Boletín de la Sociedad Matemática Mexicana Vol. 33 No. 1, 1988

ON THE STRUCTURE OF ASSOCIATIVE ALGEBRAS WITH CONJUGATION

JERZY F. PLEBAŃSKI[1)] AND MACIEJ PRZANOWSKI[2)]

Introduction

Let A be an algebra with unity element e_0 of dimension $n+1$ ($n \geq 0$) over a field F and let $c : A \to A$ be a linear mapping, called a conjugation on A, such that

$$c(c(x)) = x, \tag{1.1}$$

$$x + c(x) = t(x)e_0, \quad t(x) \in F, \tag{1.2}$$

$$xc(x) = c(x)x = n(x)e_0, \quad n(x) \in F, \tag{1.3}$$

$$c(xy) = c(y)c(x) \tag{1.4}$$

for all $x, y \in A$.

Then the pair (A, c) is called an *algebra with conjugation* of dimension $n+1$ over F.

The elements $t(x), n(x) \in F$ are called the *trace* and the *norm* of x, respectively.

Algebras with conjugation have been examined extensively by Albert [1], Schafer [2], [3], and Adem [4] in the context of the Cayley-Dickson process

---

[1)] On leave of absence from the University of Warsaw, Warsaw, Poland.

[2)] This autor is grateful to all the members of the Departament of Physics of the Centro de Investigación y de Estudios Avanzados del IPN for their warm hospitality during his stay at the Centro.
Permanent address: Instytut Fizyki, Politechnika Łódzca, Wólczańska 219, 93-005 Łódź, Poland.
This work was supported in part by the Consejo Nacional de Ciencia y Tecnología (CONACYT) and by the Centro de Investigación y de Estudios Avanzados del IPN (CINVESTAV), México, D.F., México.

which is, roughly speaking, an iterative process leading from one algebra with conjugation to another. More precisely, if (A, c) is an algebra with conjugation of dimension $n+1$ $(n \geq 0)$ over F, then one defines an algebra $A(\Theta)$, where $0 \neq \Theta \in F$, of dimension $2(n+1)$ over F which consists of all ordered pairs $w = (x, y)$, $x, y \in A$ with addition and mutltiplication by scalars defined componentwise, and with the multiplication of pairs defined as follows

$$w_1 w_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 + \Theta c(y_2) y_1, y_2 x_1 + y_1 c(x_2)).$$

Define then a linear mapping $c_1 : A(\Theta) \to A(\Theta)$ by

$$c_1 : A(\Theta) \ni w = (x, y) \mapsto c_1(w) = (c(x), -y) \in A(\Theta).$$

One can easily verify that $(A(\Theta), c_1)$ is an algebra with conjugation of dimension $2(n+1)$ over F. Thus, starting with some algebra with conjugation we can construct new algebras with conjugation by iteration, using the above described method. This is just the Cayley-Dickson process. If one begins with the algebra (Fe_0, c), where $c : Fe_0 \to Fe_0$ is now the identity transformation on Fe_0, i.e., $c(x) = x$ for every $x \in Fe_0$, then having k non-zero scalars $\Theta_1, \ldots, \Theta_k$ one constructs k new algebras over $F : A_1 = Fe_0(\Theta_1), A_2 = A_1(\Theta_2), \ldots, A_k = A_{k-1}(\Theta_k)$.

It is well known that A_1, A_2 and A_3 are the algebras of generalized complex, quaternion and Cayley numbers, respectively (see [1-5]). Therefore, a natural question is if there exists such a transparent interpretation when the initiating algebra is not, as before, (Fe_0, c) but some other algebra with conjugation. It is evident that at first we shoud have an algebra which we are to start with. This is the main theme of the present note.

We intend to analyse the properties of associative algebras with conjugation and then we give the possible structures of these algebras. An algebra with conjugation (A, c) is said to be *associative* if A is associative.

