices, and a column vector x of F^r. Clearly, as a function $N = N(x)$ is an $n \times s$ matrix determined by x. Then, for $y \in F^s$, set

$$\Phi(x, y) = (\Delta \circ x)y = Ny. \tag{1.6}$$

And, from (1.2) and (1.3), it follows that Φ constructed in this form defines an orthogonal pairing.

Consequently, Φ and $\Delta = (N_1, \cdots, N_r)$ can be considered as equivalent in the sense that any one of them determines the other.

The following notation is introduced in order to have a better setting for some of the results.

Let $\mathrm{O}_F(r, s, n)$ denote the set of all orthogonal pairings Φ of type $[r, s, n]$ over F.

Let $\mathrm{H}_F(r; n \times s)$ denote the totality of r-tuples $\Delta = (N_1, \cdots, N_r)$ of $n \times s$ matrices N_i over F, fulfilling the Hurwitz equations.

Let the map

$$\theta\colon \mathrm{H}_F(r; n \times s) \to \mathrm{O}_F(r, s, n) \tag{1.7}$$

be defined by $\theta(\Delta) = \Phi$, where Δ determines Φ according to (1.6).

The next lemma gives in a condensed from the above statements of this section.

LEMMA (1.8). *The map θ is bijective (one to one and onto) and the explicit values of θ^{-1} and θ are determined by* (1.4) *and* (1.6), *respectively.*

Proof. It follows directly from the definitions of θ and θ^{-1}.

Let $O(n) = O_n(F)$ be the orthogonal group or group of isometries of (F^n, q). According to Yuzvinsky ([7; *p* 139]), two orthogonal pairings, Φ, Ψ: $F^r \times F^s \to F^n$ are said to be *equivalent*, in symbols $\Phi \simeq \Psi$, if there exist orthogonal operations $S \in O(r)$, $Q \in O(s)$ and $P \in O(n)$ such that

$$P\Phi(Sx, Qy) = \Psi(x, y), \tag{1.9}$$

for all $x \in F^r$, $y \in F^s$. We can easily verify that $\simeq$ is an equivalence relation.

Let $\mathrm{EO}_F(r, s, n)$ denote the set of all equivalence classes of orthogonal pairings $\Phi : F^r \times F^s \to F^n$, where the symbol $\{\Phi\}$ represents the class determined by Φ.

Keeping r, s and n fixed, write briefly $\mathrm{H}_F = \mathrm{H}_F(r; n \times s)$. Let $\Delta = (N_1, \cdots, N_r)$ be an element of H_F and let the matrix $S = (u_1, \cdots, u_r)$, where the u_j's denote the columns. The product $\Delta \circ S$ is defined as the obvious extension of $\Delta \circ x$, introduced in (1.5). That is, $\Delta \circ S = (\Delta \circ u_1, \cdots, \Delta \circ u_r)$. A direct verification gives

$$(\Delta \circ S) \circ x = \Delta \circ (Sx), \quad \text{for} \quad x \in F^r. \tag{1.10}$$

Set $P\Delta Q = (PN_1Q, \cdots, PN_rQ)$. It is easy to prove that

$$[(P\Delta Q) \circ z]y = P(\Delta \circ z)Qy, \quad \text{for} \quad z \in F^r, y \in F^s. \tag{1.11}$$

Now, we have the following

LEMMA (1.12). *Let $S \in O(r)$, $Q \in O(s)$ and $P \in O(n)$. Then $\Delta \in \mathrm{H}_F$ if and only if $(P\Delta Q) \circ S \in \mathrm{H}_F$.*

Proof. If $S = I_r$, it follows easily that $\Delta \in \mathrm{H}_F$ if and only if $P\Delta Q \in \mathrm{H}_F$. Hence, it is enough to prove the lemma only for $Q = I_s$ and $P = I_n$.

As above, let $\Delta \circ S = (\Delta \circ u_1, \cdots, \Delta \circ u_r)$ and set $M_i = \Delta \circ u_i$. To establish that these M_i's satisfy conditions (1.2) and (1.3), proceed as follows. The fact that $u_i{}^t u_i = 1$ and the hypothesis $\Delta \in \mathrm{H}_F$ imply that

$$M_i{}^t M_i = (\Delta \circ u_i)^t(\Delta \circ u_i) = I_s,$$

and this proves (1.2).

We have $(u_i + u_j)^t(u_i + u_j) = 2$ if $i \neq j$. And, using the same argument as above, we get

$$(M_i + M_j)^t(M_i + M_j) = [\Delta \circ (u_i + u_j)]^t[\Delta \circ (u_i + u_j)] = 2I_s.$$

Thus, expanding the term $(M_i + M_j)^t(M_i + M_j)$, we obtain

$$M_i{}^t M_j + M_j{}^t M_i = 0 \quad \text{for} \quad i \neq j.$$

Therefore, (1.3) is established and $\Delta \circ S \in \mathrm{H}_F$.

Conversely, starting with $\Delta \circ S \in \mathrm{H}_F$, consider $(\Delta \circ S) \circ S^t = \Delta \circ (SS^t) = \Delta$, consequently $\Delta \in \mathrm{H}_F$, and this ends the proof of (1.12).

Parallel to the case of orthogonal pairings, two r-tuples Δ, Σ of H_F are said to be *equivalent*, in symbols $\Delta \simeq \Sigma$ if there exist orthogonal matrices S, Q and P, as those of (1.12), such that

$$(P\Delta Q) \circ S = \Sigma. \tag{1.13}$$

From (1.12) it follows that Σ is well defined and, as for (1.9), we verify that $\simeq$ is an equivalence relation.

Given two r-tuples, Δ, Σ of H_F, let $\Phi = \theta(\Delta)$, $\Psi = \theta(\Sigma)$ be the two orthogonal pairings determined by the map θ of (1.7). The following result holds.

LEMMA (1.14). *Let Δ, Σ and Φ, Ψ be as above. Then $\Delta \simeq \Sigma$ if and only if $\Phi \simeq \Psi$.*

Proof. Suppose $\Phi \simeq \Psi$. Then there are orthogonal matrices S, Q and P such that

$$\begin{aligned} P\Phi(Sx, Qy) &= P[\Delta \circ (Sx)]Qy \\ &= ([(P\Delta Q) \circ S] \circ x)y = \Psi(x, y) = (\Sigma \circ x)y, \end{aligned}$$

for *all* $x \in F^r$ and $y \in F^s$. the second equality follows using, first (1.11) and then (1.10). Therefore, $(P\Delta Q) \circ S = \Sigma$ and $\Delta \simeq \Sigma$.

Starting with $\Delta \simeq \Sigma$ and reversing the arguments we conclude that $\Phi \simeq \Psi$. Consequently, the lemma is proved.

Let $\mathrm{EH}_F(r; n \times s)$ denote the set of all equivalence classes of r-tuples $\Delta \in \mathrm{H}_F(r; n \times s)$ where the symbol $[\Delta]$ represents the class determined by Δ.

Given $[\Delta] \in \mathrm{EH}_F(r; n \times s)$, define $\Theta([\Delta]) = \{\theta(\Delta)\}$, where θ is the map (1.7). We have the following

LEMMA (1.15). *The map*

$$\Theta\colon \mathrm{EH}_F(r; n \times s) \to \mathrm{EO}_F(r, s, n)$$

is bijective.

Proof. From (1.14) it follows that $[\Delta] = [\Sigma]$ if and only if $\{\theta(\Delta)\} = \{\theta(\Sigma)\}$. Then, Θ is well defined and one to one. To prove that Θ is onto let $\{\Phi\} \in EO_F(r, s, n)$. Then $\theta^{-1}(\Phi) = \Delta$ and $\Theta([\Delta]) = \{\Phi\}$. This ends the proof.

LEMMA (1.16). *Let* $\Delta \in H_F$ *and* $\Delta = (N_1, \cdots, N_r)$. *Then,*

$$\Delta \simeq (\epsilon_1 N_1, \cdots, \epsilon_r N_r),$$

where $\epsilon_i = \pm 1$ *for* $1 \leq i \leq r$.

Proof. It follows after substitution in (1.13) of $P = I_n$, $Q = I_s$ and $S = \operatorname{diag}(\epsilon_1, \cdots, \epsilon_r)$.

LEMMA (1.17). *Let* $(P\Delta Q) \circ S = \Sigma$ *so that* $\Delta \simeq \Sigma$. *Then, up to equivalence, we can always suppose* $\det S = 1$.

Proof. If $\det S = -1$, let $S_1 = \operatorname{diag}(I_{r-1}, -1)$ and $S' = S_1 S$. Then $\Delta \circ S = \Delta \circ (S_1 S') = (\Delta \circ S_1) \circ S'$ and, if $\Delta' = \Delta \circ S_1$, we have $(P\Delta' Q) \circ S' = \Sigma$ where $\det S' = 1$.

LEMMA (1.18). *Given* $\Delta \in H_F$ *there exists* $\Sigma = (E_1, \cdots, E_r)$ *such that* $\Delta \simeq \Sigma$, *where* $E_i = [A_i, B_i]^t$ *with* $A_1 = I_s$ *and* $B_1 = 0$ *is the zero* $s \times (n - s)$ *matrix. Moreover, for all* $2 \leq i \leq r$, *each* A_i *is an alternate (skew) matrix of order s and each* B_i *a suitable rectangular* $s \times (n - s)$ *matrix.*

Proof. A proof of (1.18) has already been given elsewhere. We refer to [1; p 33–34] for it.

Remarks.

(1) A collection $\Sigma = (E_1, \cdots, E_r)$ like the one in (1.18) is said to be *normalized* and, in terms of $E_i = [A_i, B_i]^t$, the Hurwitz equations for $2 \leq i, j \leq r$ become

$$-A_i^2 + B_i B_i^t = I_s, \tag{1.19}$$

$$A_i A_j + A_j A_i = B_i B_j^t + B_j B_i^t, \quad \text{for} \quad i \neq j. \tag{1.20}$$

In fact, using the alternate condition $A_i^t = -A_i$, these relations follow easily from (1.2) and (1.3) (see [1; p 34]).

