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GAUSS MAPS AND FIRST ORDER DEFORMATIONS OF SINGULAR HYPERSURFACES

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Summary

We consider the problem of determining the smoothing directions of a singular hypersurface moving in a family. We determine the set-theoretic tangent cone to the discriminant locus of the universal family of degree d hypersurfaces in $\mathbb{P}^n$, by using a "Gauss map" to relate the problem to a theorem of Lê-Teissier expressing a duality between the Nash blow-up and ordinary blow-up of a codimension one subvariety of an analytic space.

We want to express our deep appreciation to the organizers of the Lefschetz Centennial Conference that took place in Mexico City during December, 1984. This paper is based on the lecture given by the first named author.

§1. Motivating Question

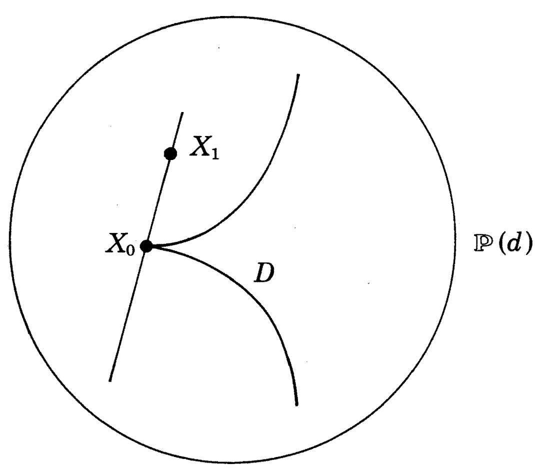
What are the smoothing directions for an arbitrary complete hypersurface X_0 moving in a family?

To get some idea, first look at plane curves $X_0 \subseteq \mathbb{P}^2$, i.e. let

$$\begin{aligned} \mathbb{P}(d) &= \{\text{space of plane curves of degree } d\} \\ &\cup\!| \\ D &= \{\text{variety of singular curves}\}. \end{aligned}$$

We are interested in the local geometry of D.

If $X_0 \in D$, then we denote the projectivized tangent space to $\mathbb{P}(d)$ at X_0 by: $\Pi = \mathbb{P}(T_{X_0}\mathbb{P}(d)) = \{\text{lines in } \mathbb{P}(d) \text{ through } X_0\} = \{\text{pencils of form } X_0 + t \cdot X_1\}$



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Question: In which direction does X_0 remain singular? A partial answer is very famous:

"BERTINI": *The generic curve in the pencil* $X_0 + t \cdot X_1$ *is non-singular except possibly at the base points* $X_0 \cap X_1$ *of the pencil.*

COROLLARY. *If* $X_1 \cap \operatorname{sing} X_0 = \phi$, *then* $X_0 + t \cdot X_1$ *is a smoothing direction for* X_0.

Remark. If X_0 is not reduced, then $\operatorname{sing} X_0$ is a curve, $X_1 \cap \operatorname{sing} X_0$ is never empty, and so the corollary tells us nothing about smoothing directions.

Question. ("Infinitesimal Bertini") In which directions does X_0 remain singular, even to first order? i.e., describe $\mathbb{P}C_0(D) \subseteq \Pi$, the projectivized tangent cone to the discriminant locus, at X_0.

THEOREM. *If* X_0 *is reduced, then as sets* $\mathbb{P}C_0(D) = \cup_{x \in \operatorname{sing} X_0} H_x$ *where*

$$H_x = \{\text{the hyperplane of pencils } X_0 + t \cdot X_1 \text{ with } x \in X_1\}.$$

In the general case of germs of isolated singularities of arbitrary hypersurfaces, this was proved by Teissier [7], but in our case, at least, this is probably classical.

Proposed Generalization. Let $X_0 \subseteq \mathbb{P}^2$ be any plane curve, $\Sigma_0 = \operatorname{sing}(X_{0,\,\mathrm{red}})$, and for $1 \leqslant i \leqslant n$ let $\{\Sigma_i\} = \{$those irreducible components of $X_{0,\,\mathrm{red}}$ which are not reduced in $X_0\}$. Then, as a set,

$$\mathbb{P}C_0(D) = \cup_{i=0}^{n} \phi_d(\Sigma_i)^* \subseteq \Pi$$

where

$$\begin{array}{rcl} \phi_d\colon X_0 & \hookrightarrow & \Pi^* \\ \cap\!\mid & & \cap\!\mid \\ \mathbb{P}^2 & \hookrightarrow & \mathbb{P}(d)^* \end{array}$$

is the d-fold Veronese map. (Thus smoothing directions would come from pencils $X_0 + t \cdot X_1$ in which X_1 neither contains any point of $\operatorname{sing}(X_{0,\,\mathrm{red}})$ nor is tangent to any one of the Σ_i, $1 \leqslant i \leqslant n$).

