Boletín de la Sociedad Matemática Mexicana Vol. 32 No. 1, 1987

ORIENTABILITY OF VECTOR BUNDLES AND FORMULAE FOR STIEFEL-WHITNEY CLASSES

BY PARAMESWARAN SANKARAN[1]

§1. Introduction

It is proved in §12 of [3] that if ξ is an orientable real vector bundle over a paracompact base space B then $w_1(\xi) = 0$. That the converse is also true for CW-complexes is left as an exercise (Problem 12A). Exercise H on page 281 of [7] deals with Stiefel-Whitney classes of sphere bundles, and 3(d) of this exercise states that a sphere bundle η is orientable if and only if $w_1(\eta) = 0$. In the case ξ possesses an Euclidean metric, one can apply this result to the associated sphere bundle of ξ. But when B is not paracompact, it is not true in general that ξ possesses an Euclidean metric. In this paper, we adopt a slightly weaker definition of orientability of a vector bundle and prove, in Theorem 2.4, that a real vector bundle ξ over an *arbitrary* base space is orientable if and only if $w_1(\xi) = 0$.

We apply this result and the splitting principle (Theorem 3.1) to obtain formulae for the Stiefel-Whitney classes of tensor products, symmetric and exterior powers of real vector bundles over an *arbitrary* base space assuming only that each of the vector bundles admits an Euclidean metric (Theorem 3.3). Theorem 3.3 (i), (ii) is well-known in case the base space is suitably restricted so that the classification theorem can be applied. (cf. [1]). We give an axiomatic proof that uses only elementary concepts. In this respect, it parallels the proof of analogous formulae for Chern classes given in page 64 of [2].

The formulae for the Stiefel-Whitney classes of symmetric powers of a vector bundle are probably new.

This paper is based on chapter 2 of the author's Ph.D. thesis [6].

§2. Orientability of Vector Bundles

Let R be a commutative ring with identity 1_R. Let ξ be a real vector bundle of rank $n > 0$ over an arbitrary base space B. Let E denote the total space and E_0 the complement of the zero cross section in E. Let $\pi : E \to B$ denote the projection of ξ and π_0 the restriction of π to E_0. For $b \in B$, F_b denotes the fibre of ξ over b and $F_{b,0}$ the non-zero elements of F_b. Let $j_b : (F_b, F_{b,0}) \subset (E, E_0)$ be the inclusion. We use the following definition of orientability throughout this paper.

[1] The author was partially supported by NSERC grant A-8225.
1980 A.M.S. subject classification: 55R25; 57R20
Keywords and Phrases: Vector bundles, orientability, Stiefel-Whitney classes.

Definition (2.1). A real vector bundle ξ of rank n is said to be R-orientable if there exists an assignment Ω, called an R-orientation on ξ, of a preferred generator u_b^R of $H^n(F_b, F_{b,0}; R) \cong R$ to each $b \in B$ satisfying the following local compatibility condition:

For each $b_0 \in B$, there exists a neighbourhood N of b_0 and an element u_N^R in $H^n(\pi^{-1}(N), \pi_0^{-1}(N); R)$ such that $u_N^R \mid (F_b, F_{b,0}) = u_b^R$ for all $b \in N$. Here $u_N^R \mid (F_b, F_{b,0}) = \mu_b^*(u_N^R)$ where μ_b is the inclusion $(F_b, F_{b,0} \subset (\pi^{-1}(N), \pi_0^{-1}(N))$. ξ is said to be orientable if it is $\mathbf{Z}$-orientable.

Example: Consider the deleted comb space

$$D = I \times 0 \cup \{\frac{1}{n} \mid n \geq 1\} \times I \cup \{(0,1)\} \subset \mathbf{R} \times \mathbf{R}$$