(2) Lemma (1.18) assures that each class $[\Delta] \in H_F$ contains a representative Σ in normalized form. If $\Phi = \theta(\Sigma)$, from (1.4), it follows that $\Phi(e_1, y) = y$ for $y \in F^s$ and such Φ is also called *normalized.*

(3) For $r = 1$, any $n \times s$ matrix N such that $N^t N = I_s$ is equivalent to $\Sigma = [I_s, 0]^t$. Therefore, if $r = 1$ there is only one class of equivalence for $n \times s$ matrices, hence

$$EH_F(1; n \times s) = \{[\Sigma]\}.$$

Finally, we have the following

LEMMA (1.21). *Let* $\Sigma \in H_F$, *where* $\Sigma = (E_1, \cdots, E_r)$ *is a normalized r-tuple of* $n \times s$ *matrices and let* $E_i^t = [A_i, B_i]$ *for* $i \geq 2$. *If* $Q \in O(s)$, $R \in O(n - s)$, *then*

$$P = \begin{bmatrix} Q^t & 0 \\ 0 & R^t \end{bmatrix} \in O(n),$$

and $P\Sigma Q = \Sigma' = (E_1', \cdots, E_r')$ *is also normalized and obviously* $\Sigma \simeq \Sigma'$. *Moreover,*

$$E_i'^{\,t} = [Q^t A_i Q, \; Q^t B_i R]$$

for $i \geq 2$. *In particular, if* $Q = I_s$ *then* $E_i'^{\,t} = [A_i, B_i R]$.

Proof. It is a direct verification and we omit it.

2. The case $r = 2$

Let $\Sigma = (E_1, E_2)$ and $\Delta = (N_1, N_2)$ be two systems of $n \times s$ matrices, each of them an element of $\mathrm{H}_F(2; n \times s)$. Clearly, to study their equivalence: $\Sigma \simeq \Delta$, it is enough to consider normalized systems. Then let $E_1 = N_1 = [I_s, 0]^t$, $E_2 = [A, B]^t$ and $N_2 = [C, D]^t$, where as in (1.18), A and C are alternate matrices of order s and B and D are appropriate $s \times (n - s)$ rectangular matrices.

Suppose there exist orthogonal matrices S, Q and P, respectively of orders 2, s and n, such that

$$\Delta = (P\Sigma Q) \circ S.$$

Now, taking into account (1.17), we may assume $\det S = 1$, so that

$$S = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

where $a^2 + b^2 = 1$. And, the above equality becomes

$$N_1 = aPE_1Q - bPE_2Q, \tag{2.1}$$

$$N_2 = bPE_1Q + aPE_2Q. \tag{2.2}$$

Out of these expressions, consider the product

$$N_2^{\,t} N_1 = (bQ^t E_1^{\,t} P^t + aQ^t E_2^{\,t} P^t)(aPE_1Q - bPE_2Q).$$

An elementary calculation, using the identities $E_1^{\,t}E_1 = E_2^{\,t}E_2 = I_s$ and $E_1^{\,t}E_2 = -E_2^{\,t}E_1$, shows that

$$N_2^{\,t} N_1 = Q^t E_2^{\,t} E_1 Q.$$

But, we have

$$N_2^{\,t} N_1 = [C, D]\begin{bmatrix} I_s \\ 0 \end{bmatrix} = C$$

and

$$E_2^{\,t} E_1 = [A, B]\begin{bmatrix} I_s \\ 0 \end{bmatrix} = A.$$

Consequently, $C = Q^tAQ$, so that C and A are orthogonally similar. Hence, the equivalence $\Sigma \simeq \Delta$ transforms into

$$([I_s, 0]^t, [A, B]^t) \simeq ([I_s, 0]^t, [Q^tAQ, D]^t). \tag{2.3}$$

To get more precise information some relations will be established. Multiply each of the equations (2.1) and (2.2), to the right by Q^t and to the left by P^t, to have,

$$P^tN_1Q^t = aE_1 - bE_2 \quad \text{and} \quad P^tN_2Q^t = bE_1 + aE_2.$$

Then substitute the expressions of E_1, E_2, N_1 and N_2 in each relation, and take the transpose, so to obtain

$$Q[I_s, 0]P = a[I_s, 0] - b[A, B] = [aI_s - bA, -bB],$$

$$Q[C, D]P = b[I_s, 0] + a[A, B] = [bI_s + aA, aB].$$

Let us decompose the orthogonal matrix P into four blocks, as follows

$$P = \begin{bmatrix} U & L_1 \\ L_2 & R \end{bmatrix}, \tag{2.4}$$

where U, L_1, L_2 and R are, respectively, $s \times s$, $s \times (n - s)$, $(n - s) \times s$ and $(n - s) \times (n - s)$ matrices. Using (2.4), perform the products indicated below, to get

$$Q[I_s, O]P = Q[U, L_1],$$

$$Q[C, D]P = Q[CU + DL_2, CL_1 + DR].$$

Hence, from the above equalities, the following relations are obtained,

$$QU = aI_s - bA, \tag{2.5}$$

$$QL_1 = -bB, \tag{2.6}$$

$$QCU + QDL_2 = bI_s + aA, \tag{2.7}$$

$$QCL_1 + QDR = aB. \tag{2.8}$$

From (2.5) and (2.6), it follows that

$$U = Q^t(aI_s - bA) \quad \text{and} \quad L_1 = -bQ^tB,$$

therefore, the upper part $[U, L_1]$ of P is determined by the orthogonal matrix Q and $[A, B]$. The lower part of P, though not so readily, it can also be described in similar terms. However, we skip this and concentrate on some special results that will be required in section 5. With this purpose, we will establish the next two relations,

$$DL_2 = bQ^tBB^t, \tag{2.9}$$

$$DR = Q^t(aI_s + bA)B. \tag{2.10}$$

These relations are proved as follows. From $A = QCQ^t$ and (2.5), it follows that

$$QCU = (QCQ^t)(QU) = A(aI_s - bA) = (aA - bA^2).$$

Now, (1.19) implies that

$$QCU = (bI_s + aA) - bBB^t,$$

and a substitution of this in (2.7), establishes (2.9).

Again, from $A = QCQ^t$ and $L_1 = -bQ^tB$, it follows that

$$QCL_1 = -bQCQ^tB = -bAB.$$

And a substitution of this expression in (2.8) verifies (2.10).

For later reference, consider the following results.

LEMMA (2.11). *Let* ${Q_1}^t = Q^t(aI_s + bA)$. *If* $BB^t = 0$, *then* Q_1 *is orthogonal and* $D = {Q_1}^tBR^t$. *In general, R may not be orthogonal, but if we also assume* $B^tB = 0$, *then R becomes orthogonal.*

Proof. Supposing $BB^t = 0$, from (1.19) it follows that $AA^t = -A^2 = I_s$. Then

$$(aI_s + bA)(aI_s + bA)^t = (a^2 + b^2)I_s = I_s,$$

hence, $(aI_s + bA)$ and Q_1 are orthogonal.

Substitution of $BB^t = 0$ in (2.9) gives $DL_2 = 0$. Then, from

$$PP^t = \begin{bmatrix} UU^t + L_1{L_1}^t & U{L_2}^t + L_1R^t \\ L_2U^t + R{L_1}^t & L_2{L_2}^t + RR^t \end{bmatrix} = I_n,$$

we get four equalities and, the one in the right lower corner gives $RR^t = I_{(n-s)} - L_2{L_2}^t$. Multiplying (2.10) to the right by R^t and using this relation we obtain $D = {Q_1}^tBR^t$.

Similarly, from $P^tP = I_n$ we have another four equalities and the one in the right lower corner is ${L_1}^tL_1 + R^tR = I_{(n-s)}$. Now, from (2.6), it follows that ${L_1}^tL_1 = b^2B^tB$. Therefore, assuming $B^tB = 0$, we get $R^tR = I_{(n-s)}$, and this ends the proof.

LEMMA (2.12). *As before, suppose* $BB^t = 0$ *and* $B^tB = 0$ *and set* $P_1 = \mathrm{diag}({Q_1}^t, R)$. *Then*

$$\Delta = (P\Sigma Q) \circ S = (P_1\Sigma Q_1).$$

Hence, in this case, the action of S can be given through P_1 *and* Q_1.

Proof. Clearly, $E_1 = N_1 = [I_s, 0]^t$ and we have $D = {Q_1}^tBR^t$ in both systems: in $(P\Sigma Q) \circ S$ by (2.11) and in $P_1\Sigma Q_1$ directly after performing the product. Finally, since the orthogonal matrix $(aI_s + bA)$ commutes with A, it follows that $C = {Q_1}^tAQ_1 = Q^tAQ$, and this proves (2.12).

Now, we will consider equivalences that keep A fix, that is $A = Q^tAQ$. Therefore, $QA = AQ$, so we need to restrict to those Q's that commute with A. The next result is an immediate consequence of (2.12).

LEMMA (2.13). *Let* $\Sigma = (E_1, E_2)$ *and* $\Delta = (N_1, N_2)$ *be two elements of* $\mathrm{H}_F(2; n \times s)$, *each of them in normalized form. Suppose that*

$$E_2 = [A, B]^t \quad and \quad N_2 = [A, D]^t,$$

where $BB^t = 0$ *and* $B^tB = 0$. *Then,* $\Sigma \simeq \Delta$ *if and only if* $D = Q_1{}^tBR^t$ *where* $Q_1 \in O(n)$, $R \in O(n-s)$ *and* $Q_1A = AQ_1$.

If $n = s + 2$, the following result, analogous to (2.13), can be formulated without assuming $B^tB = 0$.

LEMMA (2.14). *Let* $\Sigma = (E_1, E_2)$ *and* $\Delta = (N_1, N_2)$ *be two elements of* $\mathrm{H}_F(2; (s+2) \times 2)$, *each of them in normalized form. Suppose that* $E_2{}^t = [A, B]^t$ *and* $N_2{}^t = [A, D]^t$, *where* $BB^t = 0$. *Then* $\Sigma \simeq \Delta$ *if and only if* $D = Q_1{}^tBR_1$, *where* $R_1 \in O(2)$ *and* $Q_1 \in O(s)$ *such that* $Q_1A = AQ_1$.

