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ON DUAL ALGEBRAS AND ALGEBRAIC OPERATORS

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1. Introduction

Let K be a separable complex Hilbert space of dimension less than or equal to $\aleph_0$. Denoting by $C_1(K)$ the trace class of $L(K)$ with the trace norm, we know we can identify the dual of $C_1(K)$ with $L(K)$ by the bilinear functional

$$< T, S >= Tr(TS), \ \ T \in L(K), \ \ S \in C_1(K).$$

A dual algebra is a subalgebra of $L(K)$ that contains the identity and is closed in the weak* topology on $L(K)$.

If ${}^{\perp}A$ is the preannihilator of A in $C_1(K)$, then A is the dual space of $Q = C_1(K)/{}^{\perp}A$ (Bercovici et al [1] Prop 1.19). The duality is given by

$$< S, [\mathrm{L}] >= Tr(S\mathrm{L}), \ \ S \in A, [\mathrm{L}] \in Q,$$

where [L] is the class Q of an element $\mathrm{L} \in C_1(K)$.

If A is a dual algebra, $M_n(A)$ is the subalgebra of $L(K^n)$ whose elements are $n \times n$ matrices with entries in A. By Bercovici et al ([1].2.2), $M_n(A)$ is a dual algebra and its predual $Q_{M_n}(A)$ can be identified with the space $M_n(Q_A)$ of matrices with entries in $Q_{A'}$ where Q_A denotes the quotient $C_1(K)/{}^{\perp}A$.

We denote by A_T the dual algebra generated by $T \in L(K)$.

If x, y are vectors in K, and we denote by $x \otimes y$ the rank-one operator defined by

$$(x \otimes y)(u) =< u, y > x, \ \ u \in K.$$

For each $\mathrm{A} \in A$, we have

$$< A, [x \otimes y] >= Tr(A(x \otimes y)) =< Ax, y > .$$

Scott Brown [2] showed that for some subnormal operator T, the predual Q_T of A_T consists enterely of elements of the form $[\mathrm{L}] = [x \otimes y]$. Many results on invariant subspaces were originated by this idea.

We say that an operator T is algebraic if $P(T) = 0$ for some polynomial P.

Definition 1. ([1] 2.01) Let $A \subset L(K)$ be a dual algebra and let n be a cardinal number such that $1 \le n \le \aleph_0$. Then A is to be said to have property $(\mathbf{A}_n)$ provided every $n \times n$ system of simultaneous equations of the form

$$[x_i \otimes y_j] = [\mathrm{L}_{ij}], \ \ 0 \le i, j < n$$

(where the $[\mathrm{L}_{ij}]$ are arbitrary fixed elements from Q_A) has solution $\{x_i\}_{0 \le i < n}$, $\{y_1\}_{0 \le i < n}$ consisting of a pair of sequences of vectors from K.

The main objective of this paper is to characterize the algebraic operators which have property $(\mathbf{A}_n)$, but not property $(\mathbf{A}_{n+1})$.

2. Main Theorem

THEOREM 1. *Let $T \in L(K)$ be an algebraic operator whose minimal polynomial is $P(\lambda) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_S)^{m_S}$, let $K_i = \ker(T - \lambda_i)^{m_i}$ and $r_i = \dim(K_i \ominus (T - \lambda_i)^{m_i - 1}(K_i)$. If $n = \text{Min}\{r_1, \ldots, r_S\} < \infty$, then A_T has property $(\mathbf{A}_n)$ and not property $(\mathbf{A}_{n+1})$. In particular this is true for every operator on a finite dimensional space.*

We will give some preliminary results before proving the theorem.

Definition 2. Let A be a subalgebra of $L(K)$ and let $\{x_1, \ldots, x_n\}$ be a subset of K. Then $\{x_1, \ldots, x_n\}$ will be said to be A-independent if

$$\sum_{i=1}^{n} T_i x_i = 0, \;\; T_i \in A$$

implies $T_i = 0 \;\; \forall i = 1, \ldots, n$.

If $x \in K$ and A is an algebra, write $Ax = \{TX : T \in A\}$.

LEMMA 1. *Suppose that $T \in L(K)$ is nilpotent of order m, and $r = \dim(\ker T^{m-1})^{\perp} < \infty$, then there is no A_T-independent subset of K with $r + 1$ vectors.*

Proof. Let $\{x_1, \ldots, x_{r+1}\} \subset K$, we can write $K = \text{Ker } T^{m-1} \oplus (\text{Ker } T^{m-1})^{\perp}$, so $x_i = y_i + z_i$ with $y_i \in \text{Ker } T^{m-1}$ and $z_i \in (\text{Ker } T^{m-1})^{\perp}$, for $i = 1, \ldots, r+1$.

