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ON THE K-THEORY AND PARALLELIZABILITY OF PROJECTIVE STIEFEL MANIFOLDS

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§1. Introduction

Let $V_{n,s}$ be the Stiefel manifold of orthonormal s-frames in R^n and let $X_{n,s}$ be the projective Stiefel manifold obtained by identifying each s-frame in $V_{n,s}$ with its negative. The double covering $V_{n,s} \to X_{n,s}$ determines a line bundle ξ over $X_{n,s}$ that we will call the Hopf bundle.

In this paper we study the question of the parallelizability of $X_{n,s}$ and obtain the following results:

Parallelizable	Undecided	Nonstably parallelizable
$X_{n,n-1}$, $n \geq 1$ $X_{2n,2n-2}$, $n \geq 1$ $X_{n,s}$, $n = 2, 4, 8$, $1 \leq s < n$ $X_{16,8}$	$X_{12,8}$	All other $X_{n,s}$

The results in the first-column are obtained by a study of the tangent bundle of $X_{n,s}$, and are stated in Theorem 2.1.

On the other hand, we will apply the Hodgkin spectral sequence to compute almost completely the K-theoretical ring for $X_{4n,s}$ (Theorem 6.6). In particular we will compute the order of the complexification of the Hopf bundle (Theorem 6.8). When this is added to the fact that the tangent bundle over $X_{n,s}$ is stably isomorphic to ns-times the Hopf bundle, it will follow that any nonparallelizable projective Stiefel manifold $X_{n,s} \neq X_{12,8}$ is not stably parallelizable (Theorem 7.1). In Section 2 we prove the positive results on parallelizability of projective Stiefel manifolds. In Section 3 we give a brief description of the Hodgkin spectral sequence in the form we will use it in §5. Section 4 contains a description of the representation rings of the different groups and some homomorphisms between them. In particular we describe the homomorphism induced by the inclusion

$$Z_2 \times \mathrm{Spin}(4n - 2k + 1) \to \mathrm{Spin}(4n)$$

which in turn is used to describe the Tor term in the spectral sequence. In Section 5 we compute the E_2 term of the Hodgkin spectral sequence for $X_{4n,2k-1}$. We prove that the spectral sequence collapses and so obtain $E_\infty = E_2$. In Section 6 we discuss the extension problem in the spectral sequence for $K^*(X_{4n,2k-1})$ and as a consequence obtain the order of the Hopf bundle over $X_{4n,s}$. In Section 7 we state and prove the nonparallelizability results. Finally, in Section 8 we prove Proposition 7.2.

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§2. The positive parallelizability results

In this section we prove that certain projective Stiefel manifolds are parallelizable.

THEOREM (2.1). *The projective Stiefel manifold $X_{n,r}$ is parallelizable in the following cases.*

(a) $(n, r) = (16, 8)$
(b) $r = n$, $r = n - 1$; *or* $r = n - 2$ *with* n *even*
(c) $n = 2, 4, 8$.

The proofs of (a) and (b) are based on Lam's methods [L], which (c) will be proved using another construction for the tangent bundles of $V_{n,r}$ and $X_{n,r}$.

Proof of (a). By [L, Corollary 3.3], $\tau(X_{16,8} \sim 2^7\xi$ where τ is the tangent bundle, $\sim$ denotes stable equivalence and ξ is the Hopf bundle over $X_{16,8}$. Under the projection $\pi\colon X_{16,8} \to X_{16,1} = RP^{15}$, the Hopf bundle ξ_{15} over RP^{15} satisfies $\pi^*(\xi_{15}) = \xi$. Since $2^7\xi_{15} \sim 0$ by [A], it follows that $2^7\xi \sim 0$ and $X_{16,8}$ is stably parallelizable. Now $\dim X_{16,8} = 92$ and span $(S^{92}) = 0$ whereas span $(X_{16,8}) \geq \binom{8}{2} = 28$ by [L, Theorem 3.2]. The Bredon-Kosinski Theorem [BK] completes the proof for $X_{16,8}$.

Proof of (b). First consider $X_{n,n-2}$ with n even. Let $G = G_R(1, 1, \cdots, 1, \tilde{2})$ be the flag of manifold of $n - 2$ unoriented lines and 1 oriented 2-plane, all mutually orthogonal in R^n. Because n is even, there is an obvious covering map $X_{n,n-2} \to G$ with fibre ${Z_2}^{n-3}$, induced by the covering $V_{n,n-2} = G_R(\tilde{1}, \cdots, \tilde{1}, \tilde{2},) \to G$. To show $X_{n,n-2}$ is stably parallelizable it will therefore suffice to show G is stably parallelizable (in fact G is parallelizable but we do not prove this here). By [1, Corollary 12],

$$\tau(G) \simeq \textstyle\sum_{1\leq i<j\leq n-1} \xi_i \otimes \xi_j,$$

where $\xi_1, \cdots, \xi_{n-2}$ are line bundles and ξ_{n-1} an oriented 2-plane bundle constructed over G as in [L]. Let us apply the second exterior power λ^2 to the bundle isomorphism

$$n \simeq (\xi_1 \oplus \cdots \oplus \xi_{n-2}) \oplus \xi_{n-1}$$

noting that $\lambda^2(\xi_i) = 0$, $1 \leq i \leq n - 2$ and $\lambda^2(\xi_{n-1}) = 1$ since ξ_{n-1} is an oriented 2-plane bundle. We thus obtain

$$\binom{n}{2} \simeq (\textstyle\sum_{1\leq i<j\leq n-2} \xi_i \otimes \xi_j) \oplus ((\textstyle\sum_{1\leq i\leq n-2} \xi_i) \otimes \xi_{n-1}) \oplus 1 \simeq \tau(G) \oplus 1$$

whence G and $X_{n,n-2}$ are stably parallelizable. We again apply the Bredon-Kosinski Theorem to show $X_{n,n-2}$ parallelizable. By [L, Theorem 3.2], span

$X_{n,n-2} \geq \binom{n-2}{2}$. Now

$$1 + \dim X_{n,n-2} = \binom{n}{2} = m \cdot (odd), \quad m = n/2$$

so we see from the Radon-Hurwitz formula that

$$\text{span } S^{\binom{n}{2}-1} = -1 + \rho\binom{n}{2} = -1 + \rho(m) = \text{span } S^{m-1} < m$$

where $\rho(m) = 2^c + 8d$ if $m = (\text{odd})2^{4d+c}$, $0 \leq c < 4$. Since $m = n/2 < \binom{n-2}{2}$ for $m \geq 3$, it follows that $X_{2m,2m-2}$ is parallelizable for $m \geq 3$. The remaining case $X_{4,2}$ is covered in (c).

