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CLASSES OF TOPOLOGICAL SPACES PRESERVED UNDER REALCOMPACTIFICATIONS

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1. Introduction and preliminary results

A well-known result in general topology states that a Tychonoff space X is a P-space if and only if νX is a P-space (see for example [G-J, problem 8A]). Few other properties seem to be preserved by the functor ν (connectedness is one of them), in particular νX is not first countable unless X is first countable and realcompact. The aim of this paper is to study two classes of topological spaces, the C-spaces and the C'-spaces, which are closed with respect to products and the functor ν, and which contain all first countable spaces (indeed all k-spaces). Strangely, these spaces are in some ways complementary to the class of P-spaces (see Corollary 2.2, below).

A subset S of a topological space X is said to be C-discrete if there exists a discrete family of distinct open sets $\{U_s: s \in S\}$ such that $s \in U_s$. Clearly every C-discrete subset of X is closed and discrete and the following easy lemma is left to the reader:

LEMMA (1.1). *Every C-discrete subset of X is C-embedded in X and every countable, closed, discrete and C-embedded subset of X is C-discrete.*

We note that the right hand edge of the Tychonoff plank (see for example [G-J, 8.20]) is an example of a closed and discrete subset which is not C-discrete; the second result of the lemma is the strongest of its type which can be proved in ZFC. Under MA($\aleph_1$), the ωth level of an $\aleph$-Cantor tree (see [R, page 21]) is closed, discrete, C-embedded (since the space is normal) and of cardinality $\aleph_1$, but is not C-discrete since the space is separable. Assuming the existence of measurable cardinals Comfort [C] has given another example of a closed discrete and C-embedded subset which is not C-discrete, but we know of no example constructed in ZFC.

A point $x \in X$ is said to be a C-point if for each countable set $\{U_n: n \in N\}$ of neighbourhoods of x it is possible to choose $x_n \in U_n$ in such a way that $\{x_n: n \in N\}$ is not C-discrete. A space is said to be a C-space if every non-isolated point is a C-point. Similar concepts (with closed and discrete replacing C-discrete) were introduced in [M], but the following example (also in [C]) is a non-discrete (pseudocompact) C-space in which every countable set is closed and discrete.

Example (1.2). It is known that in ZFC there exist weak P-points in $\beta N - N$, that is to say, points p which are not in the closure of any countable subset of $\beta N - N - \{p\}$ (see [K]). For some fixed weak P-point $p \in \beta N - N$,

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we denote the type of p by $T(p) = \{\bar{f}(p): f$ is a permutation of $N\}$ (here $\bar{f}$ denotes the Stone extension of f). $T(p)$ is pseudocompact (by [G-S, Theorem 5.3]) and hence every C-discrete subset (being C-embedded) is finite. Thus every point of $T(p)$ is a C-point. On the other hand, since p is a weak P-point of $\beta N - N$ every countable subset of $T(p)$ is closed and hence discrete.

X_p will denote the set of P-points of a topological space X and int$A$ will denote the interior of a subset $A \subset X$. A space X is said to be a C'-space if intX_p is discrete and is said to be admissible if each of its points is a C-point or a P-point.

2. Realcompactifications of C-spaces and C'-spaces

In this section all spaces are assumed to be Tychonoff spaces.

LEMMA (2.1). *A space X is a C-space if and only if it is an admissible C'-space.*

Proof. (Necessity). Let $A = \text{int}X_p$. Since the property of being a C-space is preserved under open subsets, A is a P-space and a C-space. However, in a P-space every countable subset is closed, discrete and C-embedded ([G-J, problem 4K]), and hence by Lemma (1.1), every countable subset of A is C-discrete. Thus A must be discrete.

