Boletín de la Sociedad Matemática Mexicana Vol. 31 No. 2, 1986

A NOTE ON THE REGULARIZATION THEOREM OF CONTINUOUS LINEAR RANDOM FUNCTIONALS ON $\mathscr{S}$

BY VÍCTOR PÉREZ-ABREU*

0. Introduction

Regular versions of continuous linear random functionals (c.l.r.f.) are useful in the study of nuclear spaced valued processes. In Itô and Nawata (1983) a regularization theorem is proved for c.l.r.f.'s on a vector space E with a multihilbertian topology. This result is used in Itô (1984) to prove the results of Sazonov (1958), Minlos (1963) and Kolmogorov (1959) on the characteristic functionals of probability measures on duals of vector spaces with multihilbertian topologies.

In the present note we give a proof of the regularization theorem for c.l.r.f.'s on countably Hilbertian nuclear spaces (CHNS) of a special type, of which the Schwartz space $\mathscr{S}$ is an example. The main difference with Itô and Nawata's work is that we use the Minlos Theorem as the starting point. We think that this approach is useful for those readers familiar with Minlos' work on the construction of probability measures on duals of CHNS's.

In order to establish notation, in Section 1 we present CHNS's and the Minlos Theorem. In Section 2 a proof of the Regularization Theorem is given. For applications of this theorem the reader is referred to the works by Kallianpur (1986) and Perez-Abreu (1988).

1. Nuclear Spaces and the Bochner-Minlos Theorem

Let Φ be a Fréchet space whose topology is given by an increasing sequence $\|\cdot\|_n$, $n \geqslant 0$ of Hilbertian norms. Let Φ_n be the Hilbert space completion of Φ w.r.t. $\|\cdot\|_n$ and let Φ_n' be the dual (Hilbert) space of Φ_n. For $f \in \Phi_n'$ let

$$\|f\|_{-n} = \sup_{\|\phi\|_n \geqslant 1} |f[\phi]|.$$

Since Φ is a Fréchet space then

$$\Phi = \cap_{n=1}^{\infty} \Phi_n \tag{1.1}$$

and its topology is given by the metric

$$\rho(\phi, \psi) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|\phi - \psi\|_n}{1 + \|\phi - \psi\|_n}, \quad \phi, \psi \in \Phi. \tag{1.2}$$

Since for $n < m$, $\|\phi\|_n < \|\phi\|_m$ $\forall \phi \in \Phi$, then $\Phi_m \subset \Phi_n$ and $\Phi_n' \subset \Phi_m'$. The space (Φ, ρ) is called a *Countably Hilbertian space*. It is called *nuclear* if for each $n \geqslant 0$ there exists $m > n$ such that the canonical injection

$$i: \Phi_m \hookrightarrow \Phi_n$$

* This work was done while the author was at IIMAS-UNAM and was partially supported by CONACYT Grant PCEXCNA-040651.

is a Hilbert-Schmidt operator, i.e., if $\{\phi_j\}_{j\geqslant 1}$ is a complete orthonormal system (CONS) in Φ_m the following condition holds:

$$\sum_{j=1}^{\infty} \|\phi_j\|_n^2 < \infty.$$

Let Φ' be the topological dual space of Φ with the strong topology. Then

$$\Phi' = \cup_{n=1}^{\infty} \Phi_n'.$$

An important property of CHNS's is the "validity of the Bochner property" expressed in the following theorem. It is an extension of the classical Bochner's theorem for finite dimensional spaces.

THEOREM (Bochner-Minlos). *Let* $(\Phi, \|\cdot\|_n, n \geqslant 0)$ *be a countably Hilbertian nuclear space and* $\hat{\mu}: \Phi \longrightarrow \mathbb{C}$ *be a function satisfying the following properties*:

i) $\hat{\mu}(O) = 1$
ii) $\hat{\mu}$ *is positive definite on* Φ i.e., $\forall n \geqslant 1$ *and* $a_1, \cdots, a_n \in \mathbb{C}$, $\phi_1, \cdots, \phi_n \in \Phi$

$$\sum_{i,j} a_i \bar{a}_j \hat{\mu}(\phi_i - \phi_j) \geqslant 0.$$

Then $\hat{\mu}$ is the characteristic functional of a unique probability measure μ on $(\Phi', \mathscr{B}(\Phi'))$, i.e.,

$$\hat{\mu}(\phi) = \int_{\Phi'} e^{iz[\phi]}\, d\mu(z) \ \forall \phi \in \Phi, \tag{1.3}$$

if and only if $\hat{\mu}$ is continuous at zero in the topology of Φ. Moreover if $\hat{\mu}$ is Φ_p-continuous for some $p > 0$ there exists $q > p$ such that

$$\mu(\Phi_q') = 1. \tag{1.4}$$

The proof of this theorem was given in Minlos (1963). The reader is also referred to the book by Hida (1980).