Our considerations are very closely related with the results of our previous work [6] concerning the classification of the so-called quaternionlike algebras. These algebras as introduced in Ref. [6] are, so to say, of the group-theoretical origin. Namely, we have searched for real or complex local Lie groups of some simple rational composition law of their elements.It has been shown that these groups are locally isomophic to the groups consisting of elements with norm (see Sec. 2) of some associative algebras which we have called quaternionlike algebras.

It appears that one of the groups considered is locally isomorphic to the group $SU(2)$, and then the corresponding quaternionlike algebra is exactly the well-known quaternion algebra.

This fact justifies, to some extent, the proposed name: *quaternionlike algebras*.

In Ref. [6] we have analyzed the structure of quaternionlike algebras and we have constructed many of these algebras. Then, in Ref. [7] we have found a close relation between the structure of quaternionlike algebras and the form of cross products of vectors satisfying the similar conditions as the usual cross product of vectors in 3-dimensional Euclidean vector space. Both in [6] and [7] we have dealt with quaternionlike algebras over the real or complex field. In the present note we generalize the definition of these algebras on an arbitrary field F whose characteristic is not 2, and then we show that the so defined quaternionlike algebras with suitably defined conjugation of their elements appear to be exactly the asociative algebras with conjugation (Sec. 2).

We will now be able to find an interesting relation between the Clifford algebras and the associative algebras with conjugation (Sec. 3).

Then in Sec. 4 the analysis of possible structures of associative algebras with conjugation is given.

2. Quaternionlike algebras and associative algebras with conjugation

In Secs. 2, 3 and 4 we assume that the characteristics of a field F is not 2.

Let Q be an associative algebra with unity element e_0 of dimension $n+1 (n \geq 0)$ over a field F for which there exists a decomposition

$$Q = Fe_0 \oplus V, \tag{2.1}$$

where V is an n-dimensional vector subspace of Q such that

$$v^2 \in Fe_0, \qquad \text{for every } v \in V. \tag{2.2}$$

Then Q is called a *quaternionlike algebra* of dimension $n+1$ over F.

One can easily prove that the decomposition defined by (2.1) and (2.2), if it exists, is unique.

Let for any $v \in V$, $q(v) \in F$ be a scalar defined as follows

(2.3) $$v^2 = q(v)e_0$$

The mapping $q : V \ni v \mapsto q(v) \in F$ is a quadratic form on V and the pair (V, q) is a *quadratic space.*

Thus, the algebra Q is *compatible* with q (see [8]).

If B denotes the bilinear form on V associated with q, then

(2.4) $$2B(v, w)e_0 = (q(v + w) - q(v) - q(w))e_0 = vw + wv, \qquad \text{for } v, w \in V.$$

Now we would like to define a conjugation on a quaternionlike algebra Q. Let x be any element in Q. Then it can be uniquely represented in the form of

(2.5) $$x = b(x)e_0 + a(x),$$

where $b(x) \in F,\ \ a(x) \in V$. The element $\bar{x} \in Q$

(2.6) $$\bar{x} = b(x)e_0 - a(x)$$

is called the *conjugate* of x, and the linear mapping $c : Q \to Q$ defined by

(2.7) $$c(x) = \bar{x}, \qquad \text{for all } x \in Q$$

is called de *conjugation on* Q.

One finds easily that the conjugation on Q satisfies the conditions (1.1), (1.2) and (1.3) with

(2.8) $$t(x) = 2b(x),\ \ n(x) = [b(x)]^2 - q(a(x)), \qquad \text{for } x \in Q.$$

Now, if $x, y \in Q$ then , $c(xy) = c(y)c(x)$ if and only if $c(a(x)a(y)) = a(y)a(x)$. Consequently, the conjugation on Q fulfills the condition (1.4) for all $x, y \in Q$ if and only if

$$c(uv) = vu, \qquad \text{for all } u, v \in V. \tag{2.9}$$

Before we consider this problem we must examine the consequences of the assumption that Q is associative.

