Boletín de la Sociedad Matemática Mexicana Vol. 31 No. 2, 1986

A NOTE ON CURVES OF EVEN GENUS

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Summary. Let k be an algebraically closed field of characteristic zero and let C be a projective, irreducible, smooth, non-hyperelliptic curve defined over k of even genus $2n$, $n \geqslant 2$ and generic in the sense of Brill-Noether. Let W_d, $d \geqslant 1$ be the image under the Abel-Jacobi map of the d-symmetric product $C^{(d)}$ of the curve with respect to a fixed base point. In this note we show that the curve C and all its $(2n)!/(n+1)!n!$ special linear series $g_{n+1}{}^1$ can be recovered from the Gauss map on the subvariety W_n.

Let C be a projective irreducible smooth non-hyperelliptic curve of even genus $2n$, $n \geqslant 2$ over an algebraically closed field of characteristic zero which is generic in the sense of Brill-Noether. In this situation, the subvariety $W_{n+1}{}^1$ of the Jacobian variety $J(C)$ of C, has dimension zero and consists of precisely $(2n)!/(n+1)!n!$ different points (cf. [G-H], [A-C-G-H]). Furthermore, the subvariety W_n is smooth and can be identified with the n-symmetric product $C^{(n)}$ of the curve under the Abel-Jacobi map.

The beautiful Theorem of Torelli for curves, tells us among other things, how to recover the curve of genus $g > 0$ from the Gauss map on the subvariety W_{g-1}. The discussion we present here is an extension of the proof of Torelli's Theorem given by A. Andreotti (cf. [A]).

Theorem. *Let C be a curve of genus $2n$, $n \geqslant 2$ as described above. Then the curve and all its linear series $g_{n+1}{}^1$ can be recovered from the Gauss map on the subvariety W_n:*

$$G\colon W_n \to \mathbb{G}\mathrm{r}(n-1, 2n-1)$$
$$p_1 + \cdots + p_n \to \overline{p_1 + \cdots + p_n}$$

where $\overline{p_1 + \cdots + p_n}$ denotes the linear span of the divisor $p_1 + \cdots + p_n$.

Proof. Since the subvariety W_n is smooth, the Gauss map G is defined everywhere and it is holomorphic onto its image. Let B be the analytic subset of the Grassmanian $\mathbb{G}\mathrm{r}(n-1, 2n-1)$ consisting of the points where the fibre of the map G has more than one point. Thus if $H = \overline{p_1 + \cdots + p_n}$ is an element of B, by the geometric Riemann-Roch Theorem [G-H, p. 248] and the fact that C is generic in the sense of Brill-Noether, there is a unique point p_{n+1} of C such that $\overline{p_1 + \cdots + p_n + p_{n+1}} = H$, i.e., the linear series: $p_1 + \cdots + p_n + p_{n+1}$: is a $g_{n+1}{}^1$. Conversely, any element of $W_{n+1}{}^1$ determines points of B where the fibre of the map G has more than one element. Therefore, there is a

* Partially supported by NSF grant No. DMS-8317078.

** This author would like to thank S. Recillas for the helpful conversations and suggestions on the subject. Partially supported by CONACYT grant No. PCCBND-001373.

bijection between the components of B and W^1_{n+1} such that B may be thought of as the disjoint union of $(2n)!/(n+1)!n!$ copies of $\mathbb{P}_1$. Hence, $R = G^{-1}(B)$ is also the disjoint union of the (same number) curves $G^{-1}({g_{n+1}}^1)$. The Gauss map G restricted to each one of these curves is obviously of degree $n+1$ and is injective everywhere else.

$$\begin{array}{ccc}
C^{(n)} & & \mathbb{G}\mathrm{r}(n-1,\, 2n-1) \\
\int \| & & \cup \\
W_n & \longrightarrow & G(W_n) \\
\cup & & \cup \\
R & \longrightarrow & B = \bigcup \mathbb{P}_1 \\
\cup & & \cup \\
G^{-1}({g_{n+1}}^1) & \longrightarrow & \mathbb{P}_1 \approx |{g_{n+1}}^1|
\end{array}$$

In order to see that each one of the curves $G^{-1}({g_{n+1}}^1)$ is birational to the original curve C, we define

$$F\colon G^{-1}({g_{n+1}}^1) \to C,$$
$$D = p_1 + \cdots + p_n \to [\bar{D} \cdot C - D] = p_{n+1}.$$

This map is obviously birational between the two curves. Furthermore, the Gauss map $G\colon G^{-1}({g_{n+1}}^1) \to \mathbb{P}_1 \approx |{g_{n+1}}^1|$ is such that $G(D) = \bar{D} \cdot C$ is the unique divisor of the corresponding linear system ${g_{n+1}}^1$ containing the point p_{n+1}, and the following diagram is commutative

$$\begin{array}{ccc}
G^{-1}({g_{n+1}}^1) & \longleftrightarrow & C \\
{\scriptstyle n+1 \text{ to } 1} \searrow & & \swarrow {\scriptstyle n+1 \text{ to } 1} \\
& \mathbb{P}_1 \approx |{g_{n+1}}^1| &
\end{array}$$

Thus, the analytic set B of the Gauss map G on the subvariety W_n is birationally equivalent to $(2n)!/(n+1)!n!$ disjoint copies of the curve C and the restriction of G to these copies gives the corresponding rational map associated to the linear series ${g_{n+1}}^1$.

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