where $I = \{x \mid 0 \leq x \leq 1\} \subset \mathbf{R}$. D is connected and has two path components $P_1 = \{(0,1)\}$, $P_2 = D - P_1$. Therefore, $H^0(D; \mathbf{Z}) \cong \mathbf{Z} \times \mathbf{Z}$. Using the Künneth formula $H^1(D \times (\mathbf{R}, \mathbf{R}-0)) \cong (\mathbf{Z} \times \mathbf{Z}) \otimes H^1(\mathbf{R}, \mathbf{R}-0; \mathbf{Z})$. Let α be the generator of $H^1(\mathbf{R}, \mathbf{R}-0; \mathbf{Z}) \cong \mathbf{Z}$ that corresponds to the standard orientation on $\mathbf{R}$ (cf. page 95 of [3]). Let $U = (-1, 1) \otimes \alpha$. Now consider the trivial line bundle $\mathcal{E}$ over D whose total space is $D \times \mathbf{R}$. It is readily verified $U_b = U \mid (F_b, F_{b,0})$ is a generator of $h_1(F_b, F_{b,0}; \mathbf{Z})$ for each $b \in D$. Thus U defines a $\mathbf{Z}$-orientation Ω of $\mathcal{E}$. In fact, the orientation of $F_b = b \times \mathbf{R}$ given by U_b is the standard orientation of F_b if and only if $b \neq (0,1)$. Since D is connected, it follows that there is no neighbourhood of $(0,1)$ over which there exists an *orientation preserving* local trivialization of the $\mathbf{Z}$-oriented vector bundle $(\mathcal{E}, \Omega)$. Therefore, Ω is *not* an orientation in the sense of Milnor-Stasheff [3].

Remark (2.2). In case $R = \mathbf{Z}$, definition 2.1 is the cohomological formulation of the definition of orientation given in page 96 of [3]. Thus, every vector bundle oriented in the sense of Milnor-Stasheff [3] is orientable according to Definition 2.1. The above example shows that the concept of an orientation according to Definition 2.1 is strictly more general than the one used in [3].

Remark (2.3). Note that if ξ is orientable, then it is R-orientable for every commutative ring R with identity. In fact, let r denote the unique ring homomorphism $\mathbf{Z} \to R$ and let r_* be the map in cohomology induced by the homomorphism r betwen the coefficient groups. If $\Omega = \{u_b^{\mathbf{Z}}\}$ is a $\mathbf{Z}$-orientation on ξ then $r_*\Omega = \{r_*(u_b^{\mathbf{Z}})\}$ gives an R-orientation on ξ.

Let R denote $\mathbf{Z}_2$ or, if ξ is $\mathbf{Z}$-orientable, an arbitrary commutative ring with identity. One has $H^n(F_b, F_{b,0}; R) \cong R$. If $R = \mathbf{Z}_2$ let u_b^R denote the unique non-zero element of $H^n(F_b, F_{b,0}; \mathbf{Z}_2)$. If ξ is $\mathbf{Z}$-oriented with orientation $\{u_b^{\mathbf{Z}}\}$, let $u_b^R = r_*(u_b^{\mathbf{Z}})$. We have the following Thom Isomorphism Theorem. For a proof, see §10 of [3].

THEOREM (2.4). *There exists a unique cohomology class u^R in $H^n(E, E_0; R)$ such that $j_b^*(u^R) = u_b^R$ for each $b \in B$. Moreover, the correspondence $y \mapsto y \cup u^R$ maps $H^j(E; R)$ isomorphicaly onto $H^{j+n}(E, E_0; R)$ for every integer j.*

u^R is called the Thom class of ξ, and ϕ, the composition of the isomorphisms $H^j(B;R) \xrightarrow{\pi^*} H^j(E;R) \xrightarrow{\cup u^R} H^{n+j}(E,E_0;R)$, is called the Thom isomorphism.

Remark (2.5). The proof of the above theorem in the oriented case as given in §10 of [3] requires only that ξ be **Z**-oriented in the sense formulated above (2.1).

One of the consequences of the Thom Isomorphism Theorem is that every real vector bundle ξ is (uniquely) $\mathbf{Z}_2$-orientable. The following proposition gives a criterion for orientability in terms of R-orientability.

Let $U(R)$ denote the group of units in R.

