Boletín de la Sociedad Matemática Mexicana Vol. 32 No. 2, 1987

AN EQUATION OF CONTACT VECTOR FIELDS AND COMMUTATORS OF CONTACT DIFFEOMORPHISMS

By JAVIER PULIDO CEJUDO

Introduction

The problem of expressing any diffeomorphism, contact isotopic to the identity and compactly supported, as a product of commutators, is linearized; this yields an equation of contact vector fields. This equation is translated into a functional equation on the Heisenberg group and solved; the functional spaces are adequate for the Heisenberg group structure (see [F,S]). This linearized problem in the non-contact case yielded [M1] and [M2]. The same procedure holds for any odd dimensional Euclidean space but is not included here.

A contact form in $\mathbf{R}^3$ is a 1-form w that satisfies: $w \wedge dw$ is a never vanishing form. For a C^r-diffeomorphism $f : \mathbf{R}^3 \to \mathbf{R}^3$, we have the differential $Df : \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^3 \times \mathbf{R}^3$ that makes the following diagram commutative

$$\begin{array}{cccc} T\mathbf{R}^3 = & (\mathbf{R}^3 \times \mathbf{R}^3) & \xrightarrow{Df} & \mathbf{R}^3 \times \mathbf{R}^3 \\ & \Pi \downarrow & & \downarrow \Pi \\ & \mathbf{R}^3 & \xrightarrow{f} & \mathbf{R}^3 \end{array}$$

where Π is the projection on the first factor.

Given a vector field X we associate to it the vector field $f_*X = Df \circ X \circ f^{-1}$; if μ is a 1-form, we denote by

$$< \mu, X >_p = \mu_p(X_p)$$

and define the pull-back $f^*\mu$ to be the 1-form

$$< f^*\mu, X >_p = < \mu, f_*X >_{f(p)}$$

In this setting we say that a diffeomorphism f is contact if there exists a C^r map $\lambda : \mathbf{R}^3 \to \mathbf{R}^+$ such that $f^*w = \lambda w$, where $\mathbf{R}^+$ is the set of positive reals.

The support of a diffeomorphism f is defined as the closure of the set where $f(x) \neq x$, i.e.

$$\text{supp } f = \overline{\{x \in \mathbf{R}^3 | f(x) \neq x\}}.$$

Consider the contact form

$$w = dz + xdy - ydx,$$

let G denote the group of contact diffeomorphisms with compact support, for which there exists $F : [0,1] \times \mathbf{R}^3 \to \mathbf{R}^3$ and $\lambda : [0,1] \times \mathbf{R}^3 \to \mathbf{R}^+$, both in C^r and satisfying

$$F_t^* w = \lambda_t w,$$

where $F_t = F(t,\cdot)$, $\lambda_t = \lambda(t,\cdot)$, supp F_t is compact for all t, $F_0 = \text{id}$ (identity map) and $F_1 = f$. One has the following problem:

Find suitable $f_i \in G$, where $i = 1, \dots, n$ such that for all $g \in G$ there exists $u_i \in G$ with

$$[u_1, f_1] \cdots [u_n, f_n] = g \tag{1}$$

where $[u, f] = ufu^{-1}f^{-1}$.

For g =id we can choose u_i =id. What happens when we perturb g slightly?

Let u_{it} be isotopic to the identify on G, then, by direct computation we get

$$\left.\frac{\partial [u_{i_t}, f_1] \dots [u_{n_t}, f_n]}{\partial t}\right|_{t=0} = -(\sum_{i=1}^{n} f_{i_*}\dot{u}_{i0} - \dot{u}_{i0}) \tag{2}$$

where

$$\dot{u}_{i_s}(p) = \frac{\partial u_{i_t}}{\partial t}(u_{i_s}^{-1}(p)) \qquad \text{for } p \in \mathbf{R}^3. \tag{3}$$

Let X be a vector field and μ a 1-form, let $L_X\mu$ be the Lie derivative of μ with respect to X; we say that X is a contact vector field if there exists a C^r map $p : \mathbf{R}^3 \to \mathbf{R}$ such that

$$L_X w = pw.$$

It can be proved that $\dot{u}_{i_s}$ is a contact vector field if and only if u_{i_s} is a contact isotopy, hence we have the following problem:

Find suitable diffeomorphisms $f_1, \dots, f_n$ such that for any contact vector field Y there exist contact vector fields $X_1, \dots, X_n$ with

$$Y = \sum f_* X_i - X_i$$

In this paper we give a positive answer to this question precisely stated.