There are special cases of CHNS's that occur frequently in practical problems as in Neurophysiology (Kallianpur and Wolpert (1984)), Chemistry (Kotelenez (1986)), Physics (Daletski (1967) and Miyahara (1981)) and infinite particle systems (Bojdecki and Gorostiza (1986)). These spaces, of which the space of rapidly decreasing functions is an example, are constructed in the following manner (see Kallianpur (1986) for details): Let $A = -L$ be the selfadjoint infinitesimal generator of a strongly continuous contraction semigroup $(T_t)_{t\geqslant 0}$ on a separable Hilbert space H. Assume that for some $r_1 > 0$ the r_1-th power of the resolvent $R(\lambda; A)^{r_1}$ is a Hilbert-Schmidt operator on H. Then there is a complete orthonormal set $\{\psi_j\}_{j\geqslant 1}$ in H such that

$$L\psi_j = \lambda_j \psi_j, \qquad j \geqslant 1,$$

where $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ and $\sum_{j=1}^{\infty} (1 + \lambda_j)^{-2r_1} < \infty$. Define

$$\Phi = \{\phi \in H: \sum_{j=1}^{\infty} (1 + \lambda_j)^{2r} \langle \phi, \psi_j \rangle_H^2 < \infty \ \forall r \in \mathbb{R}\} \tag{1.5}$$

and for $r \in \mathbb{R}$ and $\phi, \psi \in \Phi$,

$$(1.6) \qquad \langle \phi, \psi \rangle_r = \textstyle\sum_{j=1}^{\infty} (1 + \lambda_j)^{2r} \langle \phi, \psi_j \rangle_H \langle \psi, \psi_j \rangle_H,$$

$$(1.7) \qquad \| \phi \|_r^2 = \langle \phi, \phi \rangle_r.$$

Let Φ_r be the $\| \cdot \|_r$-completion of Φ. Then $\{\Phi, \| \cdot \|_r, r \geqslant 0\}$ is a countably Hilbertian nuclear space with the following important properties:

i) Let $\Phi_{-r} = \Phi_r'$, $r > 0$ ($\Phi_0 = H$). Then Φ_{-r} and Φ_r are in duality under the pairing

$$\xi[\phi] = \textstyle\sum_{j=1}^{\infty} \langle \xi, \psi_j \rangle_{-r} \langle \phi, \psi_j \rangle_r, \qquad \xi \in \Phi_{-r}, \phi \in \Phi_r.$$

ii) The injection of Φ_s into Φ_r is a Hilbert-Schmidt map if $s > r + r_1$.

iii) Finite linear combinations of $\{\psi_j\}_{j \geqslant 1}$ are dense in Φ and in every Φ_r; moreover $\{\psi_j\}_{j \geqslant 1}$ is an orthogonal system in each Φ_r and therefore

$$(1.8) \qquad \phi_j = (1 + \lambda_j)^{-r} \psi_j, \qquad j \geqslant 1$$

is a complete orthonormal system for Φ_r, $r \in \mathbb{R}$.

A CHNS as in (1.5) will be called *special.* The Schwartz space $\mathscr{S}$ may be constructed as above using the Laplacian operator (see Kallianpur (1986) for details).

From now on we will only consider CHNS of special type, being the property (1.8) useful in the proof of the Regularization Theorem.

