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ON MAPS COBORDANT TO EMBEDDINGS

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1. Introduction

Let M^n and N^{n+k} be closed differentiable manifolds. We consider the problem of a given continuous map $f: M \to N$, whether there is an embedding cobordant to f. In this article we first compute the number of cobordism obstructions to embeddings for $k \geq (n+1)/_2$. Next we use a result of R. L. W. Brown [Br] to determine these obstructions in the cases $k = n-1, n-2$. We use throughout homology and cohomology with $\mathbb{Z}_2$-coefficients.

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2. Cobordism Groups of Maps and of Embeddings

We say that two maps $f_0: M_0{}^n \to N_0{}^{n+k}$ and $f_1: M_1{}^n \to N_1{}^{n+k}$ are cobordant if there is a map $F: V \to W$ of compact manifolds such that ∂V is diffeomorphic to the disjoint union of M_0 and M_1, ∂W is diffeomorphic to the disjoint union of N_0 and N_1, $F \mid M_0 = f_0$ and $F \mid M_1 = f_1$. Cobordism classes form an abelian group under disjoint union, which we denote by $M(n, n+k)$. In this definition, if the term "map" is replaced by "embedding" we obtain the bordism group $E(n, n+k)$ of embeddings.

These groups were first studied by Stong [St] and Wall [Wa]. Using the Pontrjagin-Thom construction they showed that

$$M(n, n+k) \cong \mathcal{N}_{n+k}(\Omega^N MO_{k+N}), \quad N \gg k, \quad \text{and} \quad E(n, n+k) \cong \mathcal{N}_{n+k}(MO_k).$$

Let $i: \Sigma^N MO_k \to MO_{k+N}$ be the map induced by the inclusion $O_k \subset O_{k+N}$, and let $\hat{\imath}: MO_k \to \Omega^N MO_{k+N}$ denote the adjoint of i. The forgetful homomorphism $E(n, n+k) \to M(n, n+k)$ is induced by $\hat{\imath}$. If $\hat{id}$ denotes the adjoint of the identity map $\Omega^N MO_{k+N} \to \Omega^N MO_{k+N}$, by the naturality property of adjointness the following diagram commutes:

$$\begin{array}{ccc} H^*(\Sigma^N MO_k) & \xleftarrow{(\Sigma^N \hat{\imath})^*} & H^*(\Sigma^N \Omega^N MO_{k+N}) \\ & {}_{i^*}\nwarrow \quad \nearrow_{\hat{id}^*} & \\ & H^*(MO_{k+N}) & \end{array}$$

Therefore, since i^* is onto, $\hat{\imath}^*: H^*(\Omega^N MO_{k+N}) \to H^*(MO_k)$ is onto, and

$$\hat{\imath}_*: H_*(MO_k) \to H_*(\Omega^N MO_{k+N}) \tag{2.1}$$

is injective. By the natural isomorphism [C.F.] $\mathcal{N}_*(X) \cong H_*(X) \otimes_{\mathbb{Z}_2} \mathcal{N}_*$, one obtains that $E(n, n+k)$ maps injectively into $M(n, n+k)$.

If $\mu = (i_1, \cdots, i_r)$ is a partition (possibly empty) define:

$l(\mu) = r$
$|\mu| = i_i + \cdots + i_r$
μ' = subset of μ consisting of integers not of the form $2^t - 1$, and
μ'' = subset of μ consisting of integers of the form $2^t - 1$.

Stong showed that $H^*(\Omega^N MO_{k+N})$ is isomorphic to the polynomial ring $\mathbb{Z}_2[\nu_\mu]$, with dim $\nu_\mu = |\mu| + k$ and μ satisfying

$$(2.2) \qquad l(\mu'') < |\mu'| + k.$$

Let $p(r)$ denote the number of partitions of r and let $[t]$ be the greatest integer less than or equal to t.