I wish to express my gratitude to John Mather and Roberto Moriyón, whose help was very important for the preparation of this paper.

§1. Preliminary

Given a vector field X, we define on the algebra of differential forms $\Lambda(\mathbf{R}^3) = \oplus\Lambda^p(\mathbf{R}^3)$ the Lie derivative L_X and the interior product $i(X)$, in the following way:

Let L_X be the only degree 0 derivation that satisfies

$$L_X\lambda = d\lambda(X)$$
$$L_X d\lambda = dL_X(\lambda)$$

for every 0-form $\lambda : \mathbf{R}^3 \to \mathbf{R}$.

The interior product $i(X)$ is the degree -1 antiderivation that, for a p-form α and $X_1, \ldots, X_{p-1}$ vector fields, associates

$$i(X)\alpha(X_1, \ldots, X_{p-1}) = \alpha(X, X_1, \ldots, X_{p-1}).$$

L_X and $i(X)$ are related by the Cartan formulae

$$L_X = di(X) + i(X)d$$
$$i[X, Y] = L_X i(Y) - i(Y)L_X$$

where Y is another vector field and $[\ ,\ ]$ the Lie bracket.

Note that for the vector field $E = \frac{\partial}{\partial z}$ we have $i(E)w = 1$ and $i(E)dw = 0$. For a $2n + 1$ manifold equipped with a contact form w, a vector field E with the properties above always exists and is called the characteristic vector field (or Reeb's vector field, see [1]).

Any vector field X can be written uniquely as

$$X = [i(X)w]E + H(X)$$

where $i(H(X))w = 0$; $[i(X)w]E$ is called the vertical part of X and $H(X)$ the horizontal part. $i(X)w$ is called the vertical component, and we denote it by $v(X)$. Note that $v(X) = < w, X >$.

Let σ denote the set of vector fields such that $v(X) = 0$ and $\mathcal{S}$ the set of 1-forms μ for which $i(E)\mu = 0$. We define $\alpha : \sigma \to \mathcal{S}$ by $X \to i(X)dw$.

Consider the subbundle F of $T\mathbf{R}^3$ whose fiber over p in $\mathbf{R}^3$ is $F_p = \{Xp \in p \times \mathbf{R}^3 | w_p(X_p) = 0\}$ and the subbundle $\mathcal{S}$ of $\Lambda^1(\mathbf{R}^3)$ whose fiber over p is $S_p = \{\mu_p \in p\Lambda^1_p(\mathbf{R}^3) | \mu_p(Ep) = 0\}$; then σ and $\mathcal{S}$ are the set of sections of F and S respectively and α defines a bundle map which is actually a bundle isomorphism. This follows from the fact that $i(X)w = 0$ and $i(X)dw = 0$ imply $i(X)(w\Lambda dw) = 0$, and Fp and S_p having the same dimension.

σ is called the set of horizontal vector fields and $\mathcal{S}$ the set of semi basic forms, so α gives a continuous one to one correspondence between these two sets.

In $[B, P]$ we prove that the vertical component of a contact vector field completly determines the field. Though it is proved there for C^∞ vector fields, the proof is also valid fot the C^r case.

Given a C^r map g from $\mathbf{R}^3$ into $\mathbf{R}$ we associate to it the C^{r-1} horizontal vector field

$$\lambda(g) = \alpha^{-1}((i(E)dg)w - dg)$$

and the C^{r-1} contact vector field

$$\beta(g) = gE + \lambda(g).$$

This follows from the fact that

$$(i(E)dg)w - dg$$

is semibasic.

Clearly $v(\beta(g)) = g$, and from the fact that the vertical component determines the contact vector field we also have

$$\beta(v(X)) = X.$$

By support of g we mean

$$\text{supp } g = \overline{\{p \in \mathbf{R}^3 | g(p) \neq 0\}},$$

and by support of X we mean

$$\text{supp } X = \overline{\{p \in \mathbf{R}^3 | X(p) \neq 0\}}.$$

β establishes a correspondence between contact vector fields with compact support and maps from $\mathbf{R}^3$ into $\mathbf{R}$ with compact support. For g in the image of $C^r(\mathbf{R}^3, \mathbf{R})$ by v, the degree of differentiability is preserved.