PROPOSITION (2.6). *Let ξ be a real vector bundle of rank n over B and R a commutative ring with the property that $U(R)$ is a group with precisely two elemets $\pm 1_R$. Then ξ is orientable if and only if ξ is R-orientable.*

Sketch of Proof: The "only if " part follows from Remark 2.3. For $b \in B$ let u_b^R be the preferred generator of $H^n(F_b, F_{b,0}; R) \cong R$. Given $b_0 \in B$ choose an open neighbourhood N of b_0 in B with $\xi \mid N$ trivial. Let $u_N^R \in H^n(\pi^{-1}(N), \pi_0^{-1}(N); R)$ chosen as in Definition 2.1. Let $\{N_\gamma\}_{\gamma\in\Gamma}$ be the path components of N. Thus $H^n(\pi^{-1}(N_\gamma),\ \pi_0^{-1}(N_\gamma); R) \cong R$ for each $\gamma \in \Gamma$ and we can identify $H^n(\pi^{-1}(N),\ \pi_0^{-1}(N); R)$ with $\prod_{\gamma\in\Gamma} H^n(\pi^{-1}(N_\gamma),\ \pi_0^{-1}(N_\gamma;)R)$. Write $u_N^R = \prod_{\gamma\in\Gamma} u_\gamma^R$ where $u_\gamma^R \in H^n(\pi^{-1}(N_\gamma),\ \pi_0^{-1}(N_\gamma); R) \cong R$ is an R-generator. Since $r : \mathbf{Z} \to R$ maps $U(\mathbf{Z})$ isomorphicaly onto $U(R)$, there exists a unique generator $u_\gamma^{\mathbf{Z}}$ for $H^n(\pi^{-1}(N_\gamma),\ \pi_0^{-1}(N_\gamma); \mathbf{Z}) \cong \mathbf{Z}$ such that $r_*(u_\gamma^{\mathbf{Z}}) = u_\gamma^R$. Similary, there exists a unique generator $u_b^{\mathbf{Z}}$ for $H^n(F_b, F_{b,0}; \mathbf{Z}) \cong \mathbf{Z}$ such that $r_*(u_b^{\mathbf{Z}}) = u_b^R$. Let $u_N^{\mathbf{Z}} = \prod_{\gamma\in\Gamma} u_\gamma^{\mathbf{Z}}$. Then one shows that $u_N^{\mathbf{Z}} \mid (F_b, F_{b,0}) = u_b^{\mathbf{Z}}$ for each $b \in N$.

Thus $\{u_b^{\mathbf{Z}}\}$ is an **Z**-orientation on ξ. For details, see [5] or §6 of [6].

THEOREM (2.7). *Let ξ be a real vector bundle of rank $n \geq 1$ over an arbitrary base space B. Then ξ is orientable if and only if $w_1(\xi) = 0$.*

Method of Proof: Since every vector bundle is uniquely $\mathbf{Z}_2$-orientable, ξ has a unique $\mathbf{Z}_2$-Thom class $u^{\mathbf{Z}_2}$. Recall that $w_1(\xi) = \phi^{-1}(\mathrm{Sq}^1 u^{\mathbf{Z}_2})$ where $\phi : H^1(B;\mathbf{Z}_2) \to H^{n+1}(E,E_0;\mathbf{Z}_2)$ is the Thom isomorphism. Thus, $w_1(\xi) = 0$ if and only if $\mathrm{Sq}^1 u^{\mathbf{Z}_2} = 0$, equivalently, $u^{\mathbf{Z}_2} = \mathcal{E}_*(\alpha)$ for some $\alpha \in H^n(E, E_0; \mathbf{Z}_4)$ since Sq^1 is the Bockstein homomorphism corresponding to the exact sequence $0 \longrightarrow \mathbf{Z}_2 \xrightarrow{s} \mathbf{Z}_4 \xrightarrow{\mathcal{E}} \mathbf{Z}_2 \longrightarrow 0$.

If $w_1(\xi) = 0$ then one shows, using an argument similar to proof of Proposition 2.6, that $\Omega = \{j_b^*(\alpha)\}$ is a $\mathbf{Z}_4$-orientation on ξ. Therefore, ξ is **Z**-orientable by Proposition 2.6. Conversely, if ξ is **Z**-orientable, then it is $\mathbf{Z}_4$-orientable. It is easy to see that $\mathcal{E}_*(u^{\mathbf{Z}_4}) = u^{\mathbf{Z}_2}$. Taking $\alpha = u^{\mathbf{Z}_4}$, we see that $w_1(\xi) = 0$.