Proof. Here B is an $s \times 2$ matrix and the condition $BB^t = 0$ implies that $B = [\mathbf{b}, \pm i\mathbf{b}]$ where $\mathbf{b} \in F^s$. If the sign of the second column of B is minus, multiplying B to the right by diag$[1, -1]$, it follows that, up to equivalence (see (1.21)), we can take $B = [\mathbf{b}, i\mathbf{b}]$. Then, from (2.6), we have that $L_1 = -bQ^t[\mathbf{b}, i\mathbf{b}] = [H, iH]$ where $H = -bQ^t\mathbf{b}$ and set $K = H^tH = b^2\mathbf{b}^t\mathbf{b}$. Let

$$R = \begin{bmatrix} c & d \\ e & f \end{bmatrix} \tag{2.15}$$

denote the component of the matrix R appearing in (2.4). Substituting these expressions on $L_1{}^tL_1 + R^tR = I_2$, gives

$$\begin{bmatrix} K & iK \\ iK & -K \end{bmatrix} + \begin{bmatrix} c^2+e^2 & cd+ef \\ cd+ef & d^2+f^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.16}$$

Then, we have

$$K + c^2 + e^2 = 1, \quad K - d^2 - f^2 = -1, \quad iK + cd + ef = 0.$$

Multiply the third relation by $2i$ and add the result to the first and second relations, to obtain $(c + id)^2 + (e + if)^2 = 0$. Thus, $e + if = \pm i(c + id)$.

From (2.16), it follows that

$$R^tR = \begin{bmatrix} 1-K & -iK \\ -iK & 1+K \end{bmatrix}.$$

Hence, $\det(R^tR) = (\det R)^2 = 1$. Therefore, $\det R = \pm 1$. We will prove that $p = c + id \neq 0$. Suppose, on the contrary, that $c = -id$, then $e = -if$ and replacing these values in (2.15), we get $\det R = 0$, then we have a contradiction and this proves our claim.

Now, according to (2.11), let us consider the right action of R^t on B. We have

$$BR^t = [\mathbf{b}, i\mathbf{b}]\begin{bmatrix} c & e \\ d & f \end{bmatrix} = [\mathbf{b}(c+id), \mathbf{b}(e+if)] = (c+id)[\mathbf{b}, \pm i\mathbf{b}].$$

Then, up to equivalence, we can consider that $BR^t = pB$, where $p = (c + id) \neq 0$.

The matrix R^t is not orthogonal, unless $K = 0$. However, its action on B can always be obtained by the following orthogonal matrix

$$R_1 = 1/2\begin{bmatrix} (p + p^{-1}) & i(p - p^{-1}) \\ -i(p - p^{-1}) & (p + p^{-1}) \end{bmatrix}. \tag{2.17}$$

In fact, a direct computation shows that $BR^t = BR_1 = pB$.

To prove the lemma first suppose that $D = Q_1{}^t BR_1$, where R_1 and Q_1 are as in (2.14) and then, define $P_1 = \mathrm{diag}[Q_1{}^t, R_1{}^t]$. It is easy to verify that $(P_1 \Sigma Q_1) = \Delta$, hence $\Sigma \simeq \Delta$. For the implication in the other direction, let $\Sigma \simeq \Delta$. Then, from the first part of (2.11) it follows that $D = Q_1{}^t BR^t$, where R is as in (2.15) and $Q_1{}^t = Q^t(aI_s + bA)$. Then, with a suitable R_1, as in (2.17), we have $BR^t = BR_1$ and $R_1 \in O(2)$. Since the matrices $(aI_s + bA)$ and Q are orthogonal and both commute with A, it follows that $Q_1 \in O(s)$ and $Q_1 A = AQ_1$. This ends the proof.

3. Canonical forms for alternate matrices

Let us recall some results about canonical forms for alternate matrices. The results to be quoted were stated in [5; Ch. XI] for the field of complex numbers. However, it readily follows that they also hold for an algebraically closed field and, in this form, they already were used in [1] and [2], where further references can be found.

From now on assume that F is an *algebraically closed field* of characteristic different from two. Let i be a *fixed* element of F such that $i^2 = -1$. Consider the column vectors

$$u = (1/2, 0, \cdots, 0, i/2)^t \quad \text{and} \quad v = (-i/2, 0, \cdots, 0, 1/2)^t, \tag{3.1}$$

where $u, v \in F^n$ and $n \geq 2$.

Let W^d and $X_a{}^q$ be alternate canonical matrices fulfilling the following properties. First, W^d is an alternate matrix of *odd* order d, such that it has a single elementary divisor λ^d, where λ is an indeterminate. And, for any $a \in F$ (including $a = 0$), $X_a{}^q$ is an alternate matrix of *even* order $2q$, such that it only has two elementary divisors: $(\lambda - a)^q$ and $(\lambda + a)^q$.

The matrices W^d and $X_a{}^q$ are constructed as follows. For $d = 1, 3$, fix $W^1 = [0]$ and

$$W^3 = 1/2\begin{bmatrix} 0 & 1+i & 0 \\ -1-i & 0 & -1+i \\ 0 & 1-i & 0 \end{bmatrix}.$$

Then, for $d = 2p + 3 \geq 5$, set by induction

$$W^{2p+3} = \begin{bmatrix} 0 & u^t & 0 \\ -u & W^{2p+1} & -v \\ 0 & v^t & 0 \end{bmatrix}, \tag{3.2}$$

where $u, v \in F^{2p+1}$ are column vector as in (3.1).

Now, for $X_a{}^q$, first define

$$X_a{}^1 = \begin{bmatrix} 0 & ia \\ -ia & 0 \end{bmatrix}. \tag{3.3}$$

Then, for $p \geq 1$, set by induction

$$X_a{}^{p+1} = \begin{bmatrix} 0 & u^t & ia \\ -u & X_a{}^p & -v \\ -ia & v^t & 0 \end{bmatrix}, \tag{3.4}$$

where $u, v \in F^{2p}$ are as in (3.1). The matrices (3.2) and (3.4) are easily identified to be the same as those constructed in [1; (4.7), (4.8)].

Let A be an alternate matrix and consider a list of all its elementary divisors that, according to [1; (4.5)], are of the form $(\lambda - a_i)^{q_i}$, $(\lambda + a_i)^{q_i}$ and λ^{d_j}, where $i = 1, \cdots, g$; $j = 1, \cdots, h$ and each d_j is an odd number. Now, from this list construct the alternate matrix

$$W = \operatorname{diag}[W^{d_1}, \cdots, W^{d_h}; X_{a_1}{}^{q_1}, \cdots, X_{a_g}{}^{q_g}]. \tag{3.5}$$

Clearly, A and W have the same elementary divisors, therefore they are similar, and since both are alternate, it follows that they are orthogonally similar (see [1; p 38]). Consequently, there exists an orthogonal matrix Q such that $W = QAQ^t$. Hence, the matrix W determined by the elementary divisors, gives a canonical form for A.

4. Canonical forms and rectangular matrices

Let (E_1, E_2) be a normalized system of two $(s + d) \times s$ matrices. Then, $E_1{}^t = [I_s, 0]$ and $E_2{}^t = [A, B]$ where A is an alternate matrix of order s and B is a $s \times d$ matrix. In this case the Hurwitz equations transform into the *single* relation (see (1.19))

$$-A^2 + BB^t = I_s, \tag{4.1}$$

and this relation implies that ([1; p 35]),

$$s \geq \operatorname{rank} A \geq s - d. \tag{4.2}$$

Moreover, since A is an alternate matrix, the rank of A must be even.

Suppose that A has the canonical form (3.5) and that $[A, B]$ satisfies (4.1). Up to equivalence, we will consider all possible forms of $E_2{}^t = [A, B]$ for $d = 1, 2$. Excepting when s is even and $d = 2$, all these cases have already been established by the author in [2; (2.5)]. To state them we introduced some notation.

Let $X = X_i{}^1$ be as in (3.3) and define the following matrices

$$A_1 = \operatorname{diag}[\underbrace{X, \quad \cdots \quad, X}_{k\text{-times}}] \quad \text{and} \quad A_2 = \operatorname{diag}[0, A_1].$$

Now, consider

$$B_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{and} \quad B_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix},$$

where B_1 is a $2k$-column of zeros, B_2 is a $(2k + 1)$-column and B_3 is a $(2k + 1) \times 2$ matrix.

Set

$$\Sigma_1 = ([I_s, 0]^t, [A_1, B_1]^t), \quad \text{for } s = 2k,$$

$$\Sigma_2 = ([I_s, 0]^t, [A_2, B_2]^t), \quad \text{for } s = 2k + 1,$$

$$\Sigma_3 = ([I_s, 0]^t, [A_2, B_3]^t), \quad \text{for } s = 2k + 1.$$

Then the mentioned results are the following:

$$\begin{aligned} \mathrm{EH}_F(2; (2k + 1) \times 2k) &= \{[\Sigma_1]\}, \\ \mathrm{EH}_F(2; (2k + 2) \times (2k + 1)) &= \{[\Sigma_2]\}, \\ \mathrm{EH}_F(2; (2k + 3) \times (2k + 1)) &= \{[\Sigma_3]\}. \end{aligned} \tag{4.3}$$

The case $s = 2k$ and $d = 2$ is not so easy as the cases above and the rest of this paper is dedicated to solve it. For this purpose, some preliminary results need first to be established.

To study this case, let

$$(E_1, E_2) \in H_F(2; (2k + 2) \times 2k)$$

be a normalized system. Then $E_2{}^t = [A, B]$, where A is an alternate matrix of order $2k$ and B is a $2k \times 2$ matrix. Let us first construct some possible matrices $[A, B]$.

To simplify the writing set

$$X_a = X_{ia}{}^1 = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} \quad \text{and} \quad X = X_1 = X_i{}^1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

where $a \in F$ and $X_a{}^q$ are in general, the matrices (3.3) and (3.4).

Now, we will define $\mathscr{A}_1$, $\mathscr{A}_2$ and $\mathscr{A}_3$, three sets of normalized systems $\Sigma = (E_1, E_2)$ of two $(s + 2) \times s$ matrices, where $s = 2k$. First, notice that $E_1{}^t = [I_s, 0]$ is the same matrix for all the systems Σ and only $E_2{}^t = [A, B]$ needs to be specified. This is done as follows.

(4.4) The set $\mathscr{A}_1$ formed with all $\Sigma = (E_1, E_2)$ where $E_2{}^t = [A_1, B_1]$ and

$$A_1 = \mathrm{diag}[\underbrace{X, \cdots, X}_{k\text{—times}}], \quad B_1 = B_1(\mathbf{b}) = [\mathbf{b}, i\mathbf{b}],$$

for $\mathbf{b} \in F^{2k}$ and $\mathbf{b} \neq 0$.