If $h : \mathbf{R}^3 \times I \to \mathbf{R}$ is an isotopy of the identity, we consider the one parameter family of vector fields on $\mathbf{R}^3$

$$\dot{h}_s(p) = \frac{\partial h_t}{\partial t}(h_s^{-1}(p))$$

and inversely, for a family of vector fields U_s, we consider the flow $\emptyset$ on $\mathbf{R}^3 \times I$ and define the isotopy

$$h_s(p) = \Pi_{\mathbf{R}^3} \emptyset(p, 0, s),$$

where $\Pi_{\mathbf{R}^3}$ is the projection on $\mathbf{R}^3$. Then, $\dot{h}_s(p) = U_s(p)$ and $h_s(p)$ is a contact diffeomorphism if and only if $U_s(p)$ is also a contact vector field.

§2. Definitions and Notation

In $\mathbf{R}^3$ we define the product

$$(x, y, z) \cdot (a, b, c) = (x + a, y + b, z + c + bx - ay),$$

$\bar{0} = (0,0,0)$ is the identify and $(x,y,z)^{-1} = (-x,-y,-z)$. $\mathbf{R}^3$ with this product becomes a nonabelian group called the Heisenberg group. We will denote it by H.

Fix p in H and consider the diffeomorphisms $T_p(q) = q \cdot p$ and ${}_pT(q) = p \cdot q$, T_p lies on G but ${}_pT$ does not lie on G.

Consider p_i^t with the i'th coordinate equal to t and the others equal to 0. Define $X_i = {}_{p_i^t}\dot{T}$ for $i = 1,2$; then $X_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$ and $X_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}$.

Let A be a real number, $A \neq 0$. Define on H the scalar product $A(x,y,z) = (Ax, Ay, A^2z)$. Also define the element in G as $A((x,y,z,)) = A(x,y,z,)$.

Let B_k denote the subset of the space of differential operators of the form $D^k = X_{i_1} \ldots X_{i_k}$, where $X_{i_j} = X_1$ or X_2 and $k = 1,2,\ldots$. Let O_k denote the subspace spanned by B_k. Moreover, define $G_0 = \{C^0$ maps from H into $\mathbf{R}$ with compact support$\}$ and $G_k = \{v \in G_0 | D^k v \in G_0$ for all D^k in $B_k\}$. On G_0 and G_k we have the norms:

$$\| v \|_0 = \sup_{p\in H} |v(p)| \text{ and}$$

$$\| v \|_k = \sup_{D^k \in B_k} \| D^k v \|_0 .$$

Note that the subspace of G_k whose elements have fixed support is a Banach space.

For f in G and v in G_k, we define $f_* v = \lambda v \circ f^{-1}$, where $f^* w = \lambda w$, in particular $T_{p*} v = v \circ T_p^{-1}$ and $A_* v = A^2 v \circ A^{-1}$.

Note that f^* preserves G_k and

$$(1) \qquad \| T_{p*} v \|_k = \| v \|_k, \quad \| A_* v \|_k = A^{2-k} \| v \|_k .$$

Since

$$(2) \quad X_i(v \circ T_p) = (X_i v) \circ Tp,\ X_i(v \circ A) = (X_i v) \circ A, \text{ for } i = 1,2,\ p \text{ in } H \text{ and } A > 0.$$

On H we define the "normlike" function

$$|\cdot|_H : H \to \mathbf{R}^+ \quad |(x,y,z)|_H = \sqrt[4]{x^4 + y^4 + z^2}.$$

This function satisfies:

(3) i) $|p|_H \geq 0$ and $|p|_H = 0$ iff $p = \bar{0}$,
ii) $|Sp|_H = |S||p|_H$ and
iii) there exists a constant C such that $|p+q|_H \leq C(|p|_H + |q|_H)$,
whenever p and q are in H and S is a real number.

Let us denote $B_\epsilon(\bar{0}) = B_\epsilon(0,0,0) = \{p \in H \| p|_H < \epsilon\}$ and $B_\epsilon(q) = T_q(B_\epsilon(\bar{0}))$.

Given $A > 0$ we define $E_{A,k} = \{v \in G_k | \int_H v = 0, \text{ supp } v \in B_A\bar{0}\}$ and for $\gamma_p^j(t) = p \cdot p_j^t$, $E^j_{A,k} = \{v \in G_k | \int v \circ \gamma_p^j(t) dt = 0 \text{ for all } p \text{ in } H\}$.

When k is fixed, we just write E_A and E^j_A, respectively.