$\{z_1, \ldots, z_{r+1}\}$ is linearly dependent, so there exist scalars $\alpha_1, \ldots, \alpha_{r+1}$, not all equal to zero, such that

$$\alpha_1 z_1 + \cdots + \alpha_{r+1} z_{r+1} = 0$$

so

$$\alpha_1 T^{m-1} x_1 + \cdots + \alpha_{r+1} T^{m-1} x_{r+1} = 0.$$

Thus $\{x_1, \ldots, x_{r+1}\}$ is A_T-dependent.

LEMMA 2. *Let A be a dual algebra of $L(K)$ and let $\{x_1, \ldots, x_n\}$ be an A-independent subset of K. Suppose $B = Ax_1 + \cdots + Ax_n$ is closed in K. Then A has property $(\mathbf{A}_n)$.*

Proof. Let $\overline{x} = (x_1, \ldots, x_n) \in K^n$, then

$$M_n(A)\overline{x} = \{(y_1, \ldots, y_n) : y_i = \sum_{j=1}^{n} T_{ij} x_j, \;\; T_{ij} \in A\} = B \oplus B \oplus \cdots \oplus B$$

is closed in K^n because B is closed in K. Let $\tilde{T} = (T_{ij}) \in M_n(A)$ and suppose $\tilde{T}(\overline{x}) = 0$, then $\sum_{j=1}^{n} T_{ij}x_j = 0 \quad \forall i = 1, \ldots, n$.

Since $\{x_1, \ldots, x_n\}$ is A-independent, we have $T_{ij} = 0$, for $i, j = 1, 2, \ldots, n$; thus, $\tilde{T} = 0$. By Bercovici et al ([3], 2.06), $M_n(A)$ has property $(\mathbf{A}_1)$ and by ([3], 2.3) A has property $(\mathbf{A}_n)$.

COROLLARY 1. *Let $T \in L(K)$ be such that $T^m = 0$ and $T^{m-1} \neq 0$. If $n =$ dim $(\text{Ker}T^{m-1})^{\perp}$ then there exists an A_T-independent set of n vectors. Hence A_T has property $(\mathbf{A}_n)$.*

Proof. Since T is nilpotent of order m, then A_T is the linear span of $\{I, T, T^2, \ldots, T^{m-1}\}$. Let $\{x_1, \ldots, x_n\}$ be a basis for $K \ominus \text{Ker } T^{m-1}$. It is well known that the set

$$\{x_1, \ldots, x_n, Tx_1, \ldots, Tx_n, \ldots T^{m-1}x_1, \ldots, T^{m-1}x_n\}$$

is linearly independent. Therefore $\{x_1, \ldots, x_n\}$ is A_T-independent. Since A_T is finite dimensional, we trivialy have that $A_T x_1 + \cdots + A_T x_n$ is closed in K. Then by lemma 2, A_T has property $(\mathbf{A}_n)$.

LEMMA 3. *let K_1 and K_2 be Hilbert spaces and $T = S \oplus 0 \in L(K_1 \oplus K_2)$. A_T has property $(\mathbf{A}_n)$ if and only if A_S has the same property.*

Proof. Observe that if $x_i = r_i \oplus s_i$, $y_j = w_j \oplus z_j$, where $r_i, w_j \in K_1$ and $s_i, z_j \in K_2$, then $< Tx_i, y_j >=< Sr_i, w_j >$. Now the result follows by some straight forward computation.

LEMMA 4. *Let $S \in L(K)$ be a nilpotent operator of order 2. If* dim $(K \ominus S(K)) = r$ *then A_S does not have property $(\mathbf{A}_{r+1})$.*

Proof. Since $S^2 = 0$, $A_S = \{\alpha I + \beta S : \alpha, \beta \in \mathbf{C}\}$; then Q_S has dimension 2, and is generated by the functionals

$$\varphi_1(\alpha I + \beta S) = \alpha,$$
$$\varphi_2(\alpha I + \beta S) = \beta.$$

Notice that $[x \otimes y] = \varphi_1$ if and only if

$$< x, y >=< Ix, y >= \varphi_1(I) = 1,$$
$$< Sx, y >= \varphi_1(S) = 0;$$
$$\text{and } [x \otimes y] = \varphi_2 \text{ if and only if}$$
$$< x, y >=< Ix, y >= \varphi_2(I) = 0,$$
$$< Sx, y >= \varphi_2(S) = 1.$$