§2

What evidence exists for this conjecture? Since it is already known in case X_0 is reduced, we check it in the other extreme case, when $X_0 = d \cdot l$ is a d-fold line in $\mathbb{P}^2$. Then we will show that, as a scheme,

$$\mathbb{P}C_0(D) = (d - 1) \cdot 143\phi_d(l)^*.$$

Recall that $\mathbb{P}C_0(D) \subseteq \Pi = \{\text{lines in } \mathbb{P}(d) \text{ through } X_0\}$. To show the desired equality as sets, it suffices to prove that any point of Π which is not in the dual of $\phi_d(l)$ also is not in $\mathbb{P}C_0(D)$. To detect when a pencil Δ in $\mathbb{P}(d)$, through X_0, is tangent to D, we want to be able to compute its intersection number with D. We use the

FORMULA: $$\#(\Delta \cap D, X_0) = \chi_{\text{top}}(X_0) - \chi_{\text{top}}(X_\eta) + \mu(\mathscr{C}_{\restriction\Delta})$$

where χ_{top} is the topological Euler characteristic, $\mathscr{C}_{\restriction\Delta}$ is the restriction of the universal plane curve of degree d over the pencil (or disc) Δ, and X_η is a general curve in the family $\mathscr{C}_{\restriction\Delta}$. Now we apply the basic tautology concerning hyperplane sections of the Veronese mapping; i.e., if $X_0 + t \cdot X_1 = \Delta$ represents a point of Π which is not in $\phi_d(l)^*$ then the corresponding hyperplane in Π^* meets $\phi_d(l)$ transversally in d distinct points and hence X_1 meets l also transversally in d distinct points. This means the singularities of $\mathscr{C}_{\restriction\Delta}$ are generic, hence also its Milnor number and by the Formula, the intersection number $\#(\Delta \cap D, X_0)$ is also generic, so Δ is not in $\mathbb{P}C_0(D)$. This proves the set-theoretic statement.

On the other hand, the formula applied to a generic Δ tells us the multiplicity of D at X_0 is equal to $2 - (2 - (d-1)(d-2)) + d(d-1) = 2(d-1)^2$. Since the Veronese curve has dual variety $\phi_d(l)^*$ of degree $2(d-1)$, we get indeed $\mathbb{P}C_0(D) = (d-1).\, \phi_d(l)^*$. Q.E.D.

For a proof which works for a d-fold hyperplane, see [5].

§3

To shed further light on the problem, and to generalize at least the set-theoretic side of it to higher dimensions, we want to bring in the very general geometric "duality theorem" of Lê and Teissier, announced in [8], and called to our attention by S. Kleiman.

THEOREM (Lê-Teissier). *Given $0 \in D \subseteq S$, D any equidimensional subspace of a smooth analytic space S,*

(i) *consider the conormal space on D, i.e., the map $\pi: \hat{D} \to D$, where $\hat{D}$ is the closure in $\mathbb{P}T^*(S)$ of the set $\{(H, x): x$ smooth point of D, $T_x(D) \subseteq H =$ a hyperplane of $T_x(S)\}$, and $\pi(H, x) = x$.*

$$\begin{array}{ccccccl}
(H, x) & \in & \hat{D} & \subseteq & \mathbb{P}(T^*(S)) & = & \textit{projectivized cotangent bundle on } S \\
\Big\downarrow & & \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow & & \\
x & \in & D & \subseteq & S & &
\end{array}$$

(ii) *Consider any point $o \in D$, the fiber over it $\pi^{-1}(o) \subseteq \hat{D}$, and the corresponding "blow-up" diagram*:

$$\begin{array}{ccccccc}
\rho^{-1}(o) & \subseteq & \tilde{\hat{D}} & \xrightarrow{\hat{\sigma}} & \hat{D} & \supseteq & \pi^{-1}(o) \\
\downarrow & & \downarrow{\scriptstyle \tilde{\pi}} & \searrow^{\rho} & \downarrow & & \downarrow \\
\mathbb{P}C_0(D) & \subseteq & \tilde{D} & \xrightarrow[\sigma]{} & D & \ni & 0
\end{array}$$

in which σ is the blowing up of $o \in D$ and $\hat{\sigma}$ is the blowing up of the subscheme $\pi^{-1}(o) \subseteq \hat{D}$.