We denote by G^k the set of compactly supported contact vector fields X for which the vertical component $v(X)$ lies in G_k. For X in G^k we define $\| X \|_k = \| v(X) \|_k$. Let C^k_c denote the C^k contact vector fields with compact support.

3. Main Result and Lemmas Needed for the Proof

In this section we will state the main result as

THEOREM (1). *Let $k > 28$, then there exist real numbers L, M such that for every Y in C^k_c, we can find Y_i in $C^{k/2}_c$ for $i = 1, \ldots, 5$ with*

$$Y = \sum_{j=1}^{3} (T_{p_j^1 *} Y_j - Y_j) + L_* Y_4 - Y_4 + M_* Y_5 - Y_5.$$

We will see that this theorem is a consequence of

PROPOSITION(1). *Let $k > 28$, then there exist real numbers L, M such that given v in G_k we can find u_i in G_k, $i = 1, \ldots, 5$ with*

$$v = \sum_{j=1}^{3} T_{p_j^1 *} u_j - u_j + L_* u_x - u_y + M_* u_5 - u_5.$$

Since $[X_1, X_2] = -2\frac{\partial}{\partial z}$, we have that G_k is contained in $C^{k/2}$. If we are given X in G^k, then $v(X)$ lies in G_k. Use Proposition (1) to find u_i in G_k such that

$$v(X) = \sum_{j=1}^{3} T_{p_j^l *} u_j - u_j + L_* u_4 - u_4 + M_* u_5 - u_5.$$

Define $Y_i = \beta(u_i)$ (see §1). Since $C^{k/2}$ lies in $G_{k/2}$ and the vertical part characterizes the contact vector field, then, from the aditivity of β and the definition $f_* u = v(f_*(\beta(u)))$, we get Theorem 1.

From the above discussion we see that our main task is to prove Proposition 1. In order to do so we will need the following lemmas whose proofs we omit here but can be found in full detail in [P].

LEMMA (3.1). (Interpolation lemma): *Let v be an element of G_k and $j \leq 1 \leq k$ non negative integers, then there exists a universal constant C (just depends on k) such that*

$$\| v \|_1 \leq C \| v \|_j^{\frac{k-1}{k-j}} \| v \|_k^{\frac{l-j}{k-j}}.$$

LEMMA (3.2). *There exists a constant C such that for all v, w in G_k*

$$\| vw \|_k < C(\| v \|_0 \| w \|_k + \| v \|_k \| w \|_0).$$

Note that if we consider

$$F_0 = \{v \in C^0(H) | \sup_{p \in H} |v(p)| < \infty\},$$

$$F_k = \{v \in C^0(H) | Xv \in F_0 \quad \text{for all } X \text{ in } O_k$$

$$\text{and } \overline{F}_k = F_k \cap \cdots \cap F_0,$$

then the same assertion follows for v in $\overline{F}_k$ and we get Lemma 3.1′ and Lemma 3.2′.

LEMMA (3.3). *If v is in E_A then we can find v_i in E^i_{4A} and a constant C such that*

i) $v = v_1 + v_2 + v_3$
ii) $\| v_i \|_k \leq C \| v \|_k$. □

LEMMA (3.4). *Let v in E^j_A and define ${}_j\Phi_v = \sum_{i \in \mathbf{Z}} v \circ T^i_{P_j}$ then ${}_j\Phi_v$ satisfies*

i) ${}_j\Phi_v = {}_j\Phi_v \circ T_{p_j}$
ii) $\| {}_j\Phi_v \| \leq CA \| v \|_k$
iii) $\int_0^1 ({}_j\Phi_v \circ \gamma^j_p)(t)dt = 0$ *for all p in H*
iv) $\| {}_j\Phi_v \|_0 \leq CA^k \| {}_j\Phi_v \|_{2k}$ and
$\| {}_j\Phi_v \|_0 \leq CA^{k+1} \| {}_j\Phi_v \|_2 \ k+1$. □

LEMMA (3.5). *The family $\{B_4(1, m, n) | (l, m, n,) \in \mathbf{Z}^3\}$ is a locally finite cover of H and if $\chi_{B_4(l,m,n)}$ denotes the characteristic function, then there exists a constant C such that $\Sigma\chi_{B_4(l,m,n)} < C$ the sum taken over all (l, m, n) in $\mathbf{Z}^3$.* □