COROLLARY (2.8). *Let ξ be a real vector bundle of rank n over an arbitrary base space B. Then $w_1(\xi) = w_1(\lambda^n(\xi))$.*

Proof: By the above theorem, it suffices to prove that $\xi \oplus \lambda^n(\xi)$ is orientable. For $b \in B$ let $v_1, \ldots, v_n$ be any basis for F_b. Then the orientation of the vector space $F_b \oplus \lambda^n(F_b)$ determined by the basis $v_1, \ldots, v_n, v_1 \wedge \ldots \wedge v_n$ is independent of the choice of the basis $v_1, \ldots, v_n$ of F_b, and hence orients $F_b \oplus \lambda^n(F_b)$ canonically. It is easily verified that these orientations on the fibres satisfy the (stronger) local compatibility condition on p. 96 of [3], which implies orientability of $\xi \oplus \lambda^n(\xi)$ according to our definition (cf. Remark 2.2).

§3. Formulae for Stiefel-Whitney Classes

In this section, we consider only real vector bundles that admit an Euclidean metric, over an arbitrary base space B. Well-known sufficient conditions for the bundle to be Euclidean are e.g., B paracompact or the bundle being of finite type.

The following theorem can be proved using induction and the Leray-Hirsch Theorem (cf. page 258 of [7]) which is true for an arbitrary base space providing we use a field of coefficients.

THEOREM (3.1). *(The Splitting Principle): Let $\xi_1, \ldots, \xi_k$ be a real vector bundles over an arbitrary base space B, each of which possesses an Euclidean metric. Then there exists a space B' and a map $f : B' \to B$ which induces a monomorphism f^* in $\mathbf{Z}_2$-cohomology such that $f^*(\xi_i)$ splits as a Whitney sum of n_i line bundles, $1 \leq i \leq k$, where $n_i = rank\ \xi_i$.*

The proof of the above theorem is similar to that of Prop. 5.5, Chapter IV of [4]. Note that we can allow the base space to be arbitrary in applying the Leray-Hirsch theorem as we are using a field of coefficients, namely $\mathbf{Z}_2$. The existence of an Euclidean metric on ξ_i guarantees that for any $g : X \to B$, $g^*(\xi_i)$ is Euclidean and that any sub-bundle of $g^*(\xi_i)$ is a Whitney summand.

We use the above theorem to obtain formulae for tensor products, exterior powers, and symmetric powers of vector bundles.

Let $\xi, \eta_1, \ldots, \eta_r$ be real vector bundles over an arbitrary base space B, each of them possesing an Euclidean metric. Choose B' and a map $f : B' \to B$ such that $f^* : H^*(B; \mathbf{Z}_2) \to H^*(B'; \mathbf{Z}_2)$ is a monomorphism and there exist line bundles $\xi_1, \ldots, \xi_n, \eta_{ij}$, $1 \leq j \leq n_i$, where $n = \text{rank } \xi$, $n_i = \text{rank } \eta_i$ and $f^*(\xi) = \xi_1 \oplus \cdots \oplus \xi_n$, $f^*(\eta_i) = \bigoplus_{1 \leq j \leq n_i} \eta_{ij}$, $1 \leq i \leq r$.

Such a choice is possible by the Splitting Principle. Let ς^m denote the m-fold tensor product $\varsigma \otimes \cdots \otimes \varsigma (\varsigma^0 \approx \mathcal{E}, \varsigma^1 \approx \varsigma)$.

LEMMA (3.2). *With the above notations, one has the following vector bundle isomorphisms:*

$$\text{(i)}\quad f^*(\eta_1 \otimes \cdots \otimes \eta_r) \approx \bigoplus_{\substack{1\le j_i\le n_i\\ 1\le i\le r}} (\eta_{1j_1} \otimes \cdots \otimes \eta_{rj_r}),$$

$$\text{(ii)}\quad \textit{For } 1 \le k \le n,\ f^*(\lambda^k(\xi)) \approx \bigoplus_{1\le i_1<\cdots<i_k\le n} (\xi_{i_1} \otimes \cdots \otimes \xi_{i_k}),$$

$$\text{(iii)}\quad \textit{For any } k \ge 1,\ f^*(S^k(\xi)) \approx \bigoplus_{\substack{0\le k_1,\dots,k_n\le k\\ k_1+\cdots+k_n=k}} \xi_1^{k_1} \otimes \cdots \otimes \xi_n^{k_n}.$$

Here, $\lambda^k(\xi)$ and $S^k(\xi)$ denote respectively the k^{th}-exterior power and k^{th}-symmetric power of ξ.