(4.5) The set $\mathscr{A}_2$ constructed with all $\Sigma = (E_1, E_2)$ where the matrices $[A_2, B_2]$, are given by

$$A_2 = A_2(c) = \operatorname{diag}[X_a, X, \cdots, X],$$
$$B_2 = B_2(c) = \begin{bmatrix} b, 0, 0, \cdots, 0 \\ 0, b, 0, \cdots, 0 \end{bmatrix}^t,$$

where $c = (a, b) \in S^1$. Here S^1 is the "unit circle" of F^2 that is, the set of all $(a, b) \in F^2$ such that $a^2 + b^2 = 1$.

(4.6) The set $\mathscr{A}_3$ consists of a single system $\Sigma_3 = (E_1, E_2)$, where $E_2{}^t = [A_3, B_3]$ is constructed as follows,

$$A_3 = \operatorname{diag}[X_i{}^2, X, \cdots, X] \quad \text{and} \quad B_3 = [\mathbf{b}_1, \mathbf{b}_2],$$

where

$$\mathbf{b}_1 = (1, i/2, -1/2, -i, 0, \cdots, 0)^t \quad \text{and} \quad \mathbf{b}_2 = (i, 1/2, i/2, 1, 0, \cdots, 0)^t.$$

Any two of the sets $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3$ are disjoint. Let $\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3$ be their union and defined the map $(k \geq 2)$

$$\kappa\colon \mathscr{A} \to \mathrm{EH}_F(2; (2k+2) \times 2k)$$

by $\kappa(\Sigma) = [\Sigma]$. We have the following

THEOREM (4.7). *The map κ is surjective.*

Proof. Let $\Sigma \in \mathscr{A}$ where $\Sigma = [E_1, E_2]$ and $E_2{}^t = [A, B]$. Then A is an alternate matrix of order $2k$ and B is a $2k \times 2$ matrix. From (4.2), we get that $2k \geq \mathit{rank}\, A \geq 2k - 2$. Therefore, there are two possibilities: either rank $A = 2k$ or rank $A = 2k - 2$.

As pointed out in the proof of [2; (2.5)], the alternate matrix A can be assumed to have one of the canonical forms given in (3.5). Suppose this is the case and let

(4.8) $$A = \begin{bmatrix} w_1 \\ \vdots \\ w_{2k} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_1 & c_1 \\ \vdots & \vdots \\ b_{2k} & c_{2k} \end{bmatrix},$$

where $w_i{}^t \in F^{2k}$ and $b_j, c_j \in F$. Conditions (4.1) become

(4.9) $$w_i w_j{}^t + b_i b_j + c_i c_j = \delta_{ij},$$

where δ_{ij} is the Kronecker delta.

Case I. Rank $A = 2k$.

Suppose rank $A = 2k$, then $\det A \neq 0$, hence, all the characteristic values of A are nonzero. Consequently, from (3.5) it follows that A has no components of the form W^d and that all other components $X_a{}^q$ are with $a \neq 0$.

To analyze $X_a{}^q$ as a possible component of A, up to equivalence, we can assume

$$A = \text{diag}[X_a{}^q, M],$$

where M denotes a suitable matrix and q is the highest value occurring in (3.5) among the q_j's. In (4.8) consider the first $2q$ rows of A and B. Forgetting some zeros, they form the matrices

$$(4.10) \qquad X_a{}^q = \begin{bmatrix} w_1 \\ \vdots \\ w_{2q} \end{bmatrix} \quad \text{and} \quad B_1{}^* = \begin{bmatrix} b_1 & c_1 \\ \vdots & \vdots \\ b_{2q} & c_{2q} \end{bmatrix},$$

and, since $X_a{}^q$ is a block in a diagonal form of A, it follows that the pair $[X_a{}^q, B_1{}^*]$ satisfies condition (4.1). Then their elements fulfill condition (4.9) and we have

$$(4.11) \qquad w_j w_j{}^t + b_j{}^2 + c_j{}^2 = 1, \quad \text{for} \quad j = 1, \cdots, 2q.$$

Now, from (3.4), the following relations are obtained by induction

$$(4.12) \qquad \begin{aligned} w_j w_j{}^t &= -a^2, \quad \text{for} \quad j = 1, \cdots, 2q, \\ w_1 w_2{}^t &= -a, \quad \text{if} \quad q > 1, \\ w_1 w_{2q} &= 0. \end{aligned}$$

Hence, from (4.11) and the first relations of (4.12), we get

$$(4.13) \qquad b_j{}^2 + c_j{}^2 = 1 + a^2, \quad \text{for} \quad j = 1, \cdots, 2q.$$

To analyze possible values of q_j and a_j, first assume that $q = 1$ is the highest value among the matrices $X_{a_j}{}^{q_j}$ (consequently, all $q_j = 1$) and that at least one of the matrices, say $X_{a_1}{}^{q_1}$, has $a_1{}^2 \neq -1$. Then,

$$[X_{a_1}{}^1, B_1{}^*] = \left[\begin{bmatrix} 0 & ia_1 \\ -ia_1 & 0 \end{bmatrix}, \begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix}\right],$$

where $b_1 b_2 + c_1 c_2 = 0$ and $b_1{}^2 + c_1{}^2 = b_2{}^2 + c_2{}^2 = 1 + a_1{}^2 \neq 0$. Let $b^2 = 1 + a_1{}^2$, then $R^t = b^{-1} B_1{}^*$ (b determined up to sign) is an orthogonal matrix and $B_1{}^* R = bI_2$. Therefore, putting $a_1 = ia$, up to convenience (see (1.21)), we have

$$(4.14) \qquad [X_{a_1}{}^1, B_1{}^*] = \left[X_a, \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}\right],$$

where $a^2 + b^2 = 1$.

This result implies that the pair $[A, B]$ of (4.8) is equivalent to the pair $[A_2, B_2]$ introduced in (4.5). To see this, let $[b_k, c_k]$ be the part of B assigned to the kth row of $[A, B]$. If $k \geq 3$, (4.9) implies that the kth row is orthogonal to the first and second rows of $[A, B]$ and the diagonal structure of A implies that $bb_k = bc_k = 0$. But $b \neq 0$, hence $b_k = c_k = 0$ for all $k \geq 3$, therefore $B = B_2$.

Now, from (4.13), it follows that $a_j^2 = -1$ and choosing $a_j = i$, we have $X_{a_j}^1 = X$ for all other components of A. Consequently, $A = A_2$ and the above assertion is proved.

To continue with the case $q = 1$, suppose $a_j^2 = -1$ for all the matrices $X_{a_j}^1$ and as before choose $a_j = i$. Then $A = A_1$ as in (4.4) and we have two possibilities for B.

(1) B is the $2k \times 2$ zero matrix. Then $[A, B]$ is the matrix $[A_2, B_2]$ of (4.5) for the special case $c = (1, 0) \in S^1$ (here $A_1 = A_2$).

(2) B is not the zero matrix. Then we can select $[X_i^1, B_1^*]$, a first component of $[A, B]$ where B_1^* is not zero. If

$$[X_i^1, B_1^*] = \left[\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix}\right],$$

then $b_1^2 + c_1^2 = b_2^2 + c_2^2 = 0$. Thus, $c_1 = \epsilon_1 i b_1$ and $c_2 = \epsilon_2 i b_2$, where $\epsilon_k = \pm 1$ for $k = 1, 2$. Since $b_1 b_2 + c_1 c_2 = 0$, it follows $b_1 b_2 (1 - \epsilon_1 \epsilon_2) = 0$. If $b_1 b_2 \neq 0$, then $\epsilon_1 = \epsilon_2$. If $b_1 b_2 = 0$ then, since B_1^* is not the zero marix, one of b_1 and b_2 is different from zero, say $b_1 \neq 0$ and $b_2 = 0$, and we can also take $\epsilon_1 = \epsilon_2$. Now, it follows from (1.21) that, in both cases, we can suppose $\epsilon_1 = \epsilon_2 = 1$. In fact, if $R = \mathrm{diag}[1, \epsilon_1]$, then

$$B_1^* R = \begin{bmatrix} b_1 & \epsilon_1 i b_1 \\ b_2 & \epsilon_2 i b_2 \end{bmatrix} R = \begin{bmatrix} b_1 & i b_1 \\ b_2 & i b_2 \end{bmatrix}. \tag{4.15}$$

As before, let $[b_k, c_k]$ be the part of B assigned to the *kth* row of $[A, B]$ and consider $k \geq 3$. Since in (4.15) we have the first two rows of B and $[b_k, c_k]$ is orthogonal to these rows, it follows that $b_1(b_k + i c_k) = 0$ and $b_1 \neq 0$. Then $c_k = i b_k$ and this proves that $B = [\mathbf{b}, i\mathbf{b}]$, is as given in (4.4).

Keeping the assumption rank $A = 2k$, proceed to the case $q = 2$. Beginning with a pair $[A, B]$ as in (4.8), suppose $A = \mathrm{diag}[X_a^2, M]$ and consider the pair (see (4.10))

$$[X_a^2, B_2^*] = \left[\begin{bmatrix} 0 & 1/2 & i/2 & ia \\ -1/2 & 0 & ia & i/2 \\ -i/2 & -ia & 0 & -1/2 \\ -ia & -i/2 & 1/2 & 0 \end{bmatrix}, \begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \\ b_3 & c_3 \\ b_4 & c_4 \end{bmatrix}\right]. \tag{4.16}$$

We will determine the possible values of a and the form of B_2^*. From (4.13), we get that

$$b_j^2 + c_j^2 = 1 + a^2 = d \quad \text{for} \quad j = 1, \cdots, 4. \tag{4.17}$$

Since the first and fourth rows of X_a^2 are orthogonal, it follows that

$$b_1 b_4 + c_1 c_4 = 0, \tag{4.18}$$

hence, $b_1^2 b_4^2 = c_1^2 c_4^2$. From (4.17), we get $b_j^2 = d - c_j^2$, then $b_1^2 b_4^2 = (d - c_1^2)(d - c_4^2) = d(d - c_1^2 - c_4^2) + c_1^2 c_4^2$. Therefore, $d(d - c_1^2 - c_4^2) = 0$

and then, either $d = 0$ or $d = c_1^2 + c_4^2$. If this last equality holds, we have

$$c_1^2 + c_4^2 = d = b_1^2 + c_1^2 = b_4^2 + c_4^2,$$

and this implies

$$b_1^2 = c_4^2 \quad \text{and} \quad b_4^2 = c_1^2. \tag{4.19}$$

In particular, $b_1 = \epsilon c_4$ where $\epsilon = \pm 1$, and a substitution of this in (4.18) gives,

$$b_1 b_4 + c_1 c_4 = \epsilon(b_4 c_4 + b_1 c_1) = 0. \tag{4.20}$$

The products, respectively, of the first and third, and of the third and fourth rows of (4.16), give

$$b_1 b_3 + c_1 c_3 = ia \quad \text{and} \quad b_3 b_4 + c_3 c_4 = a.$$

Now, square each of these expressions, add them, and use (4.20) to simplify the result, to obtain

$$(b_1^2 + b_4^2)b_3^2 + (c_1^2 + c_4^2)c_3^2 = 0.$$

Then, a substitution here, using equalities (4.19), gives

$$(b_1^2 + c_1^2)(b_3^2 + c_3^2) = d^2 = 0.$$

Hence, $d = a^2 + 1 = 0$ in all the cases. Then $a^2 = -1$ and we choose $a = i$ (notice that $a = -i$ will give the same final result, since X_i^2 is orthogonally similar to X_{-i}^2).