2. The Regularization Theorem

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space and $(\Phi, \| \cdot \|_p, p \geqslant 0)$ be a special countably Hilbertian nuclear space. Let $L_0(\Omega)$ be the linear space of all real valued random variables on Ω with the metric of convergence in probability given by

$$(2.1) \qquad d(X, Y) = E(1 \wedge |X - Y|)$$

THEOREM. (*Regularization of continuous linear random functionals*). *Let $Y(\cdot)$: $\Phi \rightarrow L_0(\Omega)$ be a continuous linear random functional, that is, if $\phi_n \rightarrow \phi$ in Φ then $Y(\phi_n) \rightarrow Y(\phi)$ in $L_0(\Omega)$. Then there exists a unique Φ'-valued random variable $\tilde{Y}$ such that*

$$(2.2) \qquad \tilde{Y}[\phi] = Y(\phi) \quad \text{a.s.} \quad \forall \phi \in \Phi.$$

$\hat{Y}$ is unique in the sense that if X is another Φ'-valued version then $\hat{Y} = X$ a.s.

Proof. For $\phi \in \Phi$ define

$$(2.3) \qquad \hat{\mu}(\phi) = E(e^{iY(\phi)}).$$

Then clearly $\hat{\mu}(\phi)$ satisfies the conditions:

i) $\hat{\mu}(0) = 1$
ii) $\hat{\mu}$ is positive definite.

For $\phi \in \Phi$ define

$$V(\phi) = E(1 \wedge |Y(\phi)|). \tag{2.4}$$

Then since $|e^{iz} - 1| \leq |z| \wedge 2 \leq 2(1 \wedge |z|)$,

$$|\hat{\mu}(\phi) - 1| \leq 2V(\phi), \qquad \phi \in \Phi. \tag{2.5}$$

Hence since $V(\phi)$ is continuous on Φ, $\hat{\mu}(\phi)$ is continuous at zero on Φ and therefore by Bochner-Minlos theorem $\hat{\mu}$ is the characteristic functional of a unique probability measure μ on $(\Phi', \mathscr{B}(\Phi'))$ i.e.

$$\mu(\Phi') = 1. \tag{2.6}$$

Thus using (2.3) and (1.3)

$$\hat{\mu}(\phi) = E(e^{iY(\phi)}) = \int_{\Phi'} e^{iZ[\phi]}\, d\mu(Z) \quad \forall \phi \in \Phi. \tag{2.7}$$

For $q > 0$ let $\{\phi_j\}_{j\geq 1}$ be as in (1.8). Then for all $n \geq 1$ and $x_1, \cdots, x_n \in \mathbb{R}$ from (2.7) we have

$$E(\exp(i \textstyle\sum_{j=1}^n x_j Y(\phi_j))) = \displaystyle\int_{\Phi'} e^{i\sum_{j=1}^n x_j Z[\phi_j]}\, d\mu(Z). \tag{2.8}$$

Next, integrating both sides of (2.8) w.r.t. to the measure

$$\prod_{j=1}^n \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-x_j^2/2\sigma^2}\, dx_1 \cdots dx_n \quad (\sigma^2 > 0)$$

and applying Fubini's theorem we have that

$$E\left(\exp\left(-\frac{\sigma^2}{2} \sum_{j=1}^n Y(\phi_j)^2\right)\right) = \int_{\Phi'} e^{-(\sigma^2/2)\sum_{j=1}^n Z[\phi_j]^2}\, d\mu(Z).$$

Hence, applying the dominated convergence theorem when $n \to \infty$

$$E\left(\exp\left(-\frac{\sigma^2}{2} \sum_{j=1}^\infty Y(\phi_j)^2\right)\right) = \int_{\Phi'} e^{-(\sigma^2/2)\sum_{j=1}^\infty Z[\phi_j]^2}\, d\mu(Z).$$

Next for each $Z \in \Phi_q'$

$$\sum_{j=1}^\infty Z[\phi_j]^2 < \infty.$$

Then applying again the dominated convergence theorem when $\sigma \to 0$

$$P(\sum_{j=1}^\infty Y(\phi_j)^2 < \infty) = \mu(\sum_{j=1}^\infty Z[\phi_j]^2 < \infty) = \mu(\Phi_q').$$