(Sufficiency). We will show that each non C-point $x \in X$ is isolated. Since X is admissible, x is a P-point and since x is not a C-point there exists a countable family of open neighbourhoods $\{U_n: n \in N\}$ of x such that for any choice of $x_n \in U_n$, the set $\{x_n: n \in N\}$ is C-discrete. Clearly, if $z \in \bigcap_{n\in N} U_n$, then z is not a C-point of X and so $\bigcap_{n\in N} U_n \subset X_p$. Since X is a C'-space, $\text{int}(\bigcap_{n\in N} U_n)$ is discrete. But since x is a P-point, $x \in \text{int}(\bigcap_{n\in N} U_n)$ and so x is an isolated point.

COROLLARY (2.2). *X is a P-space and a C-space if and only if X is discrete.*

THEOREM (2.3). *If X is the union of connected zero sets then X is a C'-space.*

Proof. Suppose that $X = \bigcup_{\alpha\in I} Z_\alpha$, where Z_α is a connected zero-set for each $\alpha \in I$. Let A be a non-empty cozero set in X whose closure is contained in X_p (if no such set exists then $\text{int}X_p = \phi$ and the theorem is proved). If $A \cap Z_\alpha \neq \phi$ then necessarily $Z_\alpha \subset A$ because otherwise the sets $A \cap Z_\alpha$ and $Z_\alpha - A$ being closed (in a P-space each cozero set is closed) would separate the connected set Z_α. But if $Z_\alpha \subset A$, then necessarily Z_α must consist of a single point since every P-space is totally disconnected; furthermore, since every G_δ-point in a P-space is isolated, $\text{int}X_p$ must consist of isolated points.

THEOREM (2.4). *Every k-space is a C-space.* (*Compare* [M, *Lemma* 2.7]).

Proof. Since each k-space is the quotient image of a locally compact space, it suffices to show that the quotient image of a locally compact space is a C-space. To this end, let X be a locally compact T_2-space and $f: X \rightarrow Y$ a quotient mapping. We will show that each non-isolated point $y \in Y$ is a C-point. Let $\{U_n: n \in N\}$ be a countable family of open neighbourhoods of y. $f^{-1}(y)$ is closed but not open and hence does not consist solely of isolated points of X. Thus it is possible to choose $x \in \mathrm{Fr}(f^{-1}(y))$. Let V be a compact neighbourhood of x, and select $x_1 \in V \cap f^{-1}(U_1) \cap (X - f^{-1}(y))$. Let $y_1 = f(x_1)$: clearly $y \neq y_1$. Having defined $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$, select $x_{n+1} \in V \cap f^{-1}(U_{n+1}) \cap (X - f^{-1}(y, y_1, \cdots, y_n))$ and define $y_{n+1} = f(x_{n+1})$. It is not hard to see that such a choice is possible, since $f^{-1}(y, y_1, \cdots, y_n)$ is closed for each $n \in N$. Since $\{x_n: n \in N\}$ is an infinite subset of V, it is not C-discrete, and hence neither is the set $\{y_n: n \in N\}$.

THEOREM (2.5). *An arbitrary product of C-spaces (respectively C′-spaces) is a C-space (respectively C′-space).*

Proof. For finite products the results are obvious. That an infinite product of non-trivial C-spaces is a C-space follows from the fact that in such a product every point is an accumulation point of a compact subspace, namely the product of two point spaces, which also shows that no point of an infinite product of non-trivial spaces is a P-point.

Our aim will now be to show that in the absence of measurable cardinals, the classes of C-spaces and C'-spaces are closed under the functor ν; indeed, X is a C'-space if and only if νX is a C'-space.

LEMMA (2.6). *If νX is a C'-space then X is a C'-space.*

Proof. Let A be any open set in νX such that $A \cap X = \mathrm{int} X_p$. Since νX is a C'-space it suffices to show that A is an open P-subspace of νX. Let Z be a non-empty zero set in νX which is contained in A. By [G-J, problem 8D], $Z = \mathrm{cl}_{\nu X}(Z \cap X)$ and since $Z \cap X$ is a zero set in X and $Z \cap X \subset X_p$, $Z \cap X$ is open in X. Thus $\mathrm{cl}_{\beta X}(Z \cap X)$ is an open and closed subset of βX. However, since $Z = \mathrm{cl}_{\nu X}(Z \cap X) = \mathrm{cl}_{\beta X}(Z \cap X) \cap \nu X$, it follows that Z is an open and closed subset of νX. Thus A is a P-space and the result follows.