From (4.17) it follows that $b_j^2 + c_j^2 = 0$ for $j = 1, \cdots, 4$. Then $c_j = \epsilon_j i b_j$ where $\epsilon_j = \pm 1$, and (4.16) becomes

$$[X_i^2, B_2^*] = \left[X_i^2, \begin{bmatrix} b_1 & \epsilon_1 i b_1 \\ b_2 & \epsilon_2 i b_2 \\ b_3 & \epsilon_3 i b_3 \\ b_4 & \epsilon_4 i b_4 \end{bmatrix} \right].$$

The elements of $[X_i^2, B_2^*]$ fulfill the conditions of (4.9) and in X_i^2 we have $w_1 w_3^t = 1$ and $w_2 w_4^t = -1$. Therefore, in B_2^* we must have $b_1 b_3(1 - \epsilon_1\epsilon_3) = -1$ and $b_2 b_4(1 - \epsilon_2\epsilon_4) = 1$. These two equations imply that $b_j \neq 0$ for $1 \leq j \leq 4$, and that $\epsilon_1\epsilon_3 = \epsilon_2\epsilon_4 = -1$. Hence, $\epsilon_1 = -\epsilon_3$ and $\epsilon_2 = -\epsilon_4$. Now, from $w_1 w_4^t = 0$ we have that $b_1 b_4(1 - \epsilon_1\epsilon_4) = 0$ and this implies that $\epsilon_1 = \epsilon_4$. Consequently, $\epsilon_1 = \epsilon_4 = -\epsilon_2 = -\epsilon_3$. Then

$$B_2^* = \begin{bmatrix} b_1 & \epsilon i b_1 \\ b_2 & -\epsilon i b_2 \\ b_3 & -\epsilon i b_3 \\ b_4 & \epsilon i b_5 \end{bmatrix},$$

and, up to equivalence, we can take $\epsilon = 1$ (consider $B_2^* R$ where $R = \text{diag}\,[1, \epsilon]$, [see (1.21)]).

With $\epsilon = 1$, consider the relations (4.9) in the pair $[X_i^2, B_2^*]$. Respectively, from $w_1 w_2^t = -i$, $w_1 w_3^t = 1$ and $w_3 w_4^t = -i$ we get $2b_1 b_2 = i$, $2b_1 b_3 = -1$ and

$2b_3b_4 = i$. Consequently, if $b = b_1$, we obtain $b_2 = i/(2b)$, $b_3 = -1/(2b)$ and $b_4 = i/(2b_3) = ib$. then

$$B_2^* = \begin{bmatrix} b & ib \\ i/(2b) & 1/(2b) \\ -1/(2b) & i/(2b) \\ -ib & b \end{bmatrix}.$$

To simplify this expression, consider the orthogonal matrix [cf. (2.17)]

$$R = 1/2\begin{bmatrix} (b + b^{-1}) & -i(b - b^{-1}) \\ i(b - b^{-1}) & (b + b^{-1}) \end{bmatrix}.$$

It is easy to verify that the product B_2^*R becomes the above expression for $b = 1$. So, by the same argument used before, we take

$$(4.21) \qquad B_2^* = \begin{bmatrix} 1 & i \\ i/2 & 1/2 \\ -1/2 & i/2 \\ -i & 1 \end{bmatrix}.$$

To analyze the form of other components, let $[b_k, c_k]$ be the part of B assigned to the kth row of $[A, B]$ where $k \geq 5$. The diagonal structure of A implies that $[b_k, c_k]$ is orthogonal to the four components of B_2^*. In particular, $b_k + ic_k = 0$ and $-(1/2)b_k + (i/2)c_k = 0$, and these relations imply that $b_k = c_k = 0$. Hence, if X_i^2 is a component of A then no other X_a^2 with $q > 1$ and $a \neq 0$ can be a component. In fact, X_a^q is excluded since it is not an orthogonal matrix [see (4.12)] and it can not be completed to fulfill (4.9) since its corresponding part in B is already zero. Therefore, in this case only $X = X_i^1$ can be a component, so $B = B_3$ and, since rank $A = 2k$, it follows that $A = A_3$, as described in (4.6).

To complete the analysis, assume that X_i^2 is not a component of A, then we will show that X_a^{q+1} for $q \geq 2$, can not be a component of A. Suppose on the contrary that X_a^{q+1} is a component of A and that $[X_a^{q+1}, B^*]$, like in (4.10), represents the first $2q + 2$ rows of $[A, B]$. Let $u, v \in F^{2q}$ be the vectors defined in (3.1). From (3.4) it follows that the $2q \times (2q + 2)$ matrix $A_4 = [-u, X_a^q, -v]$ forms the central part of X_a^{q+1}. Let B_q^* be the part of B^* assigned to the rows of A_4 and consider the pair $[A_4, B_q^*]$. The rows of $[X_a^{q+1}, B^*]$ and, consequently, the rows of $[A_4, B_q^*]$ satisfy (4.9). Now, recall

$$u = \begin{bmatrix} 1/2 \\ 0 \\ \vdots \\ 0 \\ i/2 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} -i/2 \\ 0 \\ \vdots \\ 0 \\ 1/2 \end{bmatrix}.$$

Then, it follows that these vectors only contribute with isotropic components along the rows and with orthogonal components along the columns. Hence, taking them away we get the rows of $[X_a^q, B_q^*]$ also satisfy (4.9). Then, by induction, we end with $[X_a^3, B_3^*]$ and $[X_a^2, B_2^*]$. Hence, we can choose $a = i$ and B_2^* as is (4.21). Explicitly, we have

$$[X_i^3, B_3^*] = \left[\begin{bmatrix} 0 & 1/2 & 0 & 0 & i/2 & -1 \\ -1/2 & & & & & i/2 \\ 0 & & & & & 0 \\ 0 & & X_i^2 & & & 0 \\ -i/2 & & & & & -1/2 \\ 1 & -i/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}, \begin{bmatrix} d & e \\ & \\ & B_2^* \\ & \\ f & g \end{bmatrix}\right].$$

The six rows of $[X_i^3, B_3^*]$ must satisfy (4.9) but $w_6 w_2^t + f + ig = 0$ implies $f + ig = 1$ and $w_6 w_5^t - if + g = 0$ implies $f + ig = -1$. Therefore, we get a contradiction. Consequently, X_a^{q+1} is not a component of A for $q \geq 2$, and this ends the analysis for rank $A = 2k$.

Case II. Rank $A = 2k - 2$.

Again we will study the possible components of A according to *(3.5)*. Since the W^d's are of odd order, they need to appear an even number of times (for instance, $M = \operatorname{diag}[W^3, W^3]$ is of order 6 and rank $M = 4$). Suppose W^d is a component of A and consider the pair $[W^d, B^*]$ where B^* denotes the part of B assigned to W^d. It was established in [2; p 31–32], precisely for B^* a $d \times 2$ matrix, that $[W^d, B^*]$ fails to fulfill (4.9) if $d \neq 1$. Therefore, the component W^d cannot be in A for $d \geqq 3$. Consequently, $d = 1$ and

$$\operatorname{diag}[W^1, W^1] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = X_0^1 = X_0$$

is a possible component of A.

It follows that B^* is an orthogonal matrix of order 2 and, like in (4.14), we can take $R = B^{*t}$ to obtain, up to equivalence,

$$[X_0, B^*] = \left[\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right].$$

Then, the same argument used for $[A_2(c), B_2(c)]$, where $c = (a, b) \in S^1$ and $a \neq 0$, establishes this pair as possible for $c = (0, 1)$ (and obviously for $c = (0, -1)$). This completes (4.5) for all points of S^1.

Finally, we study the possibility to have X_0^q for $q > 1$ as a component of A. Consider the following three consecutive rows of X_0^q (see (3.4)). If $q = 2$, then

$$w_1 = (0, 1/2, i/2, 0),$$

$$w_2 = (-1/2, 0, 0, i/2),$$

$$w_3 = (-i/2, 0, 0, -1/2).$$

If $q \geqq 3$, then

$$w_{q-1} = (\cdots, 1/2, 0, 1/2, i/2, 0, i/2, \cdots),$$
$$w_q = (\cdots, 0, -1/2, 0, 0, i/2, 0, \cdots),$$
$$w_{q+1} = (\cdots, 0, -i/2, 0, 0, -1/2, 0, \cdots),$$

where the dots represent symmetrically placed zeros to complete $2q$ components. Form the pair

$$\left[\begin{bmatrix} w_{q-1} \\ w_q \\ w_{q+1} \end{bmatrix}, B^*\right] \quad \text{where} \quad B^* = \begin{bmatrix} b_{q-1} & c_{q-1} \\ b_q & c_q \\ b_{q+1} & c_{q+1} \end{bmatrix}$$

is the part of B assigned to the rows. Since the rows are isotropic and any two are orthogonal (i.e., $w_j w_k{}^t = 0$ for $q - 1 \leq j, k \leq q + 1$), it follows that the rows of B^* form 3 linearly independent row vectors of F^2. And this contradiction eliminates $X_0{}^q (q \geq 2)$ as a component of A. Therefore, since all possible cases have been reduced either to (4.4), (4.5) or (4.6) this ends the proof of theorem (4.7).

5. Main result

Our final aim is to describe the classes of $\mathrm{EO}_F(2, 2k, 2k + 2)$ or equivalently of $\mathrm{EH}_F(2; (2k + 2) \times 2k)$. The set $\mathscr{A}$ of theorem (4.7) gives a good approximation, however some of its elements may represent the same equivalent class. We proceed to characterize unique representatives of the different equivalent classes.