LEMMA (3.6). *There is a partition of unity $\{\psi_h\}_{h=1}^{\infty}$ on H satisfying:*
i) For each h in $\mathbf{N}$ there exists $q_h = (l_h, m_h, n_h)$ in $\mathbf{Z}^3$ such that supp $\psi_n \subset B_4(q_h)$.
ii) Let $V_A = \{h \in \mathbf{N} | |q_h| < 4A\}$ then $B_A(\overline{0})$ is contained in the union, taken over all h in V_A, of $B_4(q_h)$.
iii) $\{\psi_h\}$ can be indexed in such a way that the intersection of $B_4(q_h)$ with $B_A(\overline{0})$ is empty whenever $h > CA^4$ for some constant C; moreover, if the former intersection is nonempty for some h, then it is so for all h' less than h.
iv) For each k there exists C_k such that for all h in $\mathbf{N}$ we have $\| \psi_h \|_k \leq C_k$. □

§4. Proof of Theorem 1.

We have pointed out in §3, that Theorem 1 follows from Proposition 1. Hence we will prove Proposition 1. This in turn will follow by the Observation and by modifying the proof of the Claim below.

Observation: Let v be an element in G_k with support contained in $B_A(\overline{0})$ and $\int_{\mathbf{R}^3} v \neq 0$, then there exist $\overline{\overline{v}}$ in G_k and $\overline{v}$ in E_A such that $v = A_*\overline{\overline{v}} - \overline{\overline{v}} + \overline{v}$.

Proof: Let $h_0 : \mathbf{R}^3 \to \mathbf{R}$ with supp h_0 contained in $[-1,1]^3$ and $\int_{\mathbf{R}^3} h_0 \neq 0$, then $\int_{\mathbf{R}^3} A^2 h_0 \circ A^{-1} = A^6 \int_{\mathbf{R}^3} h_0$.

Define

$$h = \frac{h_0}{(A^6 - 1)\int_{\mathbf{R}^3} h_0}$$

then $\int_{\mathbf{R}^3} (A^2 h \circ A^{-1} - h) = 1$. Let $a = \int_{\mathbf{R}^3} v$, $\overline{\overline{v}} = ah$ and $\overline{v} = v - A_*\overline{\overline{v}} - \overline{\overline{v}}$, then supp $\overline{v}$ is contained in $B_A(\overline{0})$ and $\int_{\mathbf{R}^3} \overline{v} = 0$.

Claim: Given v in E_A, there exist constants C, $\mathcal{C}$ and $u_1, \ldots, u_5, e$ in G_k such that

a) $\sum_{i=1}^3 T_{p_i *} u_i - u_i + (A/4)_* u_4 - u_4 + 2_* u_5 - u_5 = v - e$

b) $\| u_i \|_k \leq C \| v \|_k$, $i = 1, \ldots, 5$

c) $\| e \|_k \leq CA^{12-[k/2]} \| v \|_k$, moreover e lies in E_A.

Proof of Claim: From Lemma (3.3) we can write $v = v_1 + v_2 + v_3$ with v_j in E^j_{4A}. Let us fix a C^∞ function $b : R \to R$ satisfying supp b contained in $[-1,1]$ and $b(t) + b(t+1) = 1$ for all t in the interval $[-1,0]$. Define

$$b_1((x,y,z)) = b(x)$$

$$b_2((x,y,z)) = b(y)$$

$$b_3((x,y,z)) = b(z)$$

and construct $e_j = b_j \; {}_j\psi_{v_j}$. It follows from Lemmas 3.2, 3.3. and 3.4 that

(1) $\| e_j \|_k \leq CA^{2+[k/2]} \| v \|_k$

and supp e_j is contained in $B_{4A}(\overline{0})$, furthermore, if we define $\tilde{v}_j = v_j - e_j$, then ${}_j\psi_{\tilde{v}_j} = 0$; it is not difficult to see that this last condition implies the existence of $\tilde{u}_j$ in G_k such that $(T_{p_j})_*\tilde{u}_j - \tilde{u}_j = \tilde{v}_j$ and

(2) $\| \tilde{u}_j \|_k \leq CA \| \tilde{v}_j \|_k$,

therefore

(3) $\| \tilde{u}_j \|_k \leq CA^{3+[k/2]} \| v \|_k$.

Note that supp $\tilde{u}_j$ is contained in supp $\tilde{v}_j$, which in turn is contained in supp v_j.