Both $X_{n,n-1}$ and $X_{n,n}$ can be mapped onto $G' = G_R(1, 1, \cdots, 1)$ as finite coverings and G' is easily seen to be parallelizable by the λ^2 construction. So $X_{n,n-1}$ and $X_{n,n}$ are also parallelizable. These two cases also follow from the theorem that the quotient of a Lie group by a finite subgroup is parallelizable [B, p. 502].

Proof of (c). We first derive some results for the Stiefel manifolds $V_{n,r}$ and their tangent bundles. For convenience a point in $V_{n,r}$ is denoted $v = (v_1, \cdots, v_r)$, where $v_i \in R^n$ and $\langle v_i, v_j \rangle = \delta_{ij}$.

LEMMA (2.2). *The tangent space $T_v(V_{n,r})$ is the R-vector space*

$$T_v(V_{n,r}) = \{(w_1, \cdots, w_r) \mid w_i \in R^n \quad \text{and} \quad \langle w_i, v_j \rangle + \langle w_j, v_i \rangle = 0\}.$$

Proof. Let $v(t) = (v_1(t), \cdots, v_r(t))$ be an arbitrary differentiable curve in $V_{n,r}$ with $v(0) = v$. Differentiating the equations $\langle v_i(t), v_j(t) \rangle = \delta_{ij}$ and setting $v'(0) = w = (w_1, \cdots, w_r)$ gives the desired result.

For the tangent space $T_{[v]}(X_{n,r})$ to $X_{n,r}$ at a point $[v] = \{v, -v\} \in X_{n,r}$ we have the identification

$$T_{[v]}(X_{n,r}) = \{[v, w] \mid w \in T_v(V_{n,r}), [v, w] = [-v, -w]\}.$$

Thus a tangent vector field on $X_{n,r}$ is equivalent to an odd (or skew) vector field s on $V_{n,r}$, i.e., $s(-v) = -s(v)$.

LEMMA (2.3). *Span $X_{n,r} \geq max\left\{\binom{r}{2}, \rho(n) - 1\right\}$.*

Proof. As mentioned above, the bound $\binom{r}{2}$ appears in [L, Theorem 3.2]. For the other bound, consider the $\rho(n) - 1$ Radon-Hurwitz transformations $\phi_i \in O(n)$ satisfying $\phi_i + \phi_i^t = 0$ and $\phi_i\phi_j + \phi_j\phi_i = 0$ for $i \neq j$. Setting $w^i(v) = (\phi_i v_1, \cdots, \phi_i v_r)$ for $1 \leq i < \rho(n)$ then gives a $(\rho(n) - 1)$-field on $V_{n,r}$ which is skew, hence also a $(\rho(n) - 1)$-field on $X_{n,r}$.

The completion of the proof of Theorem 3.1(c) is obtained by noting that

$X_{4,r}$ and $X_{8,r}$ are stably parallelizable (since $X_{4,1} = RP^3$ and $X_{8,1} = RP^7$) and then using the method in the proof of (a) above. By Lemma 2.3, span $X_{4,r} \geq 3$ and span $X_{8,r} \geq 7$, and so we may invoke the Bredon-Kosinski Theorem individually in each case, e.g., dim $X_{8,2} = 13$ and span $S^{13} = 3$. The remaining case $X_{2,1} = RP^1$ is trivial.

§3. The Hodgkin spectral sequence

Let H be a closed subgroup of a compact connected Lie group G with torsion-free fundamental group. Then we have [H1], [R].

THEOREM (3.1). (Hodgkin). *There is a multiplicative strongly convergent spectral sequence such that:*

(*a*) *As an algebra* $E_2{}^p = \mathrm{Tor}_{R(G)}{}^p[R(H), Z]$ (*lives in one line*).
(*b*) *The differential* d_r *is a derivation such that* $\bar{d}_r$: $E_r{}^{p-r} \to E_r{}^{p-r}$ *is zero for even* r.
(*c*) $E_\infty{}^*$ *is the graded algebra associated to a multiplicative filtration of* $K^*(G/H)$, *i.e., there is a filtration*

$$0 = F_{-2} \subset F_0 \subset F_2 \subset \cdots \subset F_{2n} = K^0(G/H)$$
$$0 = F_{-1} \subset F_1 \subset F_3 \subset \cdots \subset F_{2n+1} = K^1(G/H)$$

with $F_i \cdot F_j \subset F_{i+j}$ *under the product in* $K^*(G/H)$ *and* $E_\infty{}^p = F_p/F_{p-2}$. *Furthermore, the product in* $E_\infty{}^*$ *is that induced naturally by that in* $K^*(G/H)$.

We have the two edge-homomorphisms

(a) $\mathrm{Tor}_{R(G)}{}^0[R(H), Z] = E_2{}^0 \twoheadrightarrow E_\infty{}^0 = F_0 \to K^0(G/H)$.
(b) $\mathrm{Tor}_{R(G)}{}^1[R(H), Z] = E_2{}^1 \twoheadrightarrow E_\infty{}^1 = F_1 \to K^1(G/H)$.

which can be identified with the α and β constructions [R, Section 4].

§4. Representation Rings of Spin and S0

The following descriptions of the representation rings of Spin(n) and S0(n) can be found in Milnor [M], Gitler and Lam [GL], and Anderson, Brown and Peterson [ABP].

Let T_n and T_n' be the maximal tori of Spin($2n$) and S0($2n$). We have the following diagram of vertical inclusions and horizontal double coverings:

$$\begin{array}{ccc} T_n & \to & T_n' \\ \cap & & \cap \\ \mathrm{Spin}(2n) & \to & \mathrm{S0}(2n) \\ \cap & & \cap \\ \mathrm{Spin}(2n+1) & \to & \mathrm{S0}(2n+1) \end{array}$$

Since the induced diagram on the representation rings consists only of injections, we can identify all rings as subrings of the representation ring of T_n

$$R(T_n) = Z[u_1^2, \cdots, u_n^2, u_1^{-2}, \cdots, u_n^{-2}, (u_1 \cdots u_n)]/\sim$$

where now and in what follows, the relations $\sim$ are suggested by the notation, i.e., $u_i^2 u_i^{-2} = 1$ for $i = 1, \cdots, n$ and $(u_1 \cdots u_n)^2 = u_1^2 \cdots u_n^2$. It is convenient to think of this ring as a subring of the larger ring

$$Z[u_1, \cdots, u_n, u_1^{-1}, \cdots, u_n^{-1}]/\sim$$

in which the element $(u_1 \cdots u_n)$ is really a product.

Now we will introduce some well known elements. Let Π_k be the k-th elementary symmetric function in the variables $\{u_i^2 + u_i^{-2} - 2\}$ for $i = 1, \cdots, n$. This is called the k-th Pontrjagin class.