Hence if $\Omega_q = \{\omega: \sum_{j=1}^\infty (Y(\phi_j)(\omega))^2 < \infty\}$, $P(\Omega_q) = \mu(\Phi_q')$.
Define

$$\tilde{Y}_q(\omega) = \begin{cases} \sum_{j=1}^\infty Y(\phi_j)(\omega)\hat{\phi}_j & \omega \in \Omega_q \\ 0 & \omega \notin \Omega_q \end{cases} \tag{2.9}$$

where $\{\hat{\phi}_j\}_{j\geq 1}$ is the CONS of Φ_q' dual to $\{\phi_j\}_{j\geq 1}$. Then $\tilde{Y}_q(\omega) \in \Phi_q'$ $\forall \omega$ and

$$P(\tilde{Y}_q \in \Phi_q{}', \tilde{Y}_q[\phi_j] = Y(\phi_j) \forall_j) = \mu(\Phi_q{}').$$

Next we shall show that $P(\tilde{Y}_q[\phi] = Y[\phi]) = \mu(\Phi_q{}') \forall \phi \in \Phi$. From (2.9) we have $\tilde{Y}_q(\omega)[\phi_j] = Y(\phi_j)(\omega)$ $\omega \in \Omega_q$. Then since

$$\textstyle\sum_{j=1}^{n} \langle \phi, \phi_j \rangle_q \phi_j \to \phi \quad \text{in} \quad \Phi \quad \text{as} \quad n \to \infty$$

and $Y(\cdot)$ is continuous from Φ to $L_0(\Omega)$,

$$\textstyle\sum_{j=1}^{n} \langle \phi, \phi_j \rangle_q Y(\phi_j) \to Y(\phi) \quad \text{in probability} \quad \text{as} \quad n \to \infty, \quad \phi \in \Phi.$$

But on the other hand

$$P(\textstyle\sum_{j=1}^{n} \langle \phi, \phi_j \rangle_q Y(\phi_j) = \sum_{j=1}^{n} \hat{\phi}_j[\phi] \tilde{Y}_q[\phi_j] \underset{n\to\infty}{\longrightarrow} \tilde{Y}_q[\phi]) = \mu(\Phi_q{}').$$

Then, taking an appropriate subsequence,

$$P(\tilde{Y}_q[\phi] = Y(\phi)) = \mu(\Phi_q{}') \quad \forall \phi \in \Phi. \tag{2.10}$$

Next define $\Omega^0 = \cup\, \Omega_q$. Then since $\Phi_q{}' \subset \Phi_p{}'$, $p > q$, we can take $\Omega_q \subset \Omega_p$ and therefore by (2.6) $P(\Omega^0) = 1$.

Define

$$\tilde{Y}(\omega) = \begin{cases} \tilde{Y}_q(\omega) \quad \text{for} \quad \omega \in \Omega_q, & \omega \in \Omega^0 \\ 0 & \omega \notin \Omega^0. \end{cases}$$

Observe that $\tilde{Y}$ is well defined since we are assuming CHNS's of special type with property (1.8).

Then $P(\tilde{Y} \in \Phi') = 1$. So it remains to show that $\tilde{Y}[\phi] = Y(\phi)$ a.s. $\forall \phi \in \Phi$. But this follows from (2.10) since for each $\phi \in \Phi$

$$P(\tilde{Y}[\phi] = Y(\phi)) = P(\cup_{q=1}^{\infty} \{\tilde{Y}_q[\phi] = Y(\phi)\}),$$

$$\lim_{q\to\infty} P(\tilde{Y}_q[\phi] = Y(\phi)) = \lim_{q\to\infty} \mu(\Phi_q{}') = 1.$$

Uniqueness. Suppose that X is another Φ'-valued version. To prove that

$$\tilde{Y}[\phi] = X[\phi] \quad \text{a.s.} \quad \forall \phi \in \Phi.$$

Let $\Omega_n = \{\omega \in \Omega: \tilde{Y}(\omega)[\phi_n] = X(\omega)[\phi_n]\}$, then $P(\Omega_n) = 1$ where $\{\phi_n\}_{n\geq 1} \subset \Phi$ is a dense set in Φ. Let $\Omega^* = \cap_{n=1}^{\infty} \Omega_n$ then $P(\Omega^*) = 1$ and if $\omega \in \Omega^*$ and $\phi \in \Phi$, $\exists \{\phi_{n_k}\}_{k\geq 1} \subset \{\phi_n\}_{n\geq 1}$ such that $\phi_{n_k} \to_{k\to\infty} \phi$, then $X(\omega)[\phi_{n_k}] = \tilde{Y}(\omega)[\phi_{n_k}] \to_{k\to\infty} \tilde{Y}(\omega)[\phi]$. Hence $P(\tilde{Y} = X) = 1$. Q.E.D.

