

A NOTE ON CURVES OF EVEN GENUS

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Summary. Let k be an algebraically closed field of characteristic zero and let C be a projective, irreducible, smooth, non-hyperelliptic curve defined over k of even genus $2n$, $n \geq 2$ and generic in the sense of Brill-Noether. Let W_d , $d \geq 1$ be the image under the Abel-Jacobi map of the d -symmetric product $C^{(d)}$ of the curve with respect to a fixed base point. In this note we show that the curve C and all its $(2n)!/(n+1)!n!$ special linear series g_{n+1}^1 can be recovered from the Gauss map on the subvariety W_n .

Let C be a projective irreducible smooth non-hyperelliptic curve of even genus $2n$, $n \geq 2$ over an algebraically closed field of characteristic zero which is generic in the sense of Brill-Noether. In this situation, the subvariety W_{n+1}^1 of the Jacobian variety $J(C)$ of C , has dimension zero and consists of precisely $(2n)!/(n+1)!n!$ different points (cf. [G-H], [A-C-G-H]). Furthermore, the subvariety W_n is smooth and can be identified with the n -symmetric product $C^{(n)}$ of the curve under the Abel-Jacobi map.

The beautiful Theorem of Torelli for curves, tells us among other things, how to recover the curve of genus $g > 0$ from the Gauss map on the subvariety W_{g-1} . The discussion we present here is an extension of the proof of Torelli's Theorem given by A. Andreotti (cf. [A]).

THEOREM. *Let C be a curve of genus $2n$, $n \geq 2$ as described above. Then the curve and all its linear series g_{n+1}^1 can be recovered from the Gauss map on the subvariety W_n :*

$$G: W_n \rightarrow \mathbb{G}r(n-1, 2n-1)$$

$$p_1 + \cdots + p_n \rightarrow \overline{p_1 + \cdots + p_n}$$

where $\overline{p_1 + \cdots + p_n}$ denotes the linear span of the divisor $p_1 + \cdots + p_n$.

Proof. Since the subvariety W_n is smooth, the Gauss map G is defined everywhere and it is holomorphic onto its image. Let B be the analytic subset of the Grassmanian $\mathbb{G}r(n-1, 2n-1)$ consisting of the points where the fibre of the map G has more than one point. Thus if $H = \overline{p_1 + \cdots + p_n}$ is an element of B , by the geometric Riemann-Roch Theorem [G-H, p. 248] and the fact that C is generic in the sense of Brill-Noether, there is a unique point p_{n+1} of C such that $\overline{p_1 + \cdots + p_n + p_{n+1}} = H$, i.e., the linear series: $p_1 + \cdots + p_n + p_{n+1}$ is a g_{n+1}^1 . Conversely, any element of W_{n+1}^1 determines points of B where the fibre of the map G has more than one element. Therefore, there is a

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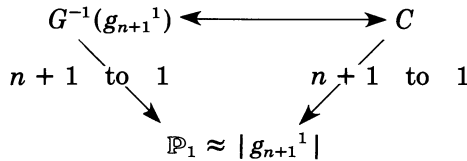
bijection between the components of B and W_{n+1}^1 such that B may be thought of as the disjoint union of $(2n)!/(n+1)!n!$ copies of \mathbb{P}_1 . Hence, $R = G^{-1}(B)$ is also the disjoint union of the (same number) curves $G^{-1}(g_{n+1}^1)$. The Gauss map G restricted to each one of these curves is obviously of degree $n+1$ and is injective everywhere else.

$$\begin{array}{c}
 C^{(n)} \quad \text{Gr}(n-1, 2n-1) \\
 \int \parallel \quad \cup \\
 W_n \rightarrow G(W_n) \\
 \cup \quad \cup \\
 R \rightarrow B = \cup \mathbb{P}_1 \\
 \cup \quad \cup \\
 G^{-1}(g_{n+1}^1) \rightarrow \mathbb{P}_1 \approx |g_{n+1}^1|
 \end{array}$$

In order to see that each one of the curves $G^{-1}(g_{n+1}^1)$ is birational to the original curve C , we define

$$\begin{aligned}
 F: G^{-1}(g_{n+1}^1) &\rightarrow C, \\
 D = p_1 + \dots + p_n &\rightarrow [\bar{D} \cdot C - D] = p_{n+1}.
 \end{aligned}$$

This map is obviously birational between the two curves. Furthermore, the Gauss map $G: G^{-1}(g_{n+1}^1) \rightarrow \mathbb{P}_1 \approx |g_{n+1}^1|$ is such that $G(D) = \bar{D} \cdot C$ is the unique divisor of the corresponding linear system g_{n+1}^1 containing the point p_{n+1} , and the following diagram is commutative



Thus, the analytic set B of the Gauss map G on the subvariety W_n is birationally equivalent to $(2n)!/(n+1)!n!$ disjoint copies of the curve C and the restriction of G to these copies gives the corresponding rational map associated to the linear series g_{n+1}^1 .

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