Let

$$\Delta_{2n}^+ = \sum_{\epsilon_1 \cdots \epsilon_n = 1} u_1^{\epsilon_1} \cdots u_n^{\epsilon_n},$$

$$\Delta_{2n}^- = \sum_{\epsilon_1 \cdots \epsilon_n = -1} u_1^{\epsilon_1} \cdots u_n^{\epsilon_n},$$

where in both sume $\epsilon_i = \pm 1$ and $\Delta_{2n+1} = \Pi_i(u_i + u_i^{-1})$.

These elements satisfy the following relations:

$$\Pi_n = (\Delta_{2n}^+ - \Delta_{2n}^-)^2, \quad \Delta_{2n}^+ \Delta_{2n}^- = \sum_{k=1}^n 4^{k-1} \Pi_{n-k} \textit{ and}$$

$$\Delta_{2n+1} = \Delta_{2n}^+ + \Delta_{2n}^-.$$

Now we can describe the other representation rings as

$$R(T_n') = Z[u_1^2, \cdots, u_n^2, u_1^{-2}, \cdots, u_n^{-2}]/\sim$$

$$R(\mathrm{Spin}(2n)) = Z[\Pi_1, \cdots, \Pi_{n-2}, \Delta_{2n}^+, \Delta_{2n}^-]$$

$$R(\mathrm{S0}(2n)) = \frac{Z[\Pi_1, \cdots, \Pi_{n-1}, (\Delta_{2n}^+)^2, (\Delta_{2n}^-)^2]}{\{(\Delta_{2n}^+)^2 (\Delta_{2n}^-)^2 = (\sum_{k=1}^n 4^{k-1} \Pi_{n-k})^2\}}$$

$$R(\mathrm{Spin}(2n+1)) = Z[\Pi_1, \cdots, \Pi_{n-1}, \Delta_{2n+1}],$$

$$R(\mathrm{S0}(2n+1)) = Z[\Pi_1, \cdots, \Pi_n].$$

Notice that $R(\mathrm{Spin}(2n))$, $R(\mathrm{Spin}(2n+1))$ and $R(\mathrm{S0}(2n+1))$ are polynomial rings.

The homomorphisms induced by the natural inclusions $\mathrm{S0}(2n-1) \subset \mathrm{S0}(2n)$ and $\mathrm{Spin}(2n-1) \subset \mathrm{Spin}(2n)$ are given by

$$\Delta_{2n}^{\pm} \mapsto \Delta_{2n-1}$$

$$\Pi_i \mapsto \begin{cases} \Pi_i, & i = 1, \cdots, n-1 \\ 0 & i = n \end{cases}$$

With this is is not difficult to describe the homomorphisms induced by the inclusions of $\mathrm{S0}(m-r) \subset \mathrm{S0}(m)$ and $\mathrm{Spin}(m-r) \subset \mathrm{Spin}(m)$.

The variables Π^i are in the kernel of the augmentation and in order to handle this kernel effectively it is convenient to make the following change of variables:

$$X_{2n} = \Delta_{2n}^{+} - \Delta_{2n}^{-}$$
$$\delta_{2n} = \Delta_{2n}^{+} - 2^{n-1}$$
$$\delta_{2n+1} = \Delta_{2n+1} - 2^{n}.$$

We now describe the homomorphisms induced by the inclusions

$$Z_2 \times \mathrm{Spin}(4n - 2k + 1) \xrightarrow{f \times j} \mathrm{Spin}(4n), \quad k = 1, \cdots, 2n.$$

where j is the usual inclusion and f is an injection of Z_2 into a central subgroup of Spin$(4n)$ that projects onto the center of S0$(4n)$ under the natural double covering. More precisely, $f \times j$ is the composition

$$Z_2 \times \mathrm{Spin}(4n - 2k + 1) \xrightarrow{f \times j} \mathrm{Spin}(4n) \times \mathrm{Spin}(4n) \xrightarrow{\mu} \mathrm{Spin}(4n)$$

where μ is the multiplication map.

To do this it is better to study first the homomorphism

$$Z_2 \times \mathrm{Spin}(4n) \xrightarrow{f \times j} \mathrm{Spin}(4n)$$

where now j is the identity map.

By restricting the double covering $\mathrm{Spin}(4n) \xrightarrow{\Pi} \mathrm{S0}(4n)$ to the maximal tori, we get the following diagram

$$\begin{array}{ccc} S^1 \times S^1 \times \cdots \times S^1 & \xrightarrow{D} & S^1 \times S^1 \times \cdots \times S^1 \\ g\downarrow\cong & & \downarrow\cong \\ T_{2n} & \xrightarrow{\Pi} & T_{2n}' \\ \cap & & \cap \\ \mathrm{Spin}(4n) & \xrightarrow{\Pi} & \mathrm{S0}(4n) \end{array}$$

where $D(z_1, \cdots, z_{2n}) = (z_1 z_2, z_2^{-1} z_3, \cdots, z_{2n}^{-1} z_1)$, z_i a complex number of absolute value 1.

With this description for D, the isomorphism induced by g on the representation rings takes the form

$$g^{\#}\colon Z[u_1^{\pm 2}, \cdots, u_{2n}^{\pm 2}, (u_1 \cdots u_{2n})]/\sim \;\overset{\cong}{\Rightarrow}\; Z[\alpha_1^{\pm 1}, \cdots, \alpha_{2n}^{\pm 1}]/\sim$$
$$u_1^2 \rightarrow \alpha_1 \alpha_2,\; u_i^2 \rightarrow \alpha_i^{-1} \alpha_{i+1} \quad \text{for} \quad 2 \le i \le 2n,$$

and $(u_1 \cdots u_{2n}) \rightarrow \alpha_1$.

Using the parametrization g for T_{2n}, we can now describe the restriction of $f \times j$ to the maximal tori as

$$\begin{array}{ccc} Z_2 \times S^1 \times S^1 x \cdots \times S^1 & \xrightarrow{f \times j} & S^1 \times S^1 \times \cdots \times S^1 \\ (t, z_1, z_2, \cdots, z_{2n}) & \longrightarrow & (z_1, tz_2, \cdots z_{2n-1}, tz_{2n}) \end{array}$$

where $Z_2 = \{-1, 1\}$.

Write $R(Z_2) = Z_2[y]/(y^2 = -2y)$, with $y = x - 1$ and $x\colon Z_2 \to S^1$ the only nontrivial 1-dimensional representation. Then we have

PROPOSITION (4.1). *The homomorphism of representation rings induced by $f \times j\colon Z_2 \times T_{2n} \to T_{2n}$ is given by*

$$(f \times j)^{\#}(u_i^{\pm 2}) = (y + 1)u_i^{\pm 2}$$

and

$$(f \times j)^{\#}(u_1 \cdots u_{2n}) = (u_1 \cdots u_{2n}).$$

Here

$$R(Z_2 \times T_{2n}) = R(Z_2) \otimes R(T_{2n}) = Z[y, u_1^{\pm 2}, \cdots, u_{2n}^{\pm 2}, (u_1 \cdots u_{2n})]/\sim.$$

Proof. A direct computation on the one dimensional representations shows that

$$(f \times j)^{\#}(\alpha_{odd}^{\pm 1}) = \alpha_{odd}^{\pm 1}$$
$$(f \times j)^{\#}(\alpha_{even}^{\pm 1}) = (y + 1)\alpha_{even}^{\pm 1}$$

and by changing variables we get the Proposition.