LEMMA (2.3). *The number of partitions μ of $k + j$ not satisfying 2.2 is $\sum_{i=0}^{[j/2]} p(i)$.*

Proof. Let $\varphi: \mathbb{N} \to \mathbb{N} \cup \{0\}$ be given by

$$\varphi(n) = \begin{cases} n, & \text{if } n \neq 2^t - 1, \\ 2^{t-1} - 1, & \text{if } n = 2^t - 1. \end{cases}$$

If $\mu = (i_1, \cdots, i_r)$, then we define $\varphi(\mu) = \Sigma\varphi(i_s)$. Observe that $2\varphi(\mu) = |\mu| + |\mu'| - l(\mu'')$. Then, if $|\mu| = k + j$, $l(\mu'') \geq |\mu'| + k$ if and only if $k + j + |\mu'| - 2\varphi(\mu) \geq |\mu'| + k$, showing that μ does not satisfy 2.2 if and only if $2\varphi(\mu) \leq j$. Let $0 \leq i \leq [j/2]$ and let $(n_1, \cdots, n_r)$ be a partition of i. There is then a unique partition of $k + j$, $\mu = (i_1, \cdots, i_r, 1, 1, \cdots, 1)$ with $\varphi(i_s) = n_s$, $1 \leq s \leq r$. Then $2\varphi(\mu) = 2i \leq j$. □

PROPOSITION (2.4). (i) *Let $0 \leq j \leq 2k$ and $N \gg k$. Then $\hat{i}_*: H_j(MO_k) \to H_j(\Omega^N MO_{k+N})$ is an isomorphism.* (ii) *Let $0 < j < k$ and $N \gg k$. There are then short exact sequences*

$$0 \to H_{2k+j}(MO_k) \to H_{2k+j}(\Omega^N MO_{k+N}) \to C_j \to 0$$

where dim $C_j = \sum_{i=0}^{[j/2]} p(i)p(j-i) + \sum_{j/2<i<j} p(i)$.

Proof. We give the proof of (ii), part (i) being entirely analogous. A basis for $H^{2k+j}(\Omega^N NO_{k+N})$ is given by $\{\nu_\mu, \nu_{\mu_1}, \nu_{\mu_2}: |\mu| = k + j, |\mu_1| + |\mu_2| = j$ and μ satisfies 2.2$\}$.

Therefore dim $H_{2k+j}(\Omega^N MO_{k+N})$ = dim $H^{2k+j}(\Omega^N MO_{k+N})$ is equal to $p(k+j) + \sum_{i=0}^{[j/2]} p(i)p(j-i) - \sum_{i=0}^{[j/2]} p(i)$. The result now follows from 2.1 and the Thom isomorphism $\tilde{H}_{2k+j}(MO_k) = H_{k+j}(BO_k)$. □

Considering the unoriented bordism exact sequence of the pair $(\Omega^N MO_{k+N}, MO_k)$ [C-F] one obtains:

COROLLARY (2.5). (i) *If $k \geq n$ then $\hat{i}_*: E(n, n+k) \to M(n, n+k)$ is an isomorphism.*

(ii) *If $n > k \geq (n+1)/2$ there are short exact sequences*

$$0 \to E(n, n+k) \to M(n, n+k) \to \textstyle\sum_{i=0}^{n-k} \mathcal{N}_i \otimes C_{n-k-1} \to 0.$$

We recall that $\mathcal{N}_*$ is isomorphic to a polynomial algebra $\mathbb{Z}_2[x_2, x_4, x_5, x_6, x_8, \cdots]$ with one generator in each dimension not of the form $2^t - 1$. Using 2.5 one may compute the cokernels of $E(n, n+k) \to M(n, n+k)$ for $2k \geq n+1$. For example the sequences

$$0 \to E(n, 2n-1) \to M(n, 2n-1) \to \mathbb{Z}_2 \to 0 \qquad (n \geq 3)$$

$$(2.6) \qquad 0 \to E(n, 2n-2) \to M(n, 2n-2) \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to 0 \qquad (n \geq 5)$$

are exact.

3. Stiefel-Whitney Numbers of Maps

Let $f: M^n \to N^{n+k}$ be a map. One has the Umkehr homomorphism $f_!: H^*(M) \to H^*(N)$ defined as follows. If $\alpha \in H^i(M)$, then $f_!(\alpha) = D_N f_*(\alpha \cap [M])$, where D_N denotes Poincaré duality for N, and $[M]$ denotes the fundamental class of M. If $\mu = (i_1, \cdots, i_p)$ is a partition, then w_μ denotes the product of Stiefel-Whitney classes $w_{i_1} \cdots w_{i_p}$. For any collection of partitions $\mu_0, \mu_1, \cdots, \mu_l$ with $|\mu_0| + |\mu_l| + \cdots + |\mu_l| + l \cdot k = n + k$, one has a Stiefel-Whitney number of f defined as

$$(3.1) \qquad \langle w_{\mu_0}(N) f_! w_{\mu_l}(M) \cdots f_! w_{\mu_l}(M), [N] \rangle$$

