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RESIDUAL CIRCUITS IN GENERALIZED MONOIDAL TRANSFORMATIONS

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Introduction

In this paper we outline a topological construction which may be applied to compute the characteristic classes of (possibly a sequence of) generalized monoidal transformations, via the characteristic ring of a principal G-bundle. Our construction may be compared with a result of I. R. Porteous [5] for computing the Chern