The following Lemma is easy and its proof is left to the reader.

LEMMA (4.2). *Let Π_k' be the* k-th *elementary function in the variables $X_1 + C, \cdots, X_{2n} + C$ and Π_k the* k-th *elementary symmetric function in the variables $X_1, \cdots, X_{2n}$. Then*

$$\Pi_k' = \sum_{j=0}^{k} \binom{2n - k + j}{j} C^j \Pi_{k-j}.$$

Since $(f \times j)^{\#}(\Pi_k)$ must be the k-th elementary symmetric function in the $2n$ variables

$$(f \times j)^{\#}(u_i^2 + u_i^{-2} - 2) = (y + 1)(u_i^2 + u_i^{-2} - 2) + 2y, \quad i = 1, \cdots, 2n,$$

and $(y + 1)y = -y$, we obtain

COROLLARY (4.3). *The homomorphism of representation rings induced by*

$$f \times j\colon Z_2 \times \mathrm{Spin}(4n - 2k + 1) \to \mathrm{Spin}(4n)$$

is given by

$$(f \times j)^{\#}(\Pi_i) = \Pi_i{}' = (-1)^{i-1}\left[\sum_{j=1}^{i} 2^{2j-1}\binom{2n-i+j}{j} y\, \Pi_{i-j}\right] + (1+y)^i \Pi_i$$

$$(f \times j)^{\#}(X_{4n} = -2^{k-1}\delta_{4n-2k+1}y - 2^{2n-1}y$$

and

$$(f \times j)^{\#}(\delta_{4n}) = 2^{k-1}\delta_{4n-2k+1}$$

where in the right side of the first formula

$$\Pi_i = \begin{cases} \delta_{4n-2k+1}{}^2 + 2^{2n-k+1}\delta_{4n-2k+1} - \sum_{j=1}^{2n-k}\Pi_{2n-k-j}, & \text{if } i = 2n-k \\ 0 & \text{if } i > 2n-k \end{cases}.$$

The matrix expressing the elements $\Pi_i{}'$ in terms of the Π_i has the following form

$$\begin{bmatrix} 1 & 2\binom{2n}{1}y & -2^3\binom{2n}{2}y & 2^5\binom{2n}{3}y & \cdots \\ 0 & 1+y & -2\binom{2n-1}{1}y & 2^3\binom{2n-1}{3}y & \cdots \\ 0 & 0 & 1 & 2\binom{2n+2}{1}y & \cdots \\ \cdot & \cdot & 0 & 1+y & \cdots \\ \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & 0 & \cdots \\ \cdot & \cdot & \cdot & & \cdots \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \end{bmatrix}$$

By Netto's formula [N, formula (2), page 51], we see that this matrix is its own inverse. Hence the new variables $\Pi_i{}'$ can equally serve as generators.

§5. Computation of $\mathrm{Tor}_{R(\mathrm{Spin}(4n))}{}^*[R(\mathrm{Spin}(4n-2k+1)\times Z_2), Z]$

We describe $\mathrm{Tor}^*_{R(\mathrm{Spin}(4n))}[R(\mathrm{Spin}(4n-2k+1)\times Z_2), Z]$ as a graded algebra. For this purpose, let $2 \le k \le 2n$ and set

$$A = R(\mathrm{Spin}(4n))/(\Pi_1, \cdots, \Pi_{2n-k}) \quad \text{and}$$

$$B = R(\mathrm{Spin}(4n-2k+1)\times Z_2)/(\Pi_i{}', \cdots, \Pi_{2n-k}{}'),$$

where $\Pi_i{}'$ is as in 4.3. Then $(f \times j)^{\#}$ induces an algebra homomorphism θ: $A \to$ B that makes B an A-algebra, and we have

PROPOSITION (5.1). *As graded algebras*:

$$\mathrm{Tor}^*_{R(\mathrm{Spin}(4n))}[R(\mathrm{Spin}(4n-2k+1)\times Z_2), Z] \cong \mathrm{Tor}_A{}^*[B, Z].$$

Proof. Let $\Lambda = Z[\Pi_1, \cdots, \Pi_{2n-k}]$ and let ϕ: $\Lambda \to R(\mathrm{Spin}(4n))$ be the inclusion. Then $R(\mathrm{Spin}(4n))$ is Λ-free and by [CE, XVI, Theorem 6.1] we have a spectral sequence:

$$\mathrm{Tor}_A[\mathrm{Tor}_\Lambda[R(\mathrm{Spin}(4n-2k+1)\times Z_2), Z], Z] \Rightarrow \mathrm{Tor}_{R(\mathrm{Spin}(4n))}[R(\mathrm{Spin}(4n-2k+1)\times Z_2, Z].$$

But since $\mathrm{Tor}_\Lambda[R(\mathrm{Spin}(4n-2k+1)\times Z_2), Z] = \mathrm{Tor}^0 = B$, the spectral sequence collapses and we get the Proposition.

In order to describe $\mathrm{Tor}_A[B, Z]$ write

$$A = Z[\Pi_{2n-k+1}, \cdots, \Pi_{2n-2}, X_{4n}, \delta_{4n}],$$

$$B = Z[\delta, y]/\left(y^2 = -2y, \delta^2 = -2^{2n-k+1}\delta + 2^{4n-2k-1}\left[1 + (-1)^{k+1}\binom{2n-1}{2n-k}\right]y\right)$$

and
$$\theta(\delta_{4n}) = 2^{k-1}\delta,$$
$$\theta(X_{4n}) = -2^{k-1}\delta y - 2^{2n-1}y$$

and
$$\theta(\Pi_i) = \Pi_i' \quad \text{for} \quad i = 2n-k+1, \cdots, 2n.$$

LEMMA (5.2). *For $i = 2n-k+1, \cdots, 2n$, we have in B*

$$\Pi_i' = 2^{2i-1}\binom{2n}{i}y - \textstyle\sum_{j=2n-k+1}^{i-1} 2^{2(i-j)}\binom{2n-j}{j}\Pi_j'$$

Proof. In $R(\mathrm{Spin}(4n)\times Z_2)$,

$$\Pi_i = (-1)^{i-1}\textstyle\sum_{j=1}^{i} 2^{2j-1}\binom{2n-i+j}{j}y\Pi'_{i-j} + (1+y)^i\Pi_i'.$$

But now in B we have $\Pi_i = 0$ for $i = 2n-k+1, \cdots, 2n$ as well as $\Pi_i^{i\,\prime} = 0$ for $i = 1, \cdots, 2n-k$ and $y\Pi_i' = -2\Pi_i'$. So we get the Lemma by substitution.