So far we have expressed v_j in the desired form module an error e_j; we seek now to express e_j in the desired form module an error supported in $B_4(\overline{0})$. Let $\{\Phi_h\}_{h=1}^\infty$ be the partition of unity of Lemma (3.6). Define $e_{jh} = \Phi_h e_j = (\Phi_h b_j)_j \psi_{v_j}$, then $\| e_{jh} \|_k \leq CA^{2+|k/2|} \| v \|_k$.

Note $e_j = \sum_h e_{jh}$, $e_{jh} = 0$ for all $h > C(4A)^4$, supp e_{jh} is contained in $B_4(q_h)$ and $|q_h| > 4A$ implies $e_{jh} = 0$. We will now "translate" each e_{jh} until we end with $\tilde{e}_{jh}$ in G_k having support in $B_4(\bar{0})$, and $\tilde{e}_{jh}$ equal to e_{jh} module a sum of terms of the form $T_{p_j *} u - u$, $j = 1, 2, 3$ and u in G_k. In order to do this we first realize that for each $q_h = (l, m, n)$ in $\mathbf{Z}^3$ and $q_h \cdot p_1^{-l} \cdot p_2^{-m} \cdot p_3^{-n-lm} = \bar{0}$. To simplify the notation we suppose l, m, n are all positive; let $N = |l| + |m| + |n| + |l||m|$ and write

$$\begin{aligned} q_{h,0} &= q_h \\ q_{h,1} &= q_h \cdot p_1^{-1} \\ &\vdots \\ q_{h,l} &= q_h \cdot p_1^{-l} \\ &\vdots \\ q_{h,l} &= q_h \cdot p_1^{-l} \\ q_{h,l+1} &= q_h \cdot p_1^{-l} \cdot p_2^{-1} \\ &\vdots \\ q_{h,l+m} &= q_h \cdot p_1^{-l} \cdot p_2^{-m} \\ q_{h,l+m+1} &= q_h \cdot p_1^{-l} \cdot p_2^{-m} \cdot p_3^{-1} \\ &\vdots \\ q_{h,N} &= q_h \cdot p_1^{-l} \cdot p_2^{-m} \cdot p_3^{-n-lm} = \bar{0}, \end{aligned}$$

let $i = 1, 2, \ldots, N$, hence $\tilde{u}_{jhi} = T^{-1}_{(q_h^{-1} q_{h,i})*} e_{jh}$ has support in $B_4(q_{h,i})$ and from (1) in section 0 we know

(4) $\| \tilde{u}_{jhi} \|_k = \| e_{jh} \|_k$, i.e.,

(5) $\| \tilde{u}_{jhi} \|_k \leq CA^{2+[k/2]} \| v \|_k$.

Observe that

$$e_{jh} = \sum_{i=1}^{l} T_{p_1 *} \tilde{u}_{jhi} - \tilde{u}_{jhi} + \sum_{i=l+1}^{l+m} T_{p_2 *} \tilde{u}_{jhi} - \tilde{u}_{jhi} + \sum_{i=l+m}^{N} T_{p_3 *} \tilde{u}_{jhN} - \tilde{u}_{jhN} + \tilde{e}_{jh},$$

with $\tilde{e}_{jh} = \tilde{u}_{jhN}$. Note that in order to solve the problem when at least one of the coordinates of (l, m, n) is negative, one considers the equality $T_p^{-1} T_p u - T_p u = T_p(-u) - (-u)$. Note also that supp $\tilde{e}_{jh}$ is contained in $B_4(\bar{0})$ and

(6) $\| \tilde{e}_{jh} \|_k \leq CA^{2+[k/2]} \| v \|_k$.

To proceed with the proof of the claim, let P be the natural with the property $B_4(q_p) \cap B_{4A}(\overline{0}) \neq \psi$ and $B_4(q_{p+1}) \cap B_4(\overline{0}) = \psi$, we know that P is less or equal to CA^4, we also know that $|q_h| < CA$ for all $h = 1, \ldots, P$, therefore $l_h < CA$, $m_h < CA$ and $n_h < CA^2$ and so $N < CA^2$. Let us define

$$u_1 = \tilde{u}_1 + \sum_{j=1}^{3} \sum_{h=1}^{P} \sum_{i=1}^{l_h} \tilde{u}_{jhi}$$

$$u_2 = \tilde{u}_2 + \sum_{j=1}^{3} \sum_{h=1}^{P} \sum_{i=l_h+1}^{l_h+m_h} \tilde{u}_{jhi}$$

$$u_3 = \tilde{u}_3 + \sum_{j=1}^{3} \sum_{h=1}^{P} \sum_{i=l_h+m_h+1}^{N} \tilde{u}_{jhi}$$