Some easy but very useful consequences of the Regularization Theorem are the following two results.

COROLLARY 1. *Let $Y(\cdot): \Phi \to L_0(\Omega)$ be a continuous linear random functional which is Φ_p-continuous for some $p \geq 1$. Then there exists $q > p$ and a unique Φ'-valued random variable $\tilde{Y}$ such that*

$$\tilde{Y}[\phi] = Y(\phi) \quad \text{a.s.} \quad \forall \phi \in \Phi$$

and $P(\tilde{Y} \in \Phi_q{}') = 1$.

Proof. It follows by the second part of the Bochner-Minlos Theorem, (1.4) and (2.10).

COROLLARY 2. *Let $r > 0$ and assume that $Y(\cdot)\colon \Phi \to L_r(\Omega)$ is a continuous linear random functional. Then there exists a unique Φ'-valued random variable $\tilde{Y}$ such that $\tilde{Y}[\phi] = Y(\phi)$* a.s. *$\forall \phi \in \Phi$ and $E(\tilde{Y}[\phi])^r < \infty$ $\forall \phi \in \Phi$. If in addition Y is Φ_p-continuous for some $p > 0$ there exists $q > p$ such that $P(\tilde{Y} \in \Phi_q') = 1$.*

Remark. In a personal communication S. Ramaswamy has shown me another proof of the Regularization Theorem using the results of L. Schwartz on Radonifying maps. I think that the proof given in the present paper is more constructive and easier to understand in the case of CHNS's of special type.

Acknowledgements. The author would like to thank Professors L. Gorostiza, G. Kallianpur and S. Ramaswamy for valuable discussions and suggestions.

CENTRO DE INVESTIGACION EN MATEMATICAS
GUANAJUATO, GTO. 36000 MÉXICO

REFERENCES

[1] T. BOJDECKI AND L. G. GOROSTIZA, *Langevin equations for S'-valued Gaussian processes and fluctuation limits of infinite particle systems*, Probab. Th. Rel. Fields, **73** (1986), 227–244.

[2] YU. L. DALETSKI, *Infinite dimensional elliptic operators and parabolic equations connected with them*, Russian Math. Surveys, 22, **4** (1967), 1–53.

[3] T. HIDA, Brownian Motion, Springer Verlag, New York, 1980.

[4] K. ITO, Foundations of Stochastic Differential Equations in Infinite Dimensional Spaces, CBMS-NSF SIAM No. 47, Philadelphia, 1984.

[5] ——— AND M. NAWATA, *Regularization of linear random functionals*, Probability Theory and Mathematical Statistics 4th. USSR-Japan Symposium, Lect. Notes in Math, 1021, Springer-Verlag, Berlin, 1983.

[6] G. KALLIANPUR, Stochastic Differential Equations in duals of Nuclear Spaces with some Applications, IMA Preprint Series #244, Institute for Mathematics and its Applications, University of Minnesota, 1986.

[7] ——— AND R. WOLPERT, *Infinite dimensional stochastic differential equation models for spatially distributed neurons*, Appl. Math. Optim., **12** (1984), 125–172.

[8] A. N. KOLMOGOROV, *A note on the papers of R. A. Minlos and V. Sazonov*, Theory Probab. Appl., **4** (1959), 221–223.

[9] P. KOTELENEZ, *Gaussian approximation to the nonlinear reaction-diffusion equation*, Report No. 146, Universität Bremen, 1986.

[10] R. A. MINLOS, *Generalized random processes and their extension to measures*, Selected Transl. Math. Statist. and Probab., **3** (1963), Amer. Math. Soc. Providence, R.I., 291–314.

[11] Y. MIYAHARA, *Infinite dimensional Langevin equation and Fokker-Plank equation*, Nagoya Math. J., **81** (1981), 177–223.

[12] V. PÉREZ-ABREU, *Decomposition of semimartingales on $\mathcal{S}'$*, J. Funct. Anal. (1988). To appear.

[13] V. SAZONOV, *On characteristic functionals*, Theory Probab. Appl., **3** (1958), 188–192.