If $l > 0$, then this number can also be expressed as a class in $H^n(M)$ evaluated on $[M]$, namely,

$$(3.2) \qquad \langle f^* w_{\mu_0}(N) f^* f_! w_{\mu_1}(M) \cdots f^* f_! w_{\mu_{l-1}}(M) w_{\mu_l}(M), [M] \rangle.$$

Stong showed that the cobordism class of f is determined by these numbers [St]. Let $W(f) = (f^*(W(N))/W(M)$ be the Stiefel-Whitney class of f in $H^*(M)$. The image of the homomorphism $E(n, n+k) \to M(n, n+k)$ is described by the following result.

THEOREM (3.3). (R. L. W. Brown) *A map $f: M^n \to N^{n+k}$ is cobordant to an embedding if and only if*

(i) *All Stiefel-Whitney numbers of f involving $w_i(f)$ are zero for $i > k$,*

and

(ii) *All numbers of the form* 3.2 *are equal to*

$$\langle f^* w_{\mu_0}(N) w_{\mu_1}(M) \cdots w_{\mu_l}(M) w_k(f)^{l-1}, [M] \rangle.$$

These conditions are somehow redundant as there are relations among the Stiefel-Whitney numbers of maps. The object of this section is to establish the relations among the numbers described in 3.3 for $k = n - 1, n - 2$. We first recall the following Riemann-Roch type theorem (See [D]).

THEOREM (3.4). *Let $f: M \to N$ be a continuous map and x a formal series in $H^*(M)$. Then*

$$\mathrm{Sq} f_!(x) = f_!(\mathrm{Sq}(x) \cdot w(f)) \quad \text{and}$$
$$\mathrm{Sq} f_!(x) \cdot \bar{w}(N) = f_!(\mathrm{Sq}(x) \cdot \bar{w}(M)).$$

According to 3.3 a map $f: M^n \to N^{2n-1}$ is cobordant to an embedding if and only if the following numbers vanish: $\langle w_n(f), [M]\rangle$, $\langle f^*w_1(N)(f^*f_!(1) + w_{n-1}(f)), [M]\rangle$, $\langle w_1(M)(f^*f_!(1) + w_{n-1}(f)), [M]\rangle$ and $\langle f^*f_!w_1(M) + w_1(M)w_{n-1}(f), [M]\rangle$.

LEMMA (3.5). *For any map $f: M^n \to N^{2n-1}$ the following relations hold:*

(i) $\langle w_n(f), [M]\rangle = 0$,

(ii) $\langle f^*w_1(N)(f^*f_!(1) + w_{n-1}(f)), [M]\rangle = 0$, *and*

(iii) $\langle f^*f_!w_1(M), [M]\rangle = \langle w_1(M)f^*f_!(1), [M]\rangle$.

Proof.

$$\begin{aligned}
\langle w_n(f), [M]\rangle &= \langle f_!w_n(f), [N]\rangle \\
&= \langle \mathrm{Sq}^n f_!(1), [N]\rangle \quad \text{by 3.4,} \\
&= 0 \quad \text{as} \quad f_!(1) \in H^{n-1}(N); \\
\langle f^*w_1(N)f^*f_!(1), [M]\rangle &= \langle w_1(N)f_!(1)f_!(1), [N]\rangle \\
&= \langle w_1(N)\mathrm{Sq}^{n-1}f_!(1), [N]\rangle \\
&= \langle w_1(N)f_!w_{n-1}(f), [N]\rangle \quad \text{by 3.4,} \\
&= \langle f^*w_1(N) \cdot w_{n-1}(f), [M]\rangle; \quad \text{and} \\
\langle f^*f_!w_1(M), [M]\rangle &= \langle f_!w_1(M)f_!(1), [N]\rangle \\
&= \langle w_1(M)f^*f_!(1), [M]\rangle. \quad \square
\end{aligned}$$

Hence, we have:

THEOREM (3.6). *A map $f: M^n \to N^{2n-1}$ is cobordant to an embedding if and only if $\langle w_1(M)(f^*f_!(1) + w_{n-1}(f)), [M]\rangle = 0$, $(n \geq 3)$.* $\square$