(iii) *Decompose the effective Cartier divisor $\rho^{-1}(o) = \cup \tilde{Z}_\alpha$ into (set-theoretic) irreducible components $\tilde{Z}_\alpha$.*

Then, these components $\tilde{Z}_\alpha$ are "conormal varieties"; i.e., *if $\hat{\sigma}(\tilde{Z}_\alpha) = Z_\alpha$, and $\tilde{\pi}(\tilde{Z}_\alpha) = W_\alpha$, then $W_\alpha = Z_\alpha^*$ (dual variety).*

Remarks. This theorem relates the *generalized components* W_α of the projectivized tangent cone of D at o to the generalized components Z_α of the fibers of the conormal map to D. Generalized components may be difficult to compute but clearly every irreducible component is a generalized component, and every irreducible component $\pi^{-1}(o)$ whose dual is a hypersurface contributes this hypersurface as an irreducible component to $\mathbb{P}C_0(D)$.

Now we will place our conjecture, which expresses a duality between $\mathbb{P}C_0(D)$ and a projective model of sing(X_0), in the context of the Lê-Teissier theorem by means of the "Gauss map". Moreover, we will consider projective hypersurfaces of arbitrary dimension.

Fix integers n, $d \geqslant 1$, and consider the universal family of hypersurfaces of degree d in $\mathbb{P}^n$:

$$\begin{array}{c}
\mathscr{X} \subseteq \mathbb{P}(n, d) \times \mathbb{P}^n \\
\downarrow{\scriptstyle \pi} \\
S = \mathbb{P}(n, d)
\end{array}$$

where $\mathbb{P}(n, d) = \{$space of homogeneous polynomials of degree d in $z_0, \cdots, z_n$, modulo $\mathbb{C}^*$-action$\}$ and $\mathscr{X} = \{F = o\}$ is the zero locus of the general polynomial of degree d.

LEMMA. *$\mathscr{X}$ is non-singular, π is surjective, proper and flat.*

Definition. (1) The critical locus $\mathscr{C} \subseteq \mathscr{X}$ of π, is the subscheme where the derivative π_* is not surjective, $\pi_{x,x}$: $T_x\mathscr{X} \to T_{\pi(x)}S$.

(2) The discriminant locus $D \subseteq S$ *is the image*, $D = \pi(\mathscr{C})$, of the critical locus. D is a hypersurface in S.

(3) The Nash blow-up of D, $\hat{D} \subseteq \mathbb{P}(T^*(S))$ is the conormal space of D as defined above.

Now we introduce the "Gauss map":

LEMMA. (1) *If* $\pi\colon \mathscr{X} \to S$ *is the family of hypersurfaces defined above, then for every critical point* $x \in \mathscr{C} \subseteq \mathscr{X}$, $\pi_*(T_x\mathscr{X})$ *is a hyperplane in* $T_{\pi(x)}(S)$, *hence defines a point* $\gamma(x) \in \mathbb{P}T_{\pi(x)}{}^*(S)$. *The map* $\gamma\colon \mathscr{C} \to \mathbb{P}T^*(S)$ *is called the Gauss map.*

(2) *If* $\phi_d\colon \mathbb{P}(n, d) \times \mathbb{P}^n \to \mathbb{P}(n, d) \times \mathbb{P}(n, d)^*$ *is the universal degree* d *Veronese map, then* ϕ_d *is an embedding and the following diagram commutes*:

$$\begin{array}{ccc}
\mathbb{P}(n, d) \times \mathbb{P}^n & \xrightarrow{\ \phi_d\ } & \mathbb{P}(n, d) \times \mathbb{P}(n, d)^* \\
\cup| & & \cup| \\
\mathscr{X} & \xrightarrow{\ \phi_d\ } & \mathbb{P}(T^*S) \\
\cup| & & \cup| \\
\mathscr{C} & \xrightarrow{\ \gamma\ } & \hat{D}
\end{array}$$

Sketch of Proof. (1) is an easy rank computation since π is a family of hypersurfaces. For (2), Sard's theorem implies $\gamma(x) = T_{\pi(x)}(D)$ for a general point $x \in \mathscr{C}$, hence $\gamma(\mathscr{C}) \subseteq \hat{D}$. For a general singular hypersurface X_0 with one node x, $\phi_d(x) = \{$the set of degree d hypersurfaces containing $x\} = T_{X_0}(D) = \gamma(x)$. Q.E.D.