and

$$\tilde{e} = \sum_{j=1}^{3} \sum_{h=1}^{P} \tilde{e}_{jh}$$

therefore we have

(7) $\| u_j \|_k \leq CA^{8+[k/2]} \| v \|_k$

with support of u_j contained in $B_{CA}(\overline{0})$ for $j = 1, 2, 3$, and

(8) $\| \tilde{e} \|_k \leq CA^{8+[k/2]} \| v \|_k$

with supp $\tilde{e}$ contained in $B_4(\overline{0})$, moreover

$$\sum_{j=1}^{3} T_{p_j*} u_j - u_j = v - \tilde{e}.$$

If we define $\overline{e} = (A/4)_*(-\tilde{e})$ we get

(9) $\| \overline{e} \|_k < CA^{11-|k/2|} \| v \|_k$

with supp $\overline{e}$ contained in $B_A(\overline{0})$ and

$$\sum_{j=1}^{3} T_{p_j*} u_j - u_j + (A/4)_*(-e) - (-e) = v - \overline{e}.$$

Note the constant in (9) is 4^{k-2} times the constant in (8).

In order to end up with the proof of the claim, we want to write $\overline{e}$ as a sum of an element in E_A plus terms of the desired form. For this purpose, fix f_1

in G_k with $\int f_1 = 1$ and supp f_1 contained in $B_1(\overline{0})$, let $D = A/2$ and define $f_D = \frac{D_* f_1}{\int D_* f_1}$ so that $\int f_D = 1$. Since $\int D_* f_1 = D^5 \int f_1 = D^5$ we get

$$\| f_D \|_k < CD^{-3-k} \| f_1 \|_k \text{ i.e.,}$$
$$\| f_D \|_k < CD^{-3-k}$$

and supp f_D is contained in $B_D(\overline{0})$. Let

$$\lambda = \frac{\int \bar{e}}{\int (2_* f_D - f_D)} = \frac{\int \bar{e}}{2^5 - 1},$$

so $|\lambda| < CA^4 \| \bar{e} \|_0$. Therefore if we define $e = \bar{e} - \lambda(2_* f_D - f_D)$ we get

$$\| e \|_k < \| \bar{e} \|_k + |\lambda| \| 2_* f_D - f_D \|_k < \| \bar{e} \|_k$$
$$+ CA^4 \| \bar{e} \| \|_0 \| f_D \|_k < \| \bar{e} \|_k + CA^{1-k} \| \bar{e} \|_0$$
$$< \| \bar{e} \|_k + CA \| \bar{e} \|_k < (1 + C)A \| \bar{e} \|_k, \text{ i.e.}$$

(10) $\| e \|_k < CA \| \bar{e} \|_k,$

hence

(11) $\| e \|_k < \mathcal{C} A^{12-|k/2|} \| v \|_k$

where $\mathcal{C}$ is the constant in (9) times the constant in (10). Define $u_4 = -\tilde{e}$ and $u_5 = \lambda f_D$. The claim is proved.

Proof of Proposition 1: We will find real numbers L, M for which the hypothesis of the Theorem in the Appendix are fulfilled. This will be achieved by a suitable modification in the proof of the claim.

Let $\mathcal{C}$ be the constant in (11) and fix $L > \max(4\mathcal{C}, 2)$. Since given A such that supp v is contained in $B_A(\overline{0})$, there exists $n \in \mathbf{N}$ such that $4L^n \geq A$. We can assume that $4L^n = A$. We copy the proof of the claim with this last assumption to get $\bar{e} = L_*^n e = L_*(L_*^{n-1} e)$, therefore if C' is the constant in (8) we obtain the inequality

$$\| \bar{e} \|_k \leq (L^n)^{2-k} (4L^n)^{8+[k/2]} C' \| v \|_k .$$

Now, since

$$\| \bar{e} \|_k \leq (L^n)^{11-[k/2]} 4^{8+[k/2]} C' \| v \|_k .$$
$$< (L^n)^{11-[k/2]} 4^{k-2} C' \| v \|_k,$$

we obtain

$$(9') \parallel \bar{e} \parallel_k < C''(L^n)^{11-[k/2]} \parallel v \parallel_k,$$

where $C'' = 4^{k-2}C'$ is the constant in (9).