Therefore, we have

$$u(vw) = (uv)w \qquad \text{for all } u, v, w \in V.$$

This implies

$$b(ua(vw)) = b(a(uv)w) \tag{2.10}$$

and

$$b(vw)u + a(ua(vw)) = b(uv)w + a(a(uv)w). \tag{2.11}$$

As $vw + wv \in Fe_0$ for any $v, w \in V$ (see (2.4)).

$$a(vw) = -a(wv). \tag{2.12}$$

Substituting $u = w$ into (2.11) and using (2.12) one obtains $b(vu)u = b(uv)u$. As this relation holds for all $u, v \in V$, we conclude that

$$b(uv) = b(vu), \qquad \text{for all } u, v \in V. \tag{2.13}$$

From (2.4), (2.8) and (2.13) it follows:

$$t(uv) = t(vu) = 2b(vu) = 2B(v, u). \tag{2.14}$$

Then it is evident that for any $x, y \in Q$

$$t(xy) = t(yx) = 2b(xy) = 2b(yx). \tag{2.15}$$

We are now in a position to prove (2.9). Indeed

$$\begin{aligned} c(uv) &= b(uv)e_0 - a(uv), \\ vu &= b(vu)e_0 + a(vu). \end{aligned}$$

Using (2.12) and (2.13) to the last formula and comparing with $c(uv)$ we find that (2.9) holds. Consequently, the condition (1.4) is satisfied for all $x, y \in Q$.

Thus we have proved the following

PROPOSITION (2.1). *If Q is an $(n+1)$-dimensional $(n \geq 0)$ quaternionlike algebra over a field F and $c : Q \to Q$ is a conjugation on Q, then (Q, c) is an associative algebra with conjugation.* □

The converse proposition is also true.

PROPOSITION (2.2) *If (A, c) is an associative algebra with conjugation of dimension $n+1$ $(n \geq 0)$ over a field F, then A is a quaternionlike algebra and $c : A \to A$ is the conjugation in the sense of this quaternionlike algebra.*

Proof. Define a vector subspace of A

$$V = \{v \in A; v + c(v) = 0\}. \tag{2.16}$$

From (1.1) and (1.4) one has $y = c(e_0 c(y)) = y c(e_0)$ for all $y \in A$. Hence $c(e_0) = e_0$, and consequently $e_0 \notin V$. Therefore,

$$Fe_0 \cap V = \{0\}. \tag{2.17}$$

Let x be any vector in A. Then

$$x = 2^{-1}[x + c(x)] + 2^{-1}[x - c(x)]. \tag{2.18}$$

As $2^{-1}[x + c(x)] \in Fe_0$ (by (1.2) and $2^{-1}[x - c(x)] \in V$ (by (2.16) and (1.1)), the formulae (2.17) and (2.18) imply

(2.19) $$A = Fe_0 \oplus V.$$

Let now v be an arbitrary vector in V. Then, by (1.3), we have

(2.20) $$v^2 = -vc(v) \in Fe_0.$$

From (2.19) and (2.20) it follows that A is a quaternionlike algebra.

For any $x \in A$

$$x = b(x)e_0 + a(x), \;\; b(x) \in F, \;\; a(x) \in V.$$

From the definition of V, (2.16), and the definition of $\bar{x}$, (2.6), we have

$$c(x) = b(x)e_0 - a(x) = \bar{x}.$$

The proof is thus completed. □

The conclusion of propositions 2.1 and 2.2 can be stated as follows: the quaternionlike algebra is in fact the same notion as the associative algebra with conjugation .

Now we intend to point out some properties of quaternionlike algebras which are of great interest from the group-theoretical point of view.

Let Q be as before, an $(n+1)$-dimensional $(n \geq 0)$ quaternionlike algebra over a field F. Define the following subset of Q

(2.21) $$H = \{x \in Q; x\bar{x} = 1e_0\}.$$

In other words, H consists of all elements of Q with the unity norm. The properties of the conjugation on Q, (1.1) (1.3) and (1.4), assure that H forms a group with the multiplication of its elements as inherited form the algebraic structure of Q.