We begin with the case (4.4) and the set $\mathscr{A}_1$. Let $\dot{F}^{2k}$ be the set of all nonzero vectors of F^{2k} and write $O(A_1)$ to denote the subgroup of $O(s)$ formed with all Q's such that $QA_1 = A_1Q$. Clearly, $A_1 \in O(A_1)$ since $A_1 \in O(s)$.

If $\Sigma \in \mathscr{A}_1$ then $\Sigma = (E_1, E_2)$ is a normalized system and $E_2{}^t = [A_1, B_1]$, where $B_1 = [\mathbf{b}, i\mathbf{b}]$ for some $\mathbf{b} \in \dot{F}^{2k}$. Observe that this sets a one to one relation between $\mathbf{b} \in \dot{F}^{2k}$ and $\Sigma \in \mathscr{A}_1$. Hence, we have a well defined bijective map

$$\alpha: \dot{F}^{2k} \rightarrow \mathscr{A}_1$$

Let $\dot{F}$ be the multiplicative group, $F - \{0\}$, of the field F.

Definition (5.1). *Two vectors* $\mathbf{b}$ *and* $\mathbf{d}$ *of* $\dot{F}^{2k}$ *are equivalent, in symbols* $\mathbf{b} \sim \mathbf{d}$, *if and only if there exist* $Q \in O(A_1)$ *and* $p \in \dot{F}$ *such that* $\mathbf{d} = pQ\mathbf{b}$.

It follows that $\sim$ is a well defined equivalence relation and its importance can be appreciated by the following

LEMMA (5.2). *Let* $\mathbf{b}, \mathbf{d} \in \dot{F}^{2k}$, *then* $\mathbf{b} \sim \mathbf{d}$ *if and only if* $\alpha(\mathbf{b}) \simeq \alpha(\mathbf{d})$.

Proof. Let $\alpha(\mathbf{b}) = \Sigma$ and $\alpha(\mathbf{d}) = \Delta$. Since $B = [\mathbf{b}, i\mathbf{b}]$, it follows that $BB^t = 0$. Then, the conditions of (2.14) are satisfied and we have that $\alpha(\mathbf{b}) \simeq \alpha(\mathbf{d})$ if and only if

$$D = [\mathbf{d},\ i\mathbf{d}] = QBR = p[Q\mathbf{b},\ iQ\mathbf{b}]$$

for some $Q \in O(A_1)$, $R \in O(2)$ and $p \in \dot{F}$, where $pB = BR$. Clearly, this is equivalent to $\mathbf{b} \sim \mathbf{d}$. Hence, (5.2) is proved.

From (5.2), it follows that the problem of determining the equivalent classes ($\simeq$) of $\mathscr{A}_1$ can be solved by finding the equivalent classes ($\sim$) of $\dot{F}^{2k}$. We will use this last procedure.

Given $\mathbf{b} \in \dot{F}^{2k}$, we have two cases: $\mathbf{b}^t\mathbf{b} \neq 0$ (anisotropic) and $\mathbf{b}^t\mathbf{b} = 0$ (isotropic). Regarding the first case, we have the following

THEOREM (5.3). *Any two anisotropic vectors of* $\dot{F}^{2k}$ *are equivalent.*

Let $\mathbf{d} \in \dot{F}^{2k}$ be an anisotropic vector where $\mathbf{d}^t\mathbf{d} = c^2$. Then, if $\mathbf{b} = c^{-1}\mathbf{d}$, we have $\mathbf{d} \sim \mathbf{b}$ and $\mathbf{b}^t\mathbf{b} = 1$. Then it easily follows that theorem (5.3) is equivalent to the next

THEOREM (5.4). *Let* $\mathbf{b} \in \dot{F}^{2k}$ *be an anisotropic vector and suppose that* $\mathbf{b}^t\mathbf{b} = 1$. *Then* $\mathbf{b} \sim e_1$ *where* $e_1 = (1, 0, \cdots, 0)^t$.

Proof of (5.4). Given $\mathbf{b}$ we need to prove that there exists $Q \in O(A_1)$ such that $Q\mathbf{b} = e_1$. First, let us characterize matrices Q that commute with $A_1 = \operatorname{diag}[X, \cdots, X]$. Given a matrix $Q = [a_{rs}]$ of order $2k$, divide it in blocks

$$Q = \begin{bmatrix} Q_{11} & \cdots & Q_{1k} \\ \vdots & & \vdots \\ Q_{k1} & \cdots & Q_{kk} \end{bmatrix},$$

where each block

$$Q_{ij} = \begin{bmatrix} a_{2i-1,2j-1} & a_{2i-1,2j} \\ a_{2i,2j-1} & a_{2i,2j} \end{bmatrix}, \tag{5.5}$$

is of order 2 and $1 \leqq i, j \leqq k$.

It is easy to verify that $QA_1 = A_1Q$ if and only if $Q_{ij}X = XQ_{ij}$ and this last relation holds if and only if $a_{2i,2j-1} = -a_{2i-1,2j}$ and $a_{2i,2j} = a_{2i-1,2j-1}$. Therefore $QA_1 = A_1Q$, if and only if

$$Q_{ij} = \begin{bmatrix} a_{2i-1,2j-1} & a_{2i-1,2j} \\ -a_{2i-1,2j} & a_{2i-1,2j-1} \end{bmatrix} \tag{5.6}$$

for each matrix (5.5).

Let $\mathbf{a}_j$ denote the j-row of Q. It follows from (5.6) that only odd rows need to be specified. So, if

$$\mathbf{a}_{2p-1} = (a_{2p-1,1},\ \cdots,\ a_{2p-1,2k})$$

define

$$\bar{\mathbf{a}}_{2p-1} = (-a_{2p-1,2},\ a_{2p-1,1},\ \cdots,\ -a_{2p-1,2k},\ a_{2p-1,2k-1})$$

and set $\mathbf{a}_{2p} = \bar{\mathbf{a}}_{2p-1}$. Clearly, this will preserve (5.6) and we will have $QA_1 = A_1Q$.

Now, given $\mathbf{b} \in \dot{F}^{2k}$, as in (5.4), in order to construct the rows of Q we proceed by induction, as follows. Set $\mathbf{a}_1 = \mathbf{b}^t$ and $\mathbf{a}_2 = \bar{\mathbf{a}}_1$. For $p < k$, suppose we have $\mathbf{a}_1, \cdots, \mathbf{a}_{2p}$ such that $\mathbf{a}_i \mathbf{a}_j{}^t = \delta_{ij}$ for $1 \leqq i, j \leqq 2p$. Let U_1 and U_2 be two subspaces of F^{2k} generated as follows: U_1 by the first $2p$ vectors $e_1, \cdots, e_{2p}$ of the standard basis $e_1, \cdots, e_{2k}$ of F^{2k} and U_2 by the vectors $\mathbf{a}_1{}^t, \cdots, \mathbf{a}_{2p}{}^t$. Define an isometry $h: U_1 \to U_2$ by setting $h(e_j) = \mathbf{a}_j{}^t$ for $1 \leqq j \leqq 2p$. Since (F^{2k}, q) is a nonsingular quadratic space, "Witt's extension theorem" implies that h can be extended to an isometry $h': F^{2k} \to F^{2k}$. Defining $\mathbf{a}_{2p+1}{}^t = h'(e_{2p+1})$ and $\mathbf{a}_{2p+2} = \bar{\mathbf{a}}_{2p+1}$, the induction step is completed and this establishes the existence of $Q \in O(A_1)$ such that $Q\mathbf{b} = e_1$. Then (5.4) is proved.

If $\mathbf{b}$ is an isotropic vector there are three equivalent classes as shown by the next

THEOREM (5.7). *Let* $\mathbf{b}^t\mathbf{b} = 0$, *where* $\mathbf{b} \in \dot{F}^{2k}$ *and* $k \geqq 2$. *Then, it holds one and only one of the following equivalences*:

(5.8) $$\mathbf{b} \sim e_1 + ie_2,$$

(5.9) $$\mathbf{b} \sim e_1 - ie_2,$$

(5.10) $$\mathbf{b} \sim e_1 + ie_4.$$

Proof. The proof is by induction and for this purpose some auxiliary results are developed.

Let $\{e_j\}$, $1 \leqq j \leqq 2k$, be the orthonormal standard basis for F^{2k}. Recall that $e_j = (0, \cdots, 1, \cdots, 0)^t$ is a column vector with $2k - 1$ zeros and a single 1 in the *jth* place.

Decompose the identity matrix I_{2k} in k rectangular $2k \times 2$ matrices J_j, as follows:

$$I_{2k} = [J_1, \cdots, J_k] \quad \text{where} \quad J_j = [e_{2j-1}, e_{2j}].$$

Let S_k be the symmetric group of degree k. If $\sigma \in S_k$, define

$$Q_\sigma = \sigma I_{2k} = [J_{\sigma(1)}, \cdots, J_{\sigma(k)}],$$

as the σ permutation of pairs of consecutive columns $J_j = [e_{2j-1}, e_{2j}]$. Clearly, Q_σ is an orthogonal matrix and it fulfills condition (5.6) on its order two required matrices. In fact, these matrices have either two O's or two 1's in the main diagonal and always O's in the other diagonal. Hence, $Q_\sigma \in O(A_1)$.

It easily follows that

$$Q_\sigma e_{2j-1} = e_{2\sigma(j)-1} \quad \text{and} \quad Q_\sigma e_{2j} = e_{2\sigma(j)}.$$

If $w \in F^{2k}$ and

$$w = \Sigma_{j=1}^k (\lambda_{2j-1} e_{2j-1} + \lambda_{2j} e_{2j})$$

where $\lambda_h \in F$, then

$$Q_\sigma w = \Sigma_{j=1}^k (\lambda_{2j-1} e_{2\sigma(j)-1} + \lambda_{2j} e_{2\sigma(j)}).$$

For simplicity, write

$$\sigma w = Q_\sigma w \tag{5.11}$$

and, for each $\sigma \in S_k$ regard $\sigma: F^{2k} \to F^{2k}$ as an operator defined by (5.11). Clearly, we have $w \sim \sigma w$. This equivalence will be useful.