Hence

$$[x \otimes y] = \varphi_2 \text{ implies } [Sx \otimes y] = \varphi_1,$$
$$[x \otimes y] = \varphi_1 \text{ implies } [Sx \otimes y] = 0.$$

Consider the $(r+1)\times(r+1)$ system of equations

$$\begin{array}{ll} [x_i\otimes y_i]=\varphi_1, & i=1,\dots,r+1, \\ [x_i\otimes y_{i+1(\operatorname{mod} r+1)}]=\varphi_2 & i=1,\dots,r+1, \\ [x_i\otimes y_j]=0, & \text{for the remaining } j\text{'s}. \end{array}$$

Since all the columns of the system are different, all the x_i's are distinct and the same can be said about the y_j's.
By lemma 1, $\{x_1,\dots,x_{r+1}\}$ is A_S-dependent, so there exists $\{S_i=\alpha_i I+\beta_i S\}_{1\leq i\leq r+1}$ not all of them zero such that

$$z=S_1x_1+\cdots+S_{r+1}x_{r+1}=0,$$

where in particular

$$[z\otimes y_j]=0,\ \ j+1,\dots,r+1;$$

but

$$[z\otimes y_j]=\alpha_{j-1}\varphi_2+\beta_{j-1}\varphi_1+\alpha_j\varphi_1,$$

where $j-1$ must be understood as $j-1(\operatorname{mod} r+1)$.

Thus

$$\alpha_{j-1}=0,$$
$$\beta_{j-1}+\alpha_j=0.$$

This contradicts the fact that some S_j is not zero. Therefore the system cannot be solved and A_S does not have property $(\mathbf{A}_{r+1})$.

Proof of the Theorem 1. T is similar to $T_1\oplus\cdots\oplus T_S$ operating on $K_1\oplus\cdots\oplus K_S$ where each T_i has a unique eigenvalue λ_i and each $(T_i-\lambda_i)$ is nilpotent of order m_i. By corollary 1 and its proof, a basis $\{x_{i1},\dots,x_{ir_i}\}$ of $K_i\ominus\operatorname{Ker}(T_i-\lambda_i)^{m_i-1}$ is a system of r_i $A_{(T_i-\lambda_i)}$-independent vectors and hence A_T-independent. Let $B_i=\{x_{i1},\dots,x_{in}\}$ $(i=1,\dots,s)$ be a subset of n vectors from such a system.

Let $B=\{y_1,\dots,y_n\}$ where $y_j=(x_{ij},\dots,x_{sj})$ $(j=1,\dots,n)$. We shall see that B is A_T-independent.

Assume that

$$\sum_{j=1}^{n}P_j(T)y_j=0;$$

then for $i=1,\dots,s$, we have

$$\sum_{j=1}^{n}P_j(T_i)x_{ij}=0.$$

By our choice of the x_{ij}'s it follows that

$$P_j(t_i) = 0 \text{ for } j = 1, \ldots, n \; i = 1, \ldots, s.$$

From lemma 2, it follows that A_T has property $(\mathbf{A}_n)$.

Now we prove that A_T does not have property $(\mathbf{A}_{n+1})$.
Without loss of generality, suppose $n = r_1$, and let $Q(\lambda) = (\lambda - \lambda_1)^{m_1 - 1} (\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_s)^{m_s}$;

$$Q(T) = \begin{pmatrix} Q(T_1) & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$Q(T_1) = (T_1 - \lambda_1)^{m_1 - 1} \cdots (T_1 - \lambda_S)^{m_S} = (T_1 - \lambda_1)^{m_1 - 1} W,$$

W being an invertible operator which commutes with T_1. $Q(T_1)$ is nilpotent of order 2, and $\dim(K_1 \ominus Q(T_1)(K_1)) = n$. From Lemmas 3 and 4 it follows that $A_{Q(T)}$ does not have property $(\mathbf{A}_{n+1})$. Since $A_{Q(T)}$ is a subalgebra of A_T, ([1] 2.04) indicates that A_T does not have that property either.

Remark. In Barria et al [4] it is proved that if $T \in T(H)$ is algebraic, the conditions of Theorem 1 are satisfied if and only if every system of equations $[\mathrm{L}_{ij}] = [x_i \otimes y_j]$ $1 \leq i, \; j \leq n$ can be solved, where each L_{ij} has finite rank (Property $\mathbf{B}_{n,n}$).
Since A_T has finite dimension the topologies weak* and WOT coincide, then, from ([3] 1.7), properties $(\mathbf{B}_{n,n})$ and $(\mathbf{A}_n)$ are equivalent

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