Moreover, let V be defined according to (2.1) and (2.2). We define the multiplication $\circ$ on V, $\circ : V \times V \to V$, as follows

$$v \circ w = vw - wv = 2a(v, w), \qquad \text{for any } v, w \in V. \tag{2.22}$$

Then the pair $(V, \circ)$ is a Lie algebra of dimension n over F. In particular, if F is a real or complex number field, then H with naturally defined topology and differential structure is, respectively, a real or complex n-dimensional Lie group, and $(V, \circ)$ appears to be isomorphic to Lie algebra of H (see Ref. [6]).

3. Clifford algebras and quaternionlike algebras

Given an $(n+1)$-dimensional ($n \geq 0$) quaternionlike algebra Q over a field F which decomposes according to (2.1) and (2.2), one defines the quadratic space (V, q), where $q : V \to F$ is the quadratic form on V defined by (2.3). Then one can construct the Clifford algebra $C(V, q)$ for (V, q). By the universal property of Clifford algebras there exists a unique homomorphism $\varphi : C(V, q) \to Q$ such that $\varphi(v) = v$ for any $v \in V$ (see Ref. [8]). We can explicity write this homomorphism. Indeed, let $\{e_1, \ldots, e_n\}$ be an orthogonal basis for (V, q). Then, $\{e_1^{\epsilon_1} \bullet \ldots \bullet e_n^{\epsilon_n};\ \epsilon_i = 0, 1;\ i = 1, \ldots, n\}$ constitutes a basis for $C(V, q)$. The homomorphism $\varphi : C(v, q) \to Q$ is defined as F-linear extension of the following mapping

$$C(V, q) \ni e_1^{\epsilon_1} \bullet \ldots \bullet e_n^{\epsilon_n} \mapsto e_1^{\epsilon_1} \ldots e_n^{\epsilon_n} \in Q, \quad \epsilon_i = 0, 1, \quad i = 1, \ldots, n. \tag{3.1}$$

(The multiplication in Clifford algebra is denoted by the fat dot "$\bullet$ ").

Now, the natural question arises: is the Clifford algebra $C(V, q)$ a quaternionlike algebra? The answer to this question reads: $C(V, q)$ is a quaternionlike algebra if and only if $n = \dim V \leq 2$.

This answer follows from a more general proposition:

PROPOSITION (3.1.) *Let V' be a vector space of dimension n over a field F and let $q' : V' \to F$ be a quadratic form on V'. Then, the Clifford algebra $C(V', q')$ for the quadratic space (V', q') is a quaternionlike algebra if and only if $n \leq 2$.*

Proof. For $n = 0$, the proposition trivially holds.

Consider $n \geq 1$.

Assume that $C(V',q')$ is a quaternionlike algebra . Therefore, there exists a decomposition

$$C(V',q') = Fe_0 \oplus W, \tag{3.2}$$

where W is a $(2^n - 1)$-dimensional vector subspace of $C(V',q')$ such that

$$w \bullet w \in Fe_0, \qquad \text{for every } w \in W. \tag{3.3}$$

Let $\{e'_1, \dots, e'_n\}$ be an orthogonal basis for (V',q'). We can prove that W is spanned by

$$\{e'^{\epsilon_1}_1 \bullet \dots \bullet e'^{\epsilon_n}_n;\ \epsilon_i = 0,1;\ i = 1,\dots,n;\ \sum_{i=1}^{n} \epsilon_i \neq 0\}. \tag{3.4}$$

Indeed, take any $x = e'^{\epsilon_1}_1 \bullet \dots \bullet e'^{\epsilon_i}_n$, $\epsilon_i = 1,0$; $i = 1,\dots,n$; $\sum_{i=1}^{n} \epsilon_i \neq 0$. According to (3.2) we can find $d \in F$ such that

$$x = de_0 + (x - de_0), \qquad \text{with } x - de_0 \in W. \tag{3.5}$$

By (3.3) and (3.5)

$$(x - de_0) \bullet (x - de_0) = x \bullet x - 2d + d^2 e_0 \in Fe_0. \tag{3.6}$$

However, (3.6) holds if and only if $d = 0$. Consequently $x \in W$ and W is spanned by the set (3.4).