Let $m = \text{g.c.d.}\left\{2^{2i-1}\binom{2n}{i} \,\middle|\, i = 2n-k+1, \cdots, 2n-2\right\}$.

COROLLARY (5.3). *A can be described as*

$$A = Z[\tau_1, \cdots, \tau_{k-3}, \rho, X_{4n}, \delta_{4n}]$$

where the elements τ_i, $i = 1, \cdots, k-3$ and ρ satisfy $\theta(\tau_i) = 0$ and $\theta(\rho) = my$.

Let $A' = A/(\tau_1, \cdots, \tau_{k-3}) = Z[\rho, X_{4n}, \delta_{4n}]$.

PROPOSITION (5.4). *As graded algebras*

$$\mathrm{Tor}_A{}^*[B, Z] \cong \Lambda_Z{}^*(t_1, \cdots, t_{k-3}) \otimes_Z \mathrm{Tor}_{A'}[B, Z]$$

where $\Lambda_Z{}^*(t_1, \cdots, t_{k-3})$ is the exterior algebra on generators of degree 1.

Proof. Since A is a polynomial ring, we can use the Koszul resolution to view $\mathrm{Tor}_A{}^*[B, Z]$ as the homology of the chain complex $\Lambda_B{}^*(t_1, \cdots, t_{k-3}, P, X, D)$ in which the differential is the derivation given on generators by

$$d(t_i) = \theta(\tau_i) = 0$$
$$d(P) = \theta(\rho)$$
$$d(X) = \theta(X_{4n})$$
$$d(D) = \theta(\Delta_{4n}).$$

But we have the isomorphism of chain complexes

$$\Lambda_B{}^*(t_1, \cdots, t_{k-3}, P, X, D) \cong \Lambda_Z{}^*(t_1, \cdots, t_{k-3}) \otimes_Z \Lambda_B{}^*(P, X, D);$$

here the right hand side has the differential $d = \epsilon \otimes d'$, where ϵ is the augmentation and d' is given by

$$d'(P) = \theta(\rho)$$
$$d'(X) = \theta(X_{4n})$$
$$d'(D) = \theta(\delta_{4n}).$$

We thus get the Proposition.

Now we are ready to describe $\mathrm{Tor}_{A'}[B, Z]$. This is the homology of the Koszul complex

$$0 \to B \xrightarrow{d} B \oplus B \oplus B \xrightarrow{d} B \oplus B \oplus B \xrightarrow{d} B \to 0$$

where d is the derivation of B-modules given by

$$d(P) = my$$
$$d(X) = -2^{k-1}\delta y - 2^{2n-1}y$$
$$d(D) = 2^{k-1}\delta.$$

Let $2^\alpha = \text{g.c.d.}\ \{m, 2^{2n-1}\}$ and let $am + b2^{2n-1} = 2^\alpha$. We then have a new basis for $\Lambda_B{}^*(P, X, D)$ given by

$$u_1 = aP - b(X - yD) \qquad\qquad d(u_1) = 2_y{}^\alpha$$
$$u_2 = D \qquad\qquad \text{where} \quad d(u_1) = 2^{k-1}\delta$$
$$u_3 = 2^{2n-1-\alpha}P + m2^{-\alpha}(X - yD) \qquad\qquad d(u_3) = 0$$

PROPOSITION (5.5). *As graded algebras*

$$\mathrm{Tor}_{A'}{}^*[B, Z] = \Lambda_Z{}^*(z_1, z_2, z_3, u) \otimes_Z Z[y, \delta]/\sim$$

where z_1, z_2, z_3, u *are of degree* 1, y *and* δ *are degree* 0 *and* $\sim$ *is given by:*

1) $y^2 = -2y,\ 2^\alpha y = 0$

2) $\delta^2 = -2^{2n-k+1}\delta + 2^{4n-2k-1}\left[1 + (-1)^{k+1}\binom{2n-1}{2n-k}\right]y,\quad 2^{k-1}\delta = 0$

3) $2^r u = 0,\quad r = \min\{\alpha, k-1\}$

4) $z_1 y = 0,\quad z_1 u = 0$

5) $z_2\delta = 2^{4n-2k-1-\alpha+r}\left[1 + (-1)^{k+1}\binom{2n-1}{2n-k}\right]u,\quad z_2 u = 0$

6) $uy = -2u - 2^{k-1-r}\delta z_1,\quad u\delta = -2^{2n-k+1}u = 2^{\alpha-r}yz_2.$

Proof. Take

$$z_1 = (y + 2)u_1,$$

$$z_2 = (\delta + 2^{k-1})u_2 - 2^{k-1-\alpha}\left[1 + (-1)^{k+1}\binom{2n-1}{2n-k}\right]u_1,$$

$$z_3 = u_3, \quad \text{and} \quad u = -2^{k-1-1}\delta u_1 + 2^{\alpha-r}yu_2$$

Now the Theorem follows by direct inspection of kernels, images and relations.

COROLLARY (5.6). *The Hodgkin spectral sequence for $X_{4n,2k-1}$ collapses.*

Proof. As we see from Propositions 5.1, 5.4 and 5.5, the E_2 term is generated as an algebra by $E_2{}^0$ and $E_2{}^1$ and since the differentials are zero there for dimensional reasons and are derivations, we obtain the Corollary.

Employing the results of this section, we can describe the E_∞ term of the Hodgkin spectral sequence for $X_{4n,2k-1}$ as

$$E_\infty{}^P = \Lambda_Z{}^p(t_1, \cdots, t_{k-3}, z_1, z_2, z_3, u) \otimes_Z Z[\delta, y]/\sim$$

where $t_1, \cdots, t_{k-3}, z_1, z_2, z_3$ and u are of degree 1, y and δ are of degree 0 and $\sim$ is exactly as in Proposition 5.5.

§6. The Extension Problem

To get $K^*(X_{4n,2k-1})$ from the $E_\infty{}^*$ term of the Hodgkin spectral sequence, we have to solve an extension problem. We will have this in mind in what follows.

We consider two categories of algebras which we will call C_K and C_E.