Proof: Isomorphism (i) follows from the fact that tensor product distributes over Whitney sums.

Isomorphism (ii) follows from the repeated application of the formula

$$\lambda^k(\alpha \oplus \beta) \approx \bigoplus_{i+j=k} \lambda^i(\alpha) \otimes \lambda^j(\beta)$$

and the property $\lambda^0(\varsigma) \approx \mathcal{E}$, $\lambda^1(\varsigma) \approx \varsigma$, while other exterior powers are zero for a line bundle ς.

Note that the symmetric power $S^m(\varsigma)$ of a line bundle ς is isomorphic to the m-fold tensor product ς^m, for $m \ge 1$. Hence, using the formula

$$S^k(\alpha \oplus \beta) \approx \bigoplus_{i+j=k} S^i(\alpha) \otimes S^j(\beta)$$

we obtain

$$\begin{aligned} f^*(S^k(\xi)) &\approx S^k(\xi_1 \oplus \cdots \oplus \xi_n) \\ &\approx \bigoplus_{\substack{0\le k_1,\dots,k_n\le k\\ k_1+\cdots+k_n=k}} \xi_1^{k_1} \otimes \cdots \otimes \xi_n^{k_n}. \end{aligned}$$

This proves (iii).

Let $q_{n,k}$, $1 \le k \le n$, denote the unique polynomial in the variables $\sigma_1, \dots, \sigma_n$ for which the following definition relation

$$q_{n,k}(\sigma_1, \dots, \sigma_n) = \prod_{1\le i_1<\dots<i_k\le n} (1 + x_{i_1} + \cdots + x_{i_k})$$

holds in $\mathbf{Z}_2[x_1, \dots, x_n]$. Here σ_i denotes the i^{th} elementary symmetric polynomial in the indeterminates $x_1, \dots, x_n$. Similary, we let $s_{n,k}$, $n, k \ge 1$, denote the unique polynomial defined by the relation

$$s_{n,k}(\sigma_1, \dots, \sigma_n) = \prod_{\substack{0\le k_1,\dots,k_n\le k\\ k_1+\cdots+k_n=k}} (1 + k_1x_1 + \cdots + k_nx_n).$$

Let $\sigma_j(i)$ denote the j^{th} elementary symmetric polynomials in the indeterminates $y_{i1}, \ldots, y_{in_i}$. Then it is easily seen that

$$\prod_{1\le i\le r}\ \prod_{1\le j_i\le n_i} (1 + y_{1j_1} + \cdots + y_{rj_r})$$

in $\mathbf{Z}_2[y_{ij} : 1 \le j \le n_i,\ 1 \le i \le r]$ is symmetric in $y_{i1}, \ldots, y_{in_i}$ for $1 \le i \le r$. Hence, it is uniquely expressible as a polynomial in the variables $\sigma_j(i)$, $1 \le j \le n_i$, $1 \le i \le r$. Let $p_{n_1,\ldots,n_r}$ denote the unique polynomial defined by the relation

$$p_{n_1,\ldots,n_r}(\sigma_1(1), \ldots, \sigma_{n_1}(1); \ldots; \sigma_1(r), \ldots, \sigma_{n_r}(r)) = \prod_{1\le i\le r}\ \prod_{1\le j_i\le n_i} (1+y_{1j_1}+\cdots+y_{rj_r}).$$

We are now ready to prove the main theorem of this section.

$$\text{Let } w_i = w_i(\xi) \text{ and } w_i(j) = w_i(\eta_j).$$

THEOREM (3.3). *With the above notations,*

(i) $w(\eta_1 \otimes \cdots \otimes \eta_r) = p_{n_1,\ldots,n_r}(w_1(1), \ldots, w_{n_1}(1); \ldots; w_1(r), \ldots w_{n_r}(r))$.