Consider now the case $n \geq 3$

From the above obtained results one infers that the vector $e'_1 + e'_2 \bullet e'_3$ belongs to W. On the other hand

$$(e'_1 + e'_2 \bullet e'_3) \bullet (e'_1 + e'_2 \bullet e'_3) = [q'(e'_1) - q'(e'_2)q'(e'_3)]e_0 + 2e'_1 \bullet e'_2 \bullet e'_3 \notin Fe_0, \tag{3.7}$$

which contradicts (3.3). Therefore, the Clifford algebra $C(v', q')$ for dim $V' = n \geq 3$ is not a quaternionlike algebra.

Consider $n = 2$.

The vector subspace W of $C(V', q')$ as defined by (3.2) and (3.3) must be spanned by $\{e'_1, e'_2, e'_1 \bullet e'_2\}$. One can easily verify that any vector being a linear combination of $e'_1, e'_2, e'_1 \bullet e'_2$ really satisfies (3.3). Therefore, for $n = 2$, $C(V', q')$ is a quaternionlike algebra for any (V', q'). In th case of $n = 1$, the proof that $C(V', q')$ is a quaternionlike algebra for any (V', q') is trivial. Thus the proof of our proposition is complete. □

If dim $V' = 1$ then the structure of the Clifford algebra $C(V', q')$ is determined by

$$e'_1 \bullet e'_1 = q'(e'_1)e_0; \tag{3.8}$$

and it is evident that *any* quaternionlike algebra of dimension 2 over F is isomorphic to some 2-dimensional Clifford algebra over F.

Of course, a 1-dimensional quaternionlike algebra over F is Fe_0 and it is also a 1-dimensional Clifford algebra over F. Let now dim $V' = 2$. If $\{e'_1, e'_2\}$ is an orthogonal basis for (V', q') then $\{e'_1, e'_2, e'_1 \bullet e'_2\}$ is a basis for the vector space W defined by (3.2) and (3.3). Define $e'_3 \in W$ as follows, $e'_3 = e'_1 \bullet e'_2$. The structure of $C(V', q')$ is now determined by

$$\begin{aligned} &e'_1 \bullet e'_2 = q'(e'_1)e_0, \qquad e'_2 \bullet e'_2 = q'(e'_2)e_0, \qquad e'_3 \bullet e'_3 = -q(e'_1)q'(e'_2)e_0, \\ &e'_1 \bullet e'_2 = e'_3, \qquad e'_3 \bullet e'_1 = -q(e'_1)e_0, \qquad e'_2 \bullet e'_3 = -q'(e'_2)e_0. \end{aligned} \tag{3.9}$$

The problem which arises is, whether every quaternionlike algebra of dimension 4 over F is isomorphic to some 4-dimensional Clifford algebra over F.

In order to solve this problem we must explore the conditions (2.10) and (2.11). We intend to make it for an arbitrary $(n+1)$-dimensional $(n \geq 1)$ quaternionlike algebra Q over F and then we consider the case $n = 3$.

First, one easily finds that (2.11) together with (2.12) imply

$$a(ua(vw)) + a(wa(uv)) + a(va(wu)) = 0 \quad \text{for all } u, v, w \in V. \tag{3.10}$$

Notice that (3.10) is exactly the Jacobi identity in the Lie algebra $(V, \circ)$ (compare (2.22)). Then (2.11) is equivalent to

(3.11) $$b(uv)w - b(vw)u = a(va(uw)), \qquad \text{for all } u, v, w \in V.$$

Let $\{e_1, \ldots, e_n\}$ constitute a basis for V. Define scalars $b_{ij}, a^i_{jk} \in F$, $i, j, k = 1, \ldots, n$ by

(3.12) $$b(e_i e_j) = b_{ij} \quad i, j = 1, \ldots, n.$$

(3.13) $$a(e_j e_k) = \sum_{i=1}^{n} a^i_{jk} e_i, \qquad j, k = 1, \ldots, n.$$

From (2.12) and (2.13) it follows that

(3.14) $$a^i_{jk} = -a^i_{kj} \text{ and } b_{ij} = b_{ji}, \qquad i, j, k = 1, \ldots, n.$$