The category C_K consists of Z_2-graded algebras $K = K^0 \oplus K^1$ with multiplicative filtration

$$0 = F_{-2} \subset F_0 \subset F_2 \subset \cdots \subset F_{2n} \subset \cdots \subset K^0 = \cup_i F_{2i},$$

$$0 = F_{-1} \subset F_1 \subset F_3 \subset \cdots \subset F_{2n+1} \subset \cdots \subset K^1 = \cup_i F_{2i+1}$$

$$F_i \cdot F_j \subset F_{i+j}.$$

The category C_E consists of Z-graded algebras:

$$E = E^0 \oplus E^1 \oplus E^2 \oplus \cdots$$

There are two natural functors between these categories of algebras:

$$T\colon C_K \to C_E,$$

$T(K) = E$, with $E^i = F_i/F_{i-2}$ and the product induced by the multiplicative filtration on K, and

$$L\colon C_E \to C_K,$$

$L(E) = K$, $F_{2i} = E^0 \oplus E^2 \oplus \cdots \oplus E^{2i}$ and $F_{2i+1} = E^1 \oplus E^3 \oplus \cdots \oplus E^{2i+1}$.

It is clear that $T \circ L\colon C_E \to C_E$ is the identity functor and we have the following extension problem:

PROBLEM (6.1). *Given E in C_E, describe all algebras in $T^{-1}(E)$.*

We will partially answer this question in some elementary cases. For this let us restrict our attention to algebras that are quotients of a universal algebra, i.e., of a cummutative algebra which is the tensor product of an exterior algebra over Z, finitely generated by elements of odd degree, with a polynomial algebra over Z, finitely generated by elements of even degree.

Any Z-graded algebra can be considered as Z_2-graded by considering its even and odd parts. In particular this is true for universal algebras, and we can speak of an object of C_K as a quotient of a universal algebra.

PROPOSITION (6.2). *Suppose $E = T(K)$ and E is generated by homogenous elements $e_1, \cdots, e_n$. For each $e_i \in E = F_j/F_{j-2}$ choose a representative $k_i \in F_j$. Then the elements $k_1, \cdots, k_n$ generate K.*

Proof. Let $k \in K$ be a homogeneous nonzero element, i.e., $k \in K^0$ or $k \in K^1$. Then $k \in F_j - F_{j-2}$ for some j. Let $\bar{k}$ be the image of k in $F_j/F_{j-2} = E_j$ and write it as a sum of products of the $e_1, \cdots, e_n$.

Let k' have the same expression as $\bar{k}$ but in the elements $k_1, \cdots, k_n$. Then $k - k' \in F_{j-2}$ and the assertion follows by induction on j.

COROLLARY (6.3). *If $E = T(K)$, then E and K are both quotients of a common universal algebra.*

Now let $K \in C_K$ be a quotient of the universal algebra U by a Z_2-graded homogeneous ideal I_K. To each element $e \in I_K$, we can associate its homogeneous part of highest degree, denoted by $h(e)$. All these homogeneous elements generate a homogeneous ideal of U denoted by $h(I_K)$.

PROPOSITION (6.4). *Let $E = T(K)$, where E and K are both quotients of the universal algebra U, by ideals I_E and I_K. Then $h(I_K) = I_E$.*

Proof. Given $u \in U$, denote by $(u)_E$ and $(u)_K$ its classes in E and K. Notice that if u is homogeneous of degree n, then $(u)_K \in F_n$.

Now, if $u \in I_E$, the $(u)_E = 0$ and so $(u)_K \in F_{n-2}$. Thus $(u)_K = (u')_K$ with u' of degree $\leq n - 2$. Then $u - u' \in I_K$ and $u = h(u - u') \in h(I_K)$, so $I_E \subset h(I_K)$.

If $u \in h(I_K)$, then $u + u' \in I_K$ for some u' of degree $<n$. Then $(u')_K \in F_{n-2}$ and $(u)_E = \overline{(u)}_K = (u + u')_K$ in $E^n = F_n/F_{n-2}$, so $u \in I_E$. This ends the proof.

COROLLARY (6.5). *If $E = T(K)$ and I_E is generated by elements of degrees 0 and 1, then $K = L(E)$.*

This is the case for the $E_\infty{}^*$ term of the Hodgkin spectral sequence for the Stiefel manifolds [R], therefore $K^* = E_\infty{}^*$ thus giving a Z-grading to K^* in this case.

THEOREM (6.6). *As algebras*

$$K^*(X_{4n,2k-1}) \cong \Lambda_Z{}^*(t_1, \cdots, t_{k-3}, z_1, z_2, z_3, u) \otimes Z|y, \delta|/\sim$$

where $\sim$ is described as in Proposition 5.5, except for the two relations in dimension two

$$z_1 u = a_1 y + a_2 \delta + a_3 \delta y$$

and

$$z_2 u = b_1 y + b_2 \delta + b_3 \delta y$$

for some integers a_i and b_i.

Proof. It follows as a Corollary of Proposition 6.4 and Corollary 5.6.

In particular for the case $k = 2n$, when we are dealing with the group $PO(4n) = X_{4n,4n-1}$, we have that

$$z_1 u = 2^{\alpha} a_3 \delta y$$

$$z_2 u = 2^{\alpha-1} b_3 \delta y$$

where a_3 and b_3 are 0 or 1. This coincides with Held and Suter [HS, Theorem 6.2] where they further determine that $a_3 = b_3 = 0$.

Now we wish to identify the complexification of the Hopf bundle over $X_{2n,2k-1}$ with the element $y + 1 \in K^*(X_{4n,2k-1})$. For this notice that the element $y + 1$ is the α construction on the one dimensional representation

$$X\colon \mathrm{Spin}(4n - 2k + 1) \times Z_2 \to Z_2 \to S^1 = U(1)$$

where the first arrow is the projection on the second factor. (See [R, 4.1] and our definition of y in section 4). We have

PROPOSITION (6.7). *The α construction on the one dimensional representation $X \in R(\mathrm{Spin}(4n - 2k + 1) \times Z_2)$ is the complexification of the Hopf bundle.*

Proof. By definition of the α construction [H] or [R, 4, 1], $\alpha(X)$ is the complex line bundle over $X_{4n,2k-1}$ associated to the principle bundle

$$\mathrm{Spin}(4n - 2k + 1) \times Z_2 \to \mathrm{Spin}(4n) \to X_{4n,2k-1}$$

when we let act $\mathrm{Spin}(4n - 2k + 1) \times Z_2$ on C via the representation X.

But since X factors through Z_2 we have

$$\begin{aligned} &\mathrm{Spin}(4n) \times_{\mathrm{Spin}(4n-2k+1)\times Z_2} C \\ &\qquad = \mathrm{Spin}(4n)/\mathrm{Spin}(4n - 2k + 1) \times_{Z_2} C = V_{4n,2k-1} \times_{Z_2} C \end{aligned}$$

and this ends the proof.