For $1 \leqq p < k$, decompose F^{2k} as the orthogonal sum of the quadratic spaces F^{2p} and $F^{2(k-p)}$. Let $(e_1, \cdots, e_{2k})$, $(e_1', \cdots, e_{2p}')$ and $(e_1'', \cdots, e_{2(k-p)}'')$ be, respectively, the orthonormal standard basis for F^{2k}, F^{2p} and $F^{2(k-p)}$. As a vector space $F^{2k} = F^{2p} \oplus F^{2(k-p)}$ is a direct sum and the three different bases are connected as follows: $e_i = (e_i', 0'')$ for $1 \leqq i \leqq 2p$, and $e_{2p+j} = (0', e_j'')$ for $1 \leq j \leq 2(k-p)$, where $0' \in F^{2p}$ and $0'' \in F^{2(k-p)}$ represent the zero vectors.

Let A_1' and A_1'' be, respectively, the matrix diag$[X, \cdots, X]$, where X appears $2p$ and $2(k-p)$ times. Then $A_1 = \text{diag}[A_1', A_1'']$.

Given $w \in F^{2k}$, write $w = (w_1, w_2)$ where $w_1 \in F^{2p}$ and $w_2 \in F^{2(k-p)}$. Suppose that $Q'w_1 = u_1$ and $Q'w_2 = u_2$ where $Q' \in O(A_1')$ and $Q'' \in O(A_1'')$. Then,

$$Q(w_1, w_2) = (u_1, u_2) \tag{5.12}$$

where $Q = \text{diag}[Q', Q'']$ and $Q \in O(A_1)$.

Consider the following special case of (5.12). Suppose that

$$u_1 = q_1(e_1' + \epsilon_1 ie_2') \text{ and } u_2 = q_2(e_1'' + \epsilon_2 ie_2''), \tag{5.13}$$

where $\epsilon_j = +1$ and $q_j \in F$, for $j = 1, 2$. Then, we have

$$(u_1, u_2) \sim (e_1 + \epsilon_1 ie_2) + (e_3 + \epsilon_2 ie_4). \tag{5.14}$$

To verify this, let

$$Q = Q(q, \epsilon) = 1/2\begin{bmatrix} (q + q^{-1}) & \epsilon i(q - q^{-1}) \\ -\epsilon i(q - q^{-1}) & (q + q^{-1}) \end{bmatrix},$$

where $q \in F$ and $\epsilon = +1$. Then $Q \in O(X)$ and $Q(qe_1 + \epsilon qie_2) = e_1 + \epsilon ie_2$. Therefore, if $Q_j = Q(q_j, \epsilon_j)$ for $j = 1, 2$, and we define

$$Q_* = \text{diag}(Q_1, I_{2(k-2)}, Q_2],$$

then $Q_* \in O(A_1)$ and $Q_*(u_1, u_2) = (e_1 + \epsilon_1 ie_2) + (e_{2k-1} + \epsilon_2 ie_{2k})$.

Finally, if σ is the transposition $\sigma(k) = 2$ and Q_σ is as in (5.11), then

$$Q_\sigma Q_*(u_1, u_2) = (e_1 + \epsilon_1 ie_2) + (e_3 + \epsilon_2 ie_4),$$

and this proves (5.14).

Forgetting zeros, the expression to the right of (5.14) can be regarded as a vector $(1, \epsilon_1 i, 1, \epsilon_2 i)^t \in F^4$. As we will see the equivalence can be simplified further, according to the values of ϵ_1 and ϵ_2.

Since $(1, i, 1, -i)^t$ and $(1, -i, 1, i)^t$ are equivalent by a simple transposition, we actually have three cases: $(1, i, 1, i)^t$, $(1, i, 1, -i)^t$ and $(1, -i, 1, -i)^t$. For each case we construct an orthogonal matrix that commutes with $A_1 = \mathrm{diag}[X, X]$ and gives the needed equivalence. Define Q_j for $j = 1, 2, 3$, as follows:

$$Q_1 = 1/2\begin{bmatrix} 2 & 0 & 1 & i \\ 0 & 2 & -i & 1 \\ 1 & -i & -2 & 0 \\ i & 1 & 0 & -2 \end{bmatrix}, \quad Q_2 = 1/2\begin{bmatrix} 2 & 0 & 1 & -i \\ 0 & 2 & i & 1 \\ 1 & i & -2 & 0 \\ -i & 1 & 0 & -2 \end{bmatrix}$$

and

$$Q_3 = 1/8\begin{bmatrix} 7 & 3i & 5 & -i \\ -3i & 7 & i & 5 \\ 5 & i & -7 & 3i \\ -i & 5 & -3i & -7 \end{bmatrix}$$

It is immediate to verify that

$$Q_1\begin{bmatrix} 1 \\ i \\ 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix}, \quad Q_2\begin{bmatrix} 1 \\ -i \\ 1 \\ -i \end{bmatrix} = \begin{bmatrix} 1 \\ -i \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad Q_3\begin{bmatrix} 1 \\ i \\ 1 \\ -i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ i \end{bmatrix}.$$

And, changing notation, these results become

(5.15) $$e_1 + ie_2 + e_3 + ie_4 \sim e_1 + ie_2,$$

(5.16) $$e_1 - ie_2 + e_3 - ie_4 \sim e_1 - ie_2,$$

(5.17) $$e_1 + ie_2 + e_3 - ie_4 \sim e_1 + ie_4.$$

We will prove theorem (5.7) for $k = 2$, namely, the starting case for the induction. As in (5.12), write $F^4 = F_1{}^2 \oplus F_2{}^2$ as orthogonal sum, where $(e_1{}', e_2{}')$ and $(e_1{}'', e_2{}'')$ are, respectively, basis for $F_1{}^2$ and $F_2{}^2$. Given $\mathbf{b} \in \dot{F}^4$ set $\mathbf{b} = (u_1, u_2)$, where $u_1 \in F_1{}^2$ and $u_2 \in F_2{}^2$. Then $\mathbf{b}^t\mathbf{b} = 0$ if and only if $u_1{}^t u_1 = 0$ and $u_2{}^t u_2 = 0$. And these last two relations hold if and only if u_1 and u_2 are of the form (5.13). Hence, assuming $\mathbf{b}^t\mathbf{b} = 0$ we have, first that $\mathbf{b}$ is equivalent to the general expression (5.14) and then to one of the expressions (5.15), (5.16) and (5.17), in agreement to the values of ϵ_1 and ϵ_2.

To complete the proof for $k = 2$, it is enough to show that not two of the vectors

$$v_1 = (1, i, 0, 0)^t, \qquad v_2 = (1, -i, 0, 0)^t \quad \text{and} \quad v_3 = (1, 0, 0, i)^t$$

are equivalent. According to (5.5), the most general matrix $Q \in O(A_1)$ is of the form

$$Q = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ -a_{12} & a_{11} & -a_{14} & a_{13} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ -a_{32} & a_{31} & -a_{34} & a_{33} \end{bmatrix}.$$

The images Qv_1 and Qv_2 are as follows:

$$(5.18)\quad Q\begin{bmatrix}1\\ i\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}(a_{11}+ia_{12})\\ i(a_{11}+ia_{12})\\ (a_{31}+ia_{32})\\ i(a_{31}+ia_{32})\end{bmatrix} \quad\text{and}\quad Q\begin{bmatrix}1\\ -i\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}(a_{11}-ia_{12})\\ -i(a_{11}-ia_{12})\\ (a_{31}-ia_{32})\\ -i(a_{31}-ia_{32})\end{bmatrix}.$$

The form of these images clearly shows that v_1 is not equivalent to v_2 and that v_2 is not equivalent to v_3. Since $\sim$ is symmetric, this proves that not two of v_1, v_2, v_3 are equivalent and establishes (5.7) for $k = 2$.

For future reference, let us state as a lemma the following special case of (5.7).

LEMMA (5.19). *Let* $\mathbf{b} = (b_1, \cdots, b_{2k})^t$ *be an isotropic vector and suppose there exists a number* $p < k$ *such that* $b_1{}^2 + \cdots + b_{2p}{}^2 = c^2 \neq 0$. *Then, as in* (5.10), *we have the* $\mathbf{b} \sim e_1 + ie_4$.

Proof. Clearly, $b_{2p+1}{}^2 + \cdots + b_{2k}{}^2 = -c^2$ and since $\mathbf{b} \sim c^{-1}\mathbf{b}$ (see (5.1)), we suppose $c^2 = 1$. Write $\mathbf{b} = (u_1, u_2)$ where $u_1 = (b_1, \cdots, b_{2p})^t \in F^{2p}$ and $u_2 = (b_{2p+1}, \cdots, b_{2k})^t \in F^{2(k-p)}$. By (5.4), we have $Q'u_1 = e_1'$ and $Q''(iu_2) = e_1''$. Then, $-Q''u_2 = ie_1''$. Set $Q = \mathrm{diag}[Q', -Q'']$ then, as in (5.12), we have

$$Q\mathbf{b} = (e_1', ie_1'') = e_1 + ie_{2p+1}.$$

If $\sigma \in S_k$ is a transposition defined by $\sigma(p + 1) = 2$, then (see (5.11))

$$Q_\sigma Q\mathbf{b} = \sigma(e_1 + ie_{2p+1}) = e_1 + ie_3.$$

Finally, if $Q_1 = \mathit{diag}[I_2, X, I_2, \cdots, I_2]$, where X is as in (4.4), then $Q_1 \in O(A_1)$ and

$$Q_1Q_\sigma Q\mathbf{b} = Q_1(e_1 + ie_3) = e_1 + ie_4.$$

Hence, $\mathbf{b} \sim e_1 + ie_4$ and this ends the proof of (5.19).

We have shown that theorem (5.7) is true for $k = 2$, and now we are ready to prove it for all k. By induction, suppose that (5.7) is true for $k = p$ and let $\mathbf{b} \in F^{2p+2}$ be an isotropic vector. Decompose $F^{2p+2} = F^{2p} \oplus \mathrm{F}^2$ as an orthogonal sum and write $\mathbf{b} = (u_1, u_2)$ where $u_1 \in F^{2p}$ and $u_2 \in F^2$. We will show that a given $\mathbf{b}$ is equivalent to one of the three vectors considered in (5.7). It follows that $\mathbf{b}^t\mathbf{b} = u_1{}^tu_1 + u_2{}^tu_2 = 0$, and the two possible cases: $u_1{}^tu_1 \neq 0$ or $u_1{}^tu_1 = 0$, will be treated apart.

If $u_1{}^tu_1 \neq 0$ then, according to (5.19), we have that $\mathbf{b} \sim e_1 + ie_4$ and this agrees with (5.10).