With (3.12), (3.13) and (3.14) we can write the conditions (3.11) and (2.10) in the form of

(3.15) $$\delta^i_j b_{k\ell} - \delta^i_\ell b_{kj} = \sum_{m=1}^{n} a^i_{km} a^m_{\ell j}, \qquad i, j, k, \ell = 1, \ldots, n,$$

and

(3.16) $$\sum_{\ell=1}^{n} (b_{i\ell} a^\ell_{jk} - b_{k\ell} a^\ell_{ij}) = 0, \qquad i, j, k = 1, \ldots, n,$$

where δ^i_j is the Kronecker delta in F.

Now, the conditions (3.15) are equivalent to the following ones:

$$b_{ij} = \sum_{m=1}^{n} a^k_{im} a^m_{jk} \quad i,j,k = 1,\dots,n, \ j \neq k, \tag{3.17}$$

$$\sum_{m=1}^{n} a^\ell_{im} a^m_{jk} = 0 \quad i,j,k,\ell = 1,\dots,n, \ \ell \neq k, \tag{3.18}$$

Then (3.17) implies

$$(n-1) b_{ij} = \sum_{m,k=1}^{n} a^k_{im} a^m_{jk}, \quad i,j = 1,\dots,n \tag{3.19}$$

Multiplying (3.15) by a^k_{ih} and summing up over indices k and i, using (3.16) and (3.19) we obtain

$$(n-3) \sum_{k=1}^{n} b_{hk} a^k_{j\ell} = 0, \quad h,j,\ell = 1,\dots,n. \tag{3.20}$$

Consider now the consequences of the above presented results for $n = 3$. Then one can put (see Ref. [6]):

$$a^i_{jk} = \sum_{\ell=1}^{3} [d^{i\ell} \in_{\ell jk} + f_\ell (\delta^\ell_j \delta^i_k - \delta^\ell_k \delta^i_j)], \quad i,j,k = 1,2,3, \tag{3.21}$$

where $d^{il} = d^{li}$ and f_l, $i,l = 1,2,3$, are elements in F and $\in_{ljk}, l,j,k = 1,2,3$, is the "totally antisymetric Levi-Civita symbol" in $F (\in_{123} = 1 \in F)$. From (3.19) and (3.21), by subtituting them into (3.15), one finds that the conditions (3.15) are satisfied if and only if

$$d^{il} = 0 \text{ or } f_\ell = 0, \quad i,\ell = 1,2,3. \tag{3.22}$$

Then the conditions (3.16) are satisfied for any d^{il} and f_l, $i,l = 1,2,3$. Assume $f_l = 0$, $l = 1,2,3$. In this case one can find a basis $\{e_1, e_2, e_3\}$ for V such that either

$$e_ie_j = 0 \quad i, j = 1, 2, 3, \tag{3.23}$$

the trivial case, or for some $d_1, d_2 \in F$

$$\begin{aligned} &e_1e_1 = -d_2e_0, \quad e_2e_2 = -d_1e_0, \quad e_3e_3 = -d_1d_2e_0, \\ &e_1e_2 = e_3, \qquad e_3e_1 = d_2e_2, \qquad e_2e_3 = d_1e_1. \end{aligned} \tag{3.24}$$

Assume now, $d^{i\ell} = 0$ $i, \ell = 1, 2, 3$, and, at least, one of scalars f_1, f_2, f_3 is non-zero. Then we can choose a basis $\{e_1, e_2, e_3\}$ for V so that

$$\begin{aligned} &e_1e_1 = 0, \quad e_2e_2 = 0, \quad e_3e_3 = 1e_0, \\ &e_1e_2 = 0, \quad e_3e_1 = e_1, \quad e_2e_3 = -e_2. \end{aligned} \tag{3.25}$$

The formulae (3.23), (3.24) and (3.25) present all strucutres for 4-dimensional quaternionlike algebras over a field F. Comparing these formulae with (3.9) we arrive at the following proposition:

PROPOSITION (3.2). *There exist only two non-isomorphic quaternionlike algebras of dimension 4 over a field F which are not isomorphic to any 4-dimensional Clifford algebra over F. The structures of these algebras are defined by (3.23) and (3.25).* □

4. The structures of associative algebras with conjugation of dimension ≠ 1,2,4

Let Q be $(n+1)$-dimensional $(n \neq 0, 1, 3)$ quaternionlike algebra over F which decomposes according to (2.1) and (2.2). We assume that the characteristic of F is neither 2 (as before) nor a divisor of $n-3$. Then from (3.20) one infers

$$\sum_{k=1}^{n} b_{hk}a^k_{j\ell} = 0; \qquad h, j, \ell = 1, \ldots, n. \tag{4.1}$$

Multiplying (3.15) by b_{hi} and summing up over index i, employing also (4.1) we obtain

(4.2) $$b_{hj}b_{\ell k} - b_{h\ell}b_{jk} = 0, \qquad h, j, k, \ell = 1, \dots, n.$$

The conditions (4.2) imply that there exists a basis $\{e_1, \dots, n\}$ for V such that the matrix $(b_{ij}, \ i, j = 1, \dots, n$, is of the following diagonal form

(4.3) $$(b_{ij} = \text{ diag } (0, \dots, b_n), \qquad b_n \in F.$$

First consider the case:

(4.4) $$(b_n = 0 \Longleftrightarrow b(uv) = 0 \qquad \text{for any } u, v \in V.$$

With the use of (4.4), (3.15) gives $\sum\limits_{m=1}^{n} a^i_{km} a^m_{\ell j} = 0, \ i, j, k, \ell = 1, \dots, n$. Thus we have: the trivial case, i.e.,

(4.5) $$a^i_{jk} = 0, \qquad i, j, k = 1, \dots, n,$$

or, there exist $f, g, h \in \{1, \dots, n\}$ such that

(4.6) $$a^f_{gh} \neq 0$$

but

(4.7) $$\sum_{m=1}^{n} a^i_{km} a^m_{\ell j} = 0, \qquad i, j, k, \ell = 1, \dots, n.$$

In terms of Lie algebra $(V, \circ)$ the trivial case (4.4), (4.5), corresponds to Abelian $(V, \circ)$; the case defined by (4.4), (4.6) and (4.7) corresponds to nilpotent Lie algebra $(V, \circ)$ of the nilpotency index 2.

Consider now

$$b_n \neq 0. \tag{4.8}$$

Similary as in Ref. [6] one can prove that, with (4.3) and (4.8) the following conditions hold

$$a^n_{ni} = 0, \quad a^n_{ij} = 0, \quad a^i_{jk} = 0, \qquad i, j, k, = 1, \dots, n-1,$$

$$\sum_{k=1}^{n-1} a^i_{nk} a^k_{nj} = b_n \delta^i_j, \qquad i, j = 1, \dots, n-1. \tag{4.9}$$

Define the matrix A of order $n-1$

$$A = (a^i_{nj}), \qquad i, j = 1, \dots, n-1. \tag{4.10}$$

Then the last formula of (4.9) can be written in the matrix form

$$A^2 = b_n I_{n-1}, \tag{4.11}$$

where I_{n-1} is the unity matrix of order $n-1$. From the general theory of canonical forms of matrices (see [9]) it follows that if a matrix A satisfies (4.11) then A is similar to the matrix of the following form

$$A' = \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_r \end{pmatrix}$$

where matices $A_\nu, \nu = 1, \dots, r$ take the form of

$$A_\nu = \begin{pmatrix} 0 & 1 \\ b_n & 0 \end{pmatrix} : \text{ or } A_\nu = (d) \text{ with } d^2 = b_n; \nu = 1, \dots, r. \tag{4.12}$$