Let

$$\alpha(4n, s) = \min\left\{2n - 1,\ 2i - 1 + \nu_2\binom{2n}{i} \,\middle|\, i \geq \left[\frac{4n - s + 2}{2}\right]\right\}$$

where $\nu_2(m)$ = maximum power of 2 dividing m and $[x]$ = integral part of x.

THEOREM (6.8). *The complexification of the Hopf bundle over $X_{4n,s}$ has order $2^{\alpha(4n,s)}$ in $\bar{K}(X_{4n,s})$.*

Proof. This is a direct consequence of Theorem 6.6 and Proposition 6.7 for $s = 2k - 1$. (See the definition of α in Section 5.)

For $s = 2k$ the fibration

$$S^{4n-2k} \to X_{4n,2k} \xrightarrow{\pi} X_{4n,2k-1}$$

is totally noncohomologous to zero in K-theory [GL] and since π pulls the Hopf bundle over $X_{4n,2k-1}$ back to the Hopf bundle over $X_{4n,2k}$, we get the theorem. One can use the same fibration to describe $K^*(X_{4n,2k})$ in terms of $K^*(X_{4n,2k-1})$ as Gitler and Lam do for Stiefel manifolds in [GL].

§7. The negative parallelizability results

The object of this section is to prove the following:

THEOREM (7.1). *The projective Stiefel manifold $X_{n,s}$ is not stably parallelizable in the following cases:*

a) m odd and $s < n - 1$

b) m even, $n > 4$, $s < 4n - 2$ and $(n, s) \neq (8, s)$, $(12, 8)$ or $(16, 8)$.

All others are parallelizable (Theorem 2.1) except possibly $X_{12,8}$.

Proof. This is the union of Corollaries 7.3, 7.4 and Proposition 7.5, 7.6 and 7.7.

Let $2^{\beta(n,s)}$ be the order of the Hopf bundle in $\overline{KO}(X_{n,s})$ and let $2^{\alpha(n,s)}$ be the order of its complexification in $\bar{K}(X_{n,s})$. Since complexification followed by realification is multiplication by two, we have

$$\beta(n, s) = \alpha(n, s) + \epsilon \quad \text{with} \quad \epsilon = 0 \quad \text{or } 1.$$

PROPOSITION (7.2). *Let $n > 2$, $1 \leq s < 4n - 2$ and $(4n, s) \neq (12, 8)$ or $(16, 8)$. Then $\nu_2(4ns) < \alpha(4n, s)$.*

The proof is in the next section.

COROLLARY (7.3). *If $n \geq 2$, $1 \leq s < 4n - 2$ and $(4n, s) \neq (12, 8)$ or $(16, 8)$, the manifold $X_{4n,s}$ is not stably parallelizable.*

Proof. The tangent bundle $\tau(X_{n,s})$ is stably isomorphic to ns times the Hopf bundle [L]. Then by Proposition 7.2, $\tau(X_{4n,s})$ is not zero in $\bar{K}(X_{4n,s})$.

COROLLARY (7.4). *If $n \geq 2$, $1 \leq s < 4n$ and $1 \leq k \leq 3$, the manifold $X_{4n+k,s}$ is not stably parallelizable, except possibly for $X_{10,4}$.*

Proof. If $n > 2$, then $\nu_2(4ns) \leq \alpha(4n, s)$. To see this one checks it directly for $(4n, s) = (12, 8)$, $(16, 8)$, $(4n, 4n - 1)$ and $(4n, 4n - 2)$, and then applies Proposition 7.2 in the other cases.

Since the natural inclusion $X_{4n,s} \to X_{4n+k,s}$ pulls the Hopf bundle back to the Hopf bundle we have that $\alpha(4n, s) \leq \alpha(4n + k, s)$. But $\nu_2((4n + k)s) \leq \nu_2(4ns)$ so than $\tau(X_{4n+k,s})$ is not zero in $\bar{K}(X_{4n+k,s})$.

For $n = 2$ we know that $\alpha(8, s) = 3$ for $1 \leq s \leq 7$ and then $\alpha(8 + k, s) \geq 3$ for $k = 1, 2, 3$ and $1 \leq s \leq 7$. Since $\nu_2((8 + k)s) < 3$ except for $(10, 4)$, this completes the proof.

PROPOSITION (7.5). *If $n > 1$ and $k = 0$ or 1, the manifold $X_{4n+3,4n+k}$ is not stably parallelizable.*

Proof. This is obvious for $k = 1$ since $\nu_2((4n + 3)(4n + 1)) = 0$ and $\alpha(4n + 3, 4n + 1) \geq 1$ because the Hopf bundle is not trivial.

Now, following the ideas of Sections 5 and 6 it is possible to show that for $n > 1$, $\alpha(4n + 2, 4n) = \nu_2(8n)$.

Since the natural inclusion $X_{4n+2,4n} \to X_{4n+3,4n}$ pulls the Hopf bundle back to the Hopf bundle, we have that $\alpha(4n + 2, 4n) \leq \alpha(4n + 3, 4n)$ and then $\tau(X_{4n+3,4n})$ is not zero in $\bar{K}(X_{4n+3,4n})$.

PROPOSITION (7.6). *The manifold $X_{10,4}$ is not stably parallelizable.*

Proof. As in the proof of Proposition 7.5, we can show that $\alpha(10, 8) = 4$. Since $\alpha(10, 4) \geq \alpha(10, 8)$, $\tau(X_{10,4})$ is not zero in $\bar{K}(X_{10,4})$.

PROPOSITION (7.7). *The manifold $X_{n,s}$ is not stably parallelizable if $(n, s) =$ (7, 4), (7, 3), (7, 2), (7, 1), (6, 3), (6, 2), (6, 1), (5, 3), (5, 2) or (5, 1).*

Proof. Take first the case (7, 4). We will make use of the Atiyah-Hirzebruch spectral sequence for KO to show that $\beta(7, 4) = 3$. We already know that $\beta(7, 4) \leq \beta(8, 4) = 3$.

Let $f\colon X_{7,4} \to P^\infty$ the classifying map of the Hopf bundle, so we have the fibration

$$V_{7,4} \to X_{7,4} \xrightarrow{f} P^\infty.$$

It is well known that $\bar{H}^i(V_{7,4}; Z) = 0$ for $i \leq 3$ and that is enough to see from the Serre spectral sequence of the above nonorientable fibration that $f^*(y^2) \neq 0$ in $H^*(X_{7,4}; Z)$ where $y^2 \in H^4(P^\infty; Z)$ is the generator.

In the Atiyah-Hirzebruch spectral sequence we have

$$0 \neq f^*(y^2) \in F_2{}^{4,-4} = H^4(X_{7,4}; KO^{-4}(*)).$$

But $f^*(y^2)$ is a permanent cycle since it comes from P^∞, and it is not a boundary for dimensional reasons, so

$$0 \neq f^*(y^2) \in E_\infty{}^{4,-4}.$$

Thus, in $\bar{K}O(X_{7,4})$, $2^2(\xi - 1) = (\xi - 1)^3$ is not zero since $(\xi - 1)^3$ is represented by $f^*(y^2) \in E_\infty{}^{4,-4}$. This shows that $\beta(7, 4) \geq 3$.