If $u_1{}^tu_1 = 0$, by the induction assumption, we have that, either

$$Q'u_1 = q_1(e_1' + ie_4') \quad\text{or}\quad Q'u_1 = q_1(e_1' + \epsilon_1 ie_2'),$$

where $Q' \in O(A_1')$ is as in (5.12), $q_1 \in F$ and $\epsilon_1 = \pm 1$.

Since $u_1{}^t u_1 = 0$ implies that $u_2{}^t u_2 = 0$ and $u_2 \in F^2$, for the component u_2, it follows that $u_2 = q_2(e_1'' + \epsilon_2 i e_2'')$, where $q_2 \in F$ and $\epsilon_2 = \pm 1$.

As in (5.12), let $Q = \mathrm{diag}[Q', I_2]$. Then $Q\mathbf{b} = (Q'u_1, u_2)$, and

$$\mathbf{b} \sim q_1(e_1 + ie_4) + q_2(e_{2p+1} + \epsilon_2 i e_{2p+2}) \tag{5.20}$$

or

$$\mathbf{b} \sim q_1(e_1 + ie_2) + q_2(e_{2p+1} + \epsilon_2 i e_{2p+2}). \tag{5.21}$$

If we suppose (5.20) then, from (5.19) it follows that $\mathbf{b} \sim e_1 + ie_4$, as it happened in the preceding case.

If we now take (5.21) then, as in (5.14) it follows that

$$\mathbf{b} \sim (e_1 + \epsilon_1 i e_2) + (e_3 + \epsilon_2 i e_4).$$

Considering the values of $\epsilon_1 = \pm 1$ and $\epsilon_2 = \pm 1$, we get

$$w_1 = e_1 + ie_2 + e_3 + ie_4,$$

$$w_2 = e_1 - ie_2 + e_3 - ie_4,$$

$$w_3 = e_1 + ie_2 + e_3 - ie_4,$$

$$w_4 = e_1 - ie_2 + e_3 + ie_4.$$

Since $\sigma w_3 = w_4$, where $\sigma(1) = 2$ is a transposition, we actually have the first three cases.

Let $Q_j{}^* = \mathrm{diag}[Q_j, I_{2(p-1)}]$, where Q_j for $j = 1, 2, 3$ are, respectively, the matrices used to prove (5.15), (5.16) and (5.17). To establish the similar cases for $\mathbf{b} \in F^{2p+1}$, set $\mathbf{b}_1 = (1, i, 0, \cdots, 0)^t$, $\mathbf{b}_2 = (1, -i, 0, \cdots, 0)^t$ and $\mathbf{b}_3 = (1, 0, 0, i, 0, \cdots, 0)^t$. It is immediate to verify that $Q_j{}^* w_j = \mathbf{b}_j$, for $j = 1, 2, 3$. Therefore, $\mathbf{b}$ is equivalent to one of the vectors $e_1 + ie_2$, $e_1 - ie_2$ or $e_1 + ie_4$, as assured in (5.7).

Finally, taking the most general matrix Q of order $2p + 2$, such that $Q \in O(A_1)$ (see (5.5)), the same argument already used in (5.18) for $k = 2$, shows that not two of the vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are equivalent and this ends the proof of theorem (5.7).

Let $\Sigma = (E_1, E_2)$ be an element of the set $\mathscr{A}_1$ defined in (4.4). Then $E_1 = [I_{2k}, O]^t$ and $E_2 = [A_1, B_1]^t$, where $B_1 = B_1(\mathbf{b}) = [\mathbf{b}, i\mathbf{b}]$ for $\mathbf{b} \in F^{2k}$. Set $E_2(\mathbf{b}_j) = [A_1, B_1(\mathbf{b}_j)]^t$, and construct the systems

$$\Sigma_j^* = (E_1, E_2(\mathbf{b}_j))^t \quad \text{for } j = 0, 1, 2, 3,$$

where $\mathbf{b}_0 = e_1$, $\mathbf{b}_1 = e_1 + ie_2$, $\mathbf{b}_2 = e_1 - ie_2$ and $\mathbf{b}_3 = e_1 + ie_4$. Now, define the set

$$\mathscr{A}_1{}^* = \{\Sigma_j^* \mid j = 0, 1, 2, 3\}. \tag{5.22}$$

It follows from (5.2), (5.4) and (5.7) that $\mathscr{A}_1$ has four equivalent classes and that $\mathscr{A}_1^*$ contains exactly one representative for each one of these classes.

Let $\mathscr{A}_2 = \{\Sigma(c) \mid c \in S^1\}$ be the set of normalized systems constructed in (4.5), where $\Sigma(c) = (E_1, E_2(c))$ and $E_2{}^t(c) = [A_2(c), B_2(c)]$. Recall that $S^1 \subset F^2$ is the "unit circle" and wherefore $a^2 + b^2 = 1$ for $c = (a, b) \in S^1$.

To determine the equivalence classes of $\mathscr{A}_2$, consider

$$A_2(c) = \mathrm{diag}[X_a, X, \cdots, X]$$

and

$$B_2(c) = \begin{bmatrix} b, & 0, & \cdots, & 0 \\ 0, & b, & \cdots, & 0 \end{bmatrix}^t.$$

All the terms $E_2(c)$ have components of this form and each $A_2(c)$ has $(\lambda - ia)$, $(\lambda + ia)$, $(\lambda - i)$, $(\lambda + i)$, $\cdots$, $(\lambda - i)$, $(\lambda + i)$ as elementary divisors. Therefore, $Q^tA_2(c)Q = A_2(c')$ if and only if $a' = \pm a$ where $c' = (a', b')$. Consequently, to find equivalent systems we only need to analyze possible change of signs of a and b.

If $c = (a, b)$, write $\Sigma(c) = \Sigma(a, b)$ and $A_2(c) = A_2(a, b)$. We have

$$\text{(5.23)} \qquad \Sigma(a, b) \simeq \Sigma(a, -b) \simeq \Sigma(-a, b) \simeq \Sigma(-a, -b).$$

These equivalences are obtained from (1.21) using suitable matrices Q and R. So, the first follows trivially with $Q = I_{2k}$ and $R = -I_2$. For the others, set

$$Q = \mathrm{diag}[H, X, \cdots, X] \quad \text{where} \quad H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and $R = \epsilon H$ for $\epsilon = \pm 1$. It is easy to verify that $Q^tA_2(a, b)Q = A_2(-a, b)$ and $Q^tB_2R = \epsilon B_2$. Hence, the second and third equivalences follow, respectively for $\epsilon = 1$ and $\epsilon = -1$.

To formalize this let $\Pi = \{1, x\}$ be the cyclic group of order two and generator x. The group Π operates on the additive part of F by $x(a) = -a$, for all $a \in F$. The direct product $\Pi \times \Pi$, componentwise, operates on F^2 and on S^1. The elements of $\Pi \times \Pi$ are: $(1, 1)$, $(1, x)$, $(x, 1)$, (x, x), and the action of them on $(a, b) \in S^1$ gives

$$\{(a, b), (a, -b), (-a, b), (-a, -b)\},$$

that is, the orbit associated to (a, b). Denote by $S^1/(\Pi \times \Pi)$ the set of orbits determined by the action of $\Pi \times \Pi$ on S^1.

Let $\mathscr{A}_2^*$ indicate the equivalence classes of $\mathscr{A}_2$. Since $(a, b) \leftrightarrow \Sigma(a, b)$ gives a one to one correspondence and (5.23) holds, it follows that $\mathscr{A}_2^*$ can be represented by the set of orbits $S^1/(\Pi \times \Pi)$. That is,

$$\text{(5.24)} \qquad \mathscr{A}_2^* \xrightarrow{\approx} S^1/(\Pi \times \Pi).$$

If we consider $\Pi \times \Pi$ operating on $\mathscr{A}_2$ then, the orbit linked to $\Sigma(a, b)$, denoted by $O(\Sigma(a, b))$, is formed by the union of the four systems that appear in (5.23). We have $O(\Sigma(a, b)) \in \mathscr{A}_2^*$.

Remark. If $F = \mathbb{R}$ is the field of real numbers, we can always take (a, b) with $0 \leq a \leq 1$ and $b = \sqrt{1 - a^2}$. Hence, in this case, $S^1/(\Pi \times \Pi)$ can be realized as a simplex of dimension one. For a more general statement of this type, concerning orthogonal pairings of size $[2, s, n]$ over $\mathbb{R}$, see [7; Remark 9, p 141].

Set $\mathscr{A}^* = \mathscr{A}_1{}^* \cup \mathscr{A}_2{}^* \cup \mathscr{A}_3$ and define, for $k \geq 2$,

$$\kappa^*: \mathscr{A}^* \longrightarrow \mathrm{EH}_F(2; (2k + 2) \times 2k),$$

as a map induced by κ of (4.7). Our main result is the following

THEOREM (5.25). *The map κ^* is bijective.*

Proof. All of the work of the proof has already been done and a summary of needed results follows. The elements of $\mathscr{A}_1$ were reduced to four nonequivalent systems with a very simple expression and they form $\mathscr{A}_1{}^*$, as shown in (5.22). If $\Sigma_j \in \mathscr{A}_1{}^*$ then $\kappa^*(\Sigma_j) = [\Sigma_j]$.

Each system of $\mathscr{A}_2$ is associated with a point $(a, b) \in S^1$. Two systems are equivalent if and only if they are associated with two points out of the four points $(\pm a, \pm b)$. These collections of four equivalent systems, denoted by $O(\Sigma(a, b))$ are the elements of $\mathscr{A}_2{}^*$ as described in (5.24). Then $\kappa^*(O(\Sigma(a, b))) = [\Sigma(a, b)]$.

Finally, recall that $\mathscr{A}_3$ consists of a single system Σ_3 and set $\kappa^*(\Sigma_3) = [\Sigma_3]$.

The map κ^* is a restriction of κ on $\mathscr{A}_1{}^* \cup \mathscr{A}_3$ and it becomes one to one, on this part of $\mathscr{A}^*$. On the other hand, each orbit $O(\Sigma(a, b))$ of four elements, have the same image under κ, and is considered as a single element of $\mathscr{A}_2{}^*$. From this and from the observation already made about elementary divisors, showing that equivalences (5.23) are the only ones possible for $\mathscr{A}_2$, it follows that κ^* is one to one on $\mathscr{A}_2{}^*$. Therefore, since κ is surjective, it follows that κ^* is bijective and this ends the proof.

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