We must consider two cases. First, assume that there exists a scalar $s \in F$ such that

$$s^2 = b_n. \tag{4.13}$$

Then one can choose a vector e_n so that

$$b_n = 1. \tag{4.14}$$

Substituting (4.14) into the first set of matrices of (4.12) we have

$$A_\nu = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ which is similar to the matrix } \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Concluding, if there exists a scalar $s \in F$ satisfying (4.13) then also exists a basis $\{e_1, \dots, e_n\}$ for V such that (4.3) holds with $b_n = 1$ and the matrix A defined by (4.10) is of the form

$$A = \begin{pmatrix} \epsilon_1 & & & 0 \\ & \epsilon_2 & & \\ & & \ddots & \\ 0 & & & \epsilon_n \end{pmatrix}, \epsilon_1 = \pm 1, i = 1, \dots, n-1. \tag{4.15}$$

Let now b_n be such that the equation (4.13) doesn't have solution in F. From the form of the matrix A' and (4.12) one infers that this can only occur if $n - 1$ is even. Then the matrices A_ν in (4.12) take the form of

$$A_\nu = \begin{pmatrix} 0 & 1 \\ b_n & 0 \end{pmatrix}, \quad \nu = 1, \dots, \frac{1}{2}(n-1). \tag{4.16}$$

Therefore, one can choose an independent set of vectors $\{e_1, \dots e_{n-1}\}$ such that the matrix A defined by (4.10) takes the form of

$$A = \begin{pmatrix} 0 & I_{\frac{1}{2}(n-1)} \\ b_n I_{\frac{1}{2}(n-1)} & 0 \end{pmatrix}, \tag{4.17}$$

where $I_{\frac{1}{2}(n-1)}$ is the unity matrix of order $\frac{1}{2}(n-1)$.

In terms of Lie algebra $(V, \circ)$ the case of (4.8) corresponds to solvable but non-nilpotent $(V, \circ)$.

Thus, for $n \neq 0, 1, 3$ and for a field F of the characteristics $\neq 2$ and a divisor of $n - 3$, an $(n+1)$-dimensional quaternionlike algebra over F is defined by one of the following conditions:

(i) (4.4) and (4.5)

(ii) (4.4), (4.6) and (4.7)

(iii) (4.3) with (4.14); (4.9) and (4.15) with (4.10),

(iv) (4.3) with $b_n \in F$ such that Eq. (4.13) does not have solution in F; (4.9) and (4.17) with (4.10)

Remark: One can easily verify that the 4-dimensional quaternionlike algebra (3.23) is of type (i), the algebra (3.25) is of type (iii), the algebra (3.24) with $d_1 = d_2 = 0$ is of type (ii) and the algebra (3.24) with $d_1 = 0,\ d_2 \neq 0 (d_1 \neq 0,\ d_2 = 0)$ is of type (iii) or (iv).

For convenience, in our considerations we have dealt with quaternionlike algebras instead of associative algebras with conjugation. It is evident from Sec. 2 that these considerations concern quaternionlike algebras as well as associative algebras with conjugation.

Centro de Investigación y de Estudios Avanzados del IPN
México, D.F., México 07000

References

[1] A.A. Albert, *Quadratic forms permitting composition,* Ann. of Math. **43** (1942), 161-77.

[2] R.D. Schafer, *On the algebras formed by the Cayley-Dickson process,* Amer. J. Math. **76** (1954), 435-46.

[3] ———, An introduction to nonassociative algebras, Academic Press, New York-London, Chap. III, 1966.

[4] J. Adem, *Construction of some normed maps,* Bol. Soc. Mat. Mexicana, **20** (1975), 59-75.

[5] L.E. Dickson, *On quaternions and their generalization and the history of the eight square theorem*, Ann. of Math. **20** (1919), 155-71.

[6] J.F. PLEBAŃSKI AND M. PRZANOWSKI, *Generalization of the quaternion algebra and Lie algebras,* J. Math. Phys. **29** (1988), 529-35.
[7] ——— ———, *Notes on cross product of vectors,* J. Math. Phys. **29** (1988), 2234-37.
[8] T. Y. LAM, *The algebraic theory of quadratic forms,* W. A. Benjamin, Inc., Massachusetts, 1973.
[9] R. M. THRALL AND L. TORENHEIM, Vector spaces and matrices, John Wiley & Sons, Inc., New York, London, 1962 (Chapter 10).