COROLLARY. (1) γ *is an embedding of* $\mathscr{C}$ *into* $\mathbb{P}T^*(S)$, *hence an isomorphism of* $\mathscr{C}$ *with* $\hat{D}$.

(2) The blow-up diagrams below are isomorphic (via the Veronese or Gauss map):

$$\begin{array}{ccc}
\tilde{\mathscr{C}} & \longrightarrow & \mathscr{C} \\
\downarrow & & \downarrow \\
\tilde{D} & \longrightarrow & D
\end{array}
\quad \cong \quad
\begin{array}{ccc}
\tilde{\hat{D}} & \longrightarrow & \hat{D} \\
\downarrow & & \downarrow \\
\tilde{D} & \longrightarrow & D
\end{array}$$

(3) *If* $\pi^{-1}_{|\mathscr{C}}(o) = \mathrm{sing}(X_0) = \cup\, Z_\alpha$ *is the decomposition of the singular locus of* X_0 *into generalized components, as in the theorem of Lê-Teissier, and if* ϕ_d *denotes the degree* d *Veronese map, then as sets*

$$\mathbb{P}C_{X_0}(D) = \cup\, \phi_d(Z_\alpha)^*.$$

§4. Further Remarks

(1) More generally, if X_0 is any hypersurface, the sheaf of first order deformations of X_0, $\mathscr{T}^1X_0$, is a line bundle supported on the singular locus

$\text{sing}(X_0)$, and any family $\pi: \mathscr{X} \to (S, o)$ with $X_0 = \pi^{-1}(0)$ defines a subspace of sections of the sheaf $\mathscr{T}^1 X_0$. The resulting rational map to projective space agrees with the Gauss map of the family, and relates the singular deformation directions for X_0 dually to the projective geometry of the corresponding projective model of $\text{sing}(X_0)$, [6].

(2) A more refined study of the local geometry, (i.e., including multiplicities), of the subvariety $\mathscr{N}_0 \subseteq \mathscr{A}_g$ parametrizing (principally polarized) abelian varieties with singular theta divisors has enabled us to characterize geometrically the components of $\mathscr{N}_0$ for $g \leq 5$. For $g = 4$, we recover Beauville's theorem [1, 3]: If $(A, \theta) \in \mathscr{A}_4$, $\text{sing}(\theta) \neq \phi$ but A has no vanishing theta-null, then (A, θ) is the jacobian of a curve. For $g = 5$, we obtain Clemen's conjecture [2, 4]: If $(A, \theta) \in \mathscr{A}_5$, $\text{sing}(\theta) \neq \phi$ and A has no vanishing theta-null, then (A, θ) is the intermediate jacobian of a (singular) quartic double solid.

We hope this line of investigation will shed some light on the intrinsic geometry (Kodaira dimension, etc.) of $\mathscr{A}_6$.

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REFERENCES

[1] A. BEAUVILLE, *Prym Varieties and the Schottky Problem*, Invent. Math. **41**(1977), 149–196.

[2] R. DONAGI, R. SMITH AND R. VARLEY, *The branch locus of the Prym map*, in preparation.

[3] R. SMITH AND R. VARLEY, *On the geometry of* N_0, Rend. Sem. Mat. Univers. Politecn. Torino, Vol. **42**, 2(1984), 29–37.

[4] ———, *Components of the locus of singular theta divisors, of genus* 5, Algebraic Geometry, Sitges 1983, Lecture Notes in Mathematics, Springer-Verlag 1124, 338–416.

[5] ———, *The tangent cone to the discriminant*, Proceedings of the 1984 Vancouver Conference in Algebraic Geometry, Amer. Math. Soc., Providence, 1986, 443–460.

[6] ———, *Tangent cones to discriminant loci for families of hypersurfaces*, To appear in Trans. A.M.S.

[7] B. TEISSIER, *Déformations à type topologique constant I, II*, Séminaire Douady-Verdier 1971–72, Astérisque **16**(1974), 215–249.

[8] ———, *Sur la classification des singularités des espaces analytiques complexes*, Proceedings of the International Congress of Mathematicians 1983 Warszawa, Vol. 1, Polish Scientific Publishers, Warsaw, 1984, 763–781.