We now consider the case $k = n - 2$. As in 3.5 (iii), it is very easy to verify that

$$\begin{aligned}
&\langle f^*f_!w_2(M), [M]\rangle = \langle w_2(M)f^*f_!(1), [M]\rangle, \\
\text{(3.7)} \quad \text{and} \quad &\langle f^*f_!w_1(M)^2, [M]\rangle = \langle w_1(M)^2 f^*f_!(1), [M]\rangle, \\
&\langle f^*w_1(N)f^*f_!w_1(M), [M]\rangle = \langle f^*w_1(N) \cdot w_1(M)f^*f_!(1), [M]\rangle.
\end{aligned}$$

Brown's theorem (3.3) now asserts that for a map $f\colon M^n \to N^{2n-2}$ to be cobordant to an embedding the following numbers must vanish:

$$\varphi_1(f) = \langle w_1(M)w_{n-1}(f), [M]\rangle$$

$$\varphi_2(f) = \langle f^*w_1(N)w_{n-1}(f), [M]\rangle$$

$$\varphi_3(f) = \langle f^*w_1(N)w_1(M)(f^*f_!(1) + w_{n-2}(f)), [M]\rangle$$

$$\varphi_4(f) = \langle f^*w_2(N)(f^*f_!(1) + w_{n-2}(f)), [M]\rangle$$

$$\varphi_5(f) = \langle f^*w_1(N)^2(f^*f_!(1) + w_{n-2}(f)), [M]\rangle$$

$$\varphi_6(f) = \langle w_1(M)f^*f_!w_1(M) + w_1(M)^2w_{n-2}(f), [M]\rangle$$

$$\varphi_7(f) = \langle w_2(M)(f^*f_!(1) + w_{n-2}(f)), [M]\rangle$$

$$\varphi_8(f) = \langle w_1(M)^2(f^*f_!(1) + w_{n-2}(f)), [M]\rangle.$$

It follows from 2.6 that five relations hold among these eight numbers. In fact,

LEMMA (3.8). *If $f\colon M^n \to N^{2n-2}$ is any map and the numbers $\varphi_i(f)$ are defined as above, then*

(i) $\varphi_2(f) = \varphi_4(f) = \varphi_5(f) = 0$,
(ii) $\varphi_1(f) = \varphi_6(f)$, and
(iii) $\varphi_3(f) = n\varphi_6(f)$.

As an immediate consequence we have:

THEOREM (3.9). *A map $f\colon M^n \to N^{2n-2}$ is cobordant to an embedding if and only if the following numbers vanish*: $(n \geq 5)$

$$\langle w_1(M) \cdot w_{n-1}(f), [M]\rangle,$$

$$\langle w_2(M) \cdot (f^*f_!(1) + w_{n-2}(f)), [M]\rangle, \quad \textit{and}$$

$$\langle w_1(M)^2(f^*f_!(1) + w_{n-2}(f)), [M]\rangle.$$

Proof of (3.8). In order to show that $\varphi_4(f) = \varphi_5(f)$ one can proceed as in 3.5 (ii). Similarly,

$$\begin{aligned}\varphi_2(f) &= \langle f^*w_1(N)w_{n-1}(f), [M]\rangle\\ &= \langle w_1(N)f_!w_{n-1}(f), [N]\rangle\\ &= \langle w_1(N)\mathrm{Sq}^{n-1}f_!(1), [N]\rangle, \quad \text{by 3.4}\\ &= 0 \quad \text{as } f_!(1) \text{ is an } (n-2)\text{-dimensional class.}\end{aligned}$$

To establish the last two relations we need:

LEMMA (3.10). *If $f\colon M^n \to N^{2n-1}$ is any map, then*

(i) $(f^*w_2(N) + w_1(M)^2)w_{n-2}(f) = w_1(M)w_{n-1}(f)$

(ii) $$f^*w_1(N)w_1(M)w_{n-2}(f) = \begin{cases} f^*w_2(N)w_{n-2}(f) & \textit{if } n \textit{ is even,} \\ w_1(M)^2 w_{n-2}(f) & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. Let $v_i(M)$ denote the i-th Wu class of M. Then $v_2(M) \cdot w_{n-2}(f) = \mathrm{Sq}^2 w_{n-2}(f) = w_2(f)w_{n-2}(f) + nw_1(f)w_{n-1}(f)$ by the Wu formulae. Note that we omit the $\binom{n}{2}w_n(f)$ term since $w_n(f) = 0$ by (i) of 3.5. But $w_2(f) = f^*w_2(N) + f^*w_1(N)w_1(M) + v_2(M)$, obtaining