We proceed with the proof of the Proposition by repeating the steps of the proof of the claim, but with (9') instead of (9), so we get

$$(11') \parallel e \parallel_k \leq 4C(L^n)^{12-[k/2]} \parallel v \parallel_k < 4CL^{12-[k/2]} \parallel v \parallel_k.$$

We know $k > 28$, if we define $\rho = 4CL^{12-[k/2]}$, then $\rho < 1$.

Let $u_4 = -(\tilde{e} + L_*\tilde{e} + \ldots + L_*^{n-1}\tilde{e})$, then since $L > 2$, $1 + L^{2-k} + \ldots + (L^{2-k})^{n-1} < 2$ for all n and we get

$$\parallel u_4 \parallel_k \leq 2 \parallel e \parallel_k$$

so by (8) we obtain

$$(12) \parallel u_4 \parallel_k \leq CA^{8+[k/2]} \parallel v \parallel_k.$$

Let $u_5 = \lambda f_D$ then

$$\parallel u_5 \parallel_k = \parallel \lambda f_D \parallel_k = |\lambda| \parallel f_D \parallel_k \leq CA^4 \parallel \bar{e} \parallel_0 D^{-3-k} \leq CA^{12-[k/2]} \parallel v \parallel_k,$$

i.e.,

$$(13) \parallel u_5 \parallel_k \leq CA^{12-[k/2]} \parallel v \parallel_k.$$

Finally let $M = 2$ and K be the largest of the constants in (7), (11) and (12), then we have proved that

$$\sum_{j=1}^{3} Tp_{j*}u_j - u_j + L_*u_4 - u_4 + M_*u_5 - u_5 = v - e, \quad j = 1, 2, 3$$

with $\parallel u_i \parallel_k \leq K \parallel v \parallel_k$ and $\parallel e \parallel_k \leq \rho \parallel v \parallel_k$, $\rho < 1$.

Appendix

THEOREM: *Let B be a Banach space with norm, $\parallel \parallel$, let $P : B \times B \to B$ be a linear operator for which, given v in B, there exist constants $\rho < 1$ and $K > 0$, and u, w elements of B such that $\parallel P(u, w) - v \parallel < \rho \parallel v \parallel$ and $\parallel u \parallel < K \parallel v \parallel$, $\parallel w \parallel < K \parallel v \parallel$. Then P is surjective.*

Proof: Let $v = e_0$ be an element in B. Find u_0, w_0 according to the hypothesis of the Theorem with respect to the constants ρ and K. Define $-e_1 = P(u_0, w_0) - e_0$. Use the same method to define inductively u_i, w_i and e_{i+1}; therefore we have $\parallel u_i \parallel < K\rho^i \parallel v \parallel$, $\parallel w_i \parallel < K\rho^i \parallel v \parallel$ and $\parallel e_{i+1} \parallel < \rho^{i+1} \parallel v \parallel$.

Let $u^n = \sum_0^n u_i$ and $w^n = \sum_0^n w_i$, then $\{u^n\}$ and $\{w^n\}$ are Cauchy sequences and $\| P(u^n, w^n) - v \|$ is bounded by $\| e_{n+1} \|$, which approaches 0 as n tends to ∞. Let u and w be the respective limits of $\{u^n\}$ and $\{w^n\}$, so we end up with $P(u, w) = v$.

DEPARTAMENTO DE MATEMÁTICAS
FACULTAD DE CIENCIAS
UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO
CIUDAD UNIVERSITARIA
MÉXICO, D.F. MÉXICO

REFERENCES

[B1] D. E. BLAIR, *Contact manifolds in Riemannian geometry*, Lecture Notes in Math. **509**(1976).

[B,P] A. BANYAGA AND J. PULIDO, *On the group of contact diffeomorphisms of* $\mathbf{R}^{2n+1}$. Bol. Soc. Mat. Mexicana, **23**, 2(1978), 43-47.

[F,S] G. B. FOLLAND AND E. M. STEIN, *Estimates for the* $\bar\partial_b$ *complex and analysis on the Heisenberg group*, Comm. on Pure and Applied Math. XXVII(1974), 429-522.

[M1] J.N. MATHER, *Commutators of diffeomorphisms* I, Comment. Math. Helv. **49**, 4(1974), 512-528.

[M2] ———, *Commutators of diffeomorphisms* II, Comment. Math. Helv. **50**,(1975), 33-40.

[P] J. PULIDO CEJUDO, *An equation of contact vector fields and the group of contact diffeomorphisms*. Ph.D Thesis, Princeton University, June 1981.