The other cases follow from the fact that $\beta(4, 3) = 2$ and $\beta(6, 2) = 3$. This also can be proved by using the Atiyah-Hirzebruch spectral sequence as above.

§8. Proof of Proposition 7.2

We begin by recalling a well known theorem of Lucas. Let $\nu_2(x)$ be the maximum power of 2 dividing x, $\alpha(x)$ the number of 1's in the dyadic expansion of x and $\binom{a}{b}$ the usual binomial coefficient.

THEOREM (8.1). (Lucas). $\nu_2\binom{a}{b} = \alpha(b) + \alpha(a-b) - \alpha(a)$.

We will also make use of the following easy consequence of 8.1:

LEMMA (8.2). *Suppose that $2^t \mid 2n$, $2^t \mid i$ and $0 < j < 2^t$. Then*

$$\nu_2\binom{2n}{i} < \nu_2\binom{2n}{i+j}.$$

For the proof of Proposition 7.2 first observe that $\nu_2(4ns) \le \nu_2(4n) < 2n - 1$ if $n > 2$. Thus it is sufficient to prove

$$(*) \qquad \nu_2(4ns) < \min\left\{2i - 1 + \nu_2\binom{2n}{i} \,\middle|\, i \ge \left|\frac{4n-s+2}{2}\right|\right\}$$

for $n > 2$ and $1 \le s < 4n - 2$.

Observe $\left|\dfrac{4n-s+2}{2}\right|$ assumes the same value for $s = 2k - 1$ and $s = 2k$.

Since $\nu_2(4n(2k+1)) < \nu_2(4n(2k))$, it is enough to prove $(*)$ for s even and for $s = 4n - 3$. Thus we must prove

PROPOSITION (8.3). *Let $n > 2$ and $1 \le k < 2n - 1$. Then*

$$\nu_2(8nk) < 2i - 1 + \nu_2\binom{2n}{i} \quad \textit{for all} \quad i \ge 2n - k + 1$$

and

PROPOSITION (8.4). *Let $n > 2$. Then $\nu_2(4n) < 2i - 1 + \nu_2\binom{2n}{i}$ for all $i \ge 2$.*

Proof of 8.4. Let $2n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_s}$, $0 < n_1 < n_2 < \cdots < n_s$. Then $\nu_2(4n) = n_1 + 1 < 2^{n_1+1} - 1$ and 8.4 follows for $i \ge 2^{n_1}$.

Now let $2 \le i < 2^{n_1}$ and write $i = 2^{i_1} + \cdots + 2^{i_t}$, $i_1 < \cdots < i_t < n_1$. Then by the Theorem of Lucas $\nu_2\binom{2n}{i} = n_1 - i_1$. But $2i - 1 + \nu_2\binom{2n}{i} \ge 2^{i_1+1} - 1 + n_1 - i_1 > n_1 + 1 = \nu_2(4n)$ for $i_1 \ge 1$, and $2i - 1 + \nu_2\binom{2n}{i} \ge 3 + n_1 > n_1 + 1 = \nu_2(4n)$ for $i_1 = 0$.

Proof of 8.3. We will consider 5 cases. Let $i_0 = 2n - k + 1$ and $2n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_s}$, $0 < n_1 < n_2 < \cdots < n_s$.

Case 1: $2 \le \nu_2(k) - k_1 < n_1$ and $\nu_2(8nk) = 2 + n_1 + k_1$. Here it suffices to prove 8.3 for $k = 2n - 2^{k_1}$. But then $i_0 = 2^{k_1} + 1$.

For $i = 2^t$, $k_1 \le t < n_1$, then, since $2^{t+1} > 2t + 3 \ge k_1 + t + 3$,

$$2i + 1 + \nu_2\binom{2n}{i} = 2^{t+1} - 1 + n_1 - t > 2 + n_1 + k_1 = \nu_2(8nk).$$

For $i \neq 2^t$ and $i < 2^{n_1}$ we simply invoke Lemma 8.2. And for $i = 2^{n_1}$ we have

$$2i - 1 = 2^{n_1+1} - 1 \geq 2 + 2n_1 > 2 + n_1 + k_1 = \nu_2(8nk).$$

Case 2: $\nu_2(k) = k_1 = 0$ or 1 and $k_1 < n_1$. The proof is analogous to the proof of 8.4.

Case 3: $\nu_2(k) = k_1 = n_1$ and $\nu_2(8nk) = 2 + 2n_1$. In this case it suffices to prove 8.3 for $k = 2n - 2^{n_1+1}$. But then $i_0 = 2^{n_1+1} + 1$ and $2i_0 - 1 = 2^{n_1+2} + 1 > 2 + 2n_1 = \nu_2(8nk)$.

Case 4: $\nu_2(k) = k_1 > n_1$ and$_1k \neq n_2, \cdots, n_s$. Then $n_{t-1} < k_1 < n_t$ for some $t = 2, \cdots, s$, and it is enough to prove 8.3 for $k = 2^{k_1} + 2^{k_1+1} + \cdots + 2^{n_t-1} + 2^{n_{t+1}} + \cdots + 2^{n_s}$. Then $i_0 = 1 + 2^{n_1} + \cdots + 2^{n_{t-1}} + 2^{k_1}$ and $2i_0 - 1 \geq 2^{k_1+1} \geq 2 + 2k_1 > 2 + n_1 + k_1 = \nu_2(8nk)$.

Case 5: $\nu_2(k) = k_1 = n_t$ for some $t = 2, \cdots, s$. In this final case it suffices to prove 8.3 for $k = 2^{n_1} + \cdots + 2^{n_s}$. So $i_0 = 1 + 2^{n_1} + \cdots + 2^{n_{t-1}}$. Observe that $2i_0 - 1 + \nu_2\binom{2n}{i_0} \geq 1 + 2^{n_1+1} + n_t > 2 + n_1 + n_t = \nu_2(8nk)$ and $2^{n_t+1} - 1 \geq 2 + 2n_t > 2 + n_1 + n_t = \nu_2(8nk)$. For $i_0 < i < 2^{n_t}$, then $i = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_q}$ with $n_{t-1} \leq i_q < n_t$ and by the Theorem of Lucas $\nu_2\binom{2n}{i} \geq n_t - i_q$. So $2i - 1 + \nu_2\binom{2n}{i} \geq$

$2^{i_q+1} - 1 + n_t - i_q \geq 2 + i_q + n_t > 2 + n_1 + n_t = \nu_2(8nk)$.

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