(3.11) $$f^*w_2(N)w_{n-2}(f) = f^*w_1(N)w_1(M)w_{n-2}(f) + nw_1(M)w_{n-1}(f).$$

Also, $0 = \mathrm{Sq}^1\mathrm{Sq}^1 w_{n-2}(f) = v_1(M) \cdot \mathrm{Sq}^1 w_{n-2}(f) = v_1(M)(w_1(f)w_{n-2}(f) + (n-1)w_{n-1}(f))$. As $w_1(f) = f^*w_1(N) + w_1(M)$, and $v_1(M) = w_1(M)$, one obtains

(3.12) $$w_1(M)^2 w_{n-2}(f) = f^*w_1(N)w_1(M)w_{n-2}(f) + (n-1)w_1(M)w_{n-1}(f).$$

The equations of 3.10 follow easily from 3.11 and 3.12. □

We prove now 3.8 (ii). Since $\mathrm{Sq}\, f_!(v(M)) = w(N) f_!(1)$ by 3.4, it follows that $\mathrm{Sq}^2 f_!(1) + \mathrm{Sq}^1 f_!(v_1(M)) + f_!(v_2(M)) = w_2(N) f_!(1)$. Hence,

$$\begin{aligned} &\langle v_2(M) f^*f_!(1) + v_1(M) f^*f_! v_1(M) + f^*f_! v_2(M), [M]\rangle \\ &\quad = \langle \mathrm{Sq}^2 f^*f_!(1) + \mathrm{Sq}^1 f^*f_!(v_1(M)) + f^*f_!(v_2(M)), [M]\rangle \\ &\quad = \langle f^*w_2(N) f^*f_!(1), [M]\rangle. \end{aligned}$$

But by 3.7 $\langle f^*f_!(v_2(M), [M]\rangle = \langle v_2(M) f^*f_!(1), [M]\rangle$, implying that

(3.13) $$\langle w_1(M) f^*f_!(w_1(M)), [M]\rangle = \langle f^*w_2(N) f^*f_!(1), [M]\rangle.$$

As $\varphi_4(f) = 0$,

$$\begin{aligned} \varphi_6(f) &= \varphi_6(f) + \varphi_4(f) \\ &= \langle w_1(M) f^*f_! w_1(M) + w_1(M)^2 w_{n-2}(f) \\ &\qquad + f^*w_2(N)(f^*f_!(1) + w_{n-2}(f)), [M]\rangle \\ &= \langle (w_1(M)^2 + f^*w_2(N))w_{n-2}(f), [M]\rangle \text{ by 3.13, and} \\ &= \langle w_1(M)w_{n-1}(f), [M]\rangle \quad \text{by 3.10 (i)} \\ &= \varphi_1(f). \end{aligned}$$

Finally, we prove 3.8 (iii). By 3.4, $\mathrm{Sq}^1 f_!(1) = f_! w_1(M) + f_!(1)w_1(N)$, so

$$\begin{aligned} w_1(M) f^*f_! w_1(M) &= \mathrm{Sq}^1 f^*f_!(1) f^*(w_1(N)) \\ &= w_1(M) f^*w_1(N) f^*w_1(N). \end{aligned}$$

Therefore, if n is odd, then

$$\begin{aligned}\varphi_3(f) &= \langle f^*w_1(N)w_1(M)(f^*f_!(1) + w_{n-2}(f)), [M]\rangle \\ &= \langle w_1(M)f^*f_!w_1(M) + f^*w_1(N)w_1(M)w_{n-2}(f), [M]\rangle \\ &= \langle w_1(M)f^*f_!w_1(M) + w_1(M)^2w_{n-2}(f), [M]\rangle \text{ by 3.10 (ii)} \\ &= \varphi_6(f).\end{aligned}$$

If n is even, then

$$\begin{aligned}&\langle f^*w_1(N)w_1(M)(f^*f_!(1) + w_{n-2}(f)), [M]\rangle \\ &\qquad = \langle f^*w_2(N)(f^*f_!(1) + w_{n-2}(f)), [M]\rangle\end{aligned}$$

by 3.13 and 3.10 (ii) showing that

$\varphi_3(f) = \varphi_4(f) = 0$ for n even. This completes the proof of 3.8. □

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