LEMMA (2.7). *If X is an admissible space then νX is also admissible.*

Proof. We will show that if z is a non C-point of νX, then z is a P-point of νX. Let $\{V_n: n \in N\}$ be a nested family of open neighbourhoods of z in νX. It suffices to show that $z \in \mathrm{int}(\bigcap_{n\in N} V_n)$. Since z is not a C-point of νX, there exists a countable family of open neighbourhoods $\{U_n: n \in N\}$ of z in νX, such that for every choice $z_n \in U_n$, the set $\{z_n: n \in N\}$ is C-discrete. Let Z be a zero-set in νX such that $z \in Z \subset \bigcap_{n\in N} V_n \cap \bigcap_{n\in N} U_n$. But $Z = \mathrm{cl}_{\nu X}(Z \cap X)$ and since $Z \cap X \subset \bigcap_{n\in N} U_n \subset X_p$ it follows that $Z \cap X$ is an open and closed subset of X and hence Z is open in νX and the result follows.

COROLLARY (2.8). *If νX is a C'-space and X is admissible, then X and νX are both C-spaces.*

Proof. This follows from Lemmas (2.1), (2.6) and (2.7).

THEOREM (2.9). *If X is a C-space (respectively C′-space) and every open and closed discrete subset of X is realcompact, then νX is a C-space (respectively C′-space).*

Proof. Assume first that X is a C'-space. We wish to show that if $A \subset \operatorname{int}((\nu X)_p)$ then A is discrete. Assume that A is open in νX. If Z is a zero-set in X which is contained in $A \cap X$ and $Z^* = \mathrm{cl}_{\nu X} Z$, then $Z^* \cap A$ is a G_δ-subset of A. Hence $Z^* \cap A$ is open in νX. Since $Z = (Z^* \cap A) \cap X$, Z is open in X. Thus we have shown that $A \cap X$ is an open P-subspace of X. Since X is a C'-space, $A \cap X$ is discrete, and so we have shown that if S is a zero-set in νX contained in A then $S \cap X$ is an open and closed discrete subset of X. However, $Z^* \cap X$ is then realcompact and so $Z^* = \mathrm{cl}_{\nu X}(Z^* \cap X) = Z^* \cap X$, and so $Z^* \subset X$. Thus $A \subset X$, so A is discrete and the result follows.

The corresponding result for C-spaces follows from Lemmas (2.1) and (2.7).

COROLLARY (2.10). *Assuming that no measurable cardinals exist, a Tychonoff space X is a C'-space if and only if νX is a C'-space.*

On the other hand, if D is a discrete space of Ulam-measurable cardinality, then D is a C'-space but νD is not. Furthermore, as the following example shows, νX may be a C-space even when X is not.

Example (2.11). Let $A \subset (\omega_2 + 1) \times (\omega + 1)$ be defined by $(\alpha, \beta) \in A$ if and only if 1) $\alpha = \omega_2$ and $\beta = \omega$, or 2) $\alpha \in \omega_2$, $\beta \in \omega$ and cof $\alpha \neq \omega$.

The space X is obtained by adjoining a sequence of distinct isolated points convergent to each non-isolated, non C-point of $A \cap (\omega_2 \times \omega)$. X is not a C-space since (ω_2, ω) is neither isolated nor a C-point. On the other hand, using standard arguments one can show that if Y denotes the space obtained from X by adjoining the points $\{(\omega_2, n) : n \in N\}$ in the natural way, then X is C-embedded in Y and hence $\nu X = \nu Y$. Since Y is clearly a C-space it follows from Theorem (2.9) that νY and hence νX is a C-space.

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