(ii) $w(\lambda^k(\xi)) = q_{n,k}(w_1, \ldots, w_n)$

(iii) $w(S^k(\xi)) = s_{n,k}(w_1, \ldots, w_n)$.

Proof of (ii): Note that $\xi_{i_1} \oplus \cdots \oplus \xi_{i_k} \approx \lambda^k(\xi_{i_1} \otimes \cdots \otimes \xi_{i_k})$.
Therefore, using Corollary 2.8 and the Whitney product formula,

$$\begin{aligned} w_1(\xi_{i_1} \otimes \cdots \otimes \xi_{i_k}) &= w_1(\xi_{i_1} \oplus \cdots \oplus \xi_{i_k}) \\ &= w_1(\xi_{i_1}) + \cdots + w_1(\xi_{i_k}). \end{aligned}$$

Thus, writing $x_i = w_1(\xi_i)$ for $1 \le i \le n$,

$$(*) \qquad w(\xi_{i_1} \otimes \cdots \otimes \xi_{i_k}) = 1 + x_{i_1} + \cdots + x_{i_k}.$$

By naturality, another use of Whitney product formula, Lemma 3.2, and (*) we now find

$$\begin{aligned} f^*(w(\lambda^k(\xi)) &= w(f^*(\lambda^k(\xi)) = w(\oplus(\xi_{i_1} \otimes \cdots \otimes \xi_{i_k})) \\ &= \prod_{1\le i_1<\cdots<i_k\le n} (1 + x_{i_1} + \cdots + x_{i_k}) \\ &= q_{n,k}(\sigma_1, \ldots, \sigma_n) \in H^*(B'; \mathbf{Z}_2). \end{aligned}$$

Since f^* is a monomorphism and since $f^*w_i = \sigma_i$, it follows that $w(\lambda^k(\xi)) = q_{n,k}(w_1, \ldots, w_n)$.

Formulae (i) and (iii) are established similarly.

Remarks: Theorem 3.3 (i), (ii) and analogous formulae for Chern classes and Pontrjagin classes are proved in [1] using group representations and the description of characteristic classes in terms of root systems of suitable Lie groups. It may be noted that their proof applies for a suitable class of base spaces for which the classification theorem for vector bundle holds. Exercise 7C in [3] gives another proof for $p_{m,n}$ that holds for paracompact base spaces. The above proof, which holds for any bundles admitting an Euclidean metric, parallels the proof of an analogous result for complex vector bundles given in §4.4 of [2].

To the best of the author's knowledge, the formula for $w(S^k(\xi))$ is not found in the standard references.

Acknowledgements: The author wishes to thank Professor P. Zvengrowski and Professor K. Varadarajan for their help and encouragement.

DEPARTAMENT OF MATHEMATICS AND STATISTICS
THE UNIVERSITY OF CALGARY
CALGARY, ALBERTA
CANADA T2N 1N4

CURRENT ADDRESS:
SCHOOL OF MATHEMATICS
SPIC SCIENCE FUNDATION
92, G.N. CHETTY ROAD
T. NAGAR, MADRAS - 600 017
INDIA

REFERENCES

[1] A. BOREL AND F. HIRZEBRUCH, *Characteristic classes and homogeneous spaces - I*, Amer. J. Math. 80(1958), 458-535.

[2] F. HIRZEBRUCH, *Topological Methods in Algebraic Geometry*, 3rd Ed. Springer-Verlag.

[3] J. MILNOR AND J. STASHEFF, *Characteristic classes*, (1974), Princeton Univ. Press, Princeton, N.J.

[4] H. OSBORN, Vector Bundles, Vol. 1 (1982) Academic Press, New York.

[5] P. SANKARAN, *On orientability of vector bundles*, Research Paper #530, (1983), Dept. of Math. & Stat., Univ. of Calgary, Calgary, Alberta, Canada.

[6] ———, *Vector Fields on Flag Manifolds*, Ph.D. Thesis (1985), The Univ. of Calgary, Calgary, Alberta, Canada.

[7] E. H. SPANIER, Algebraic Topology, (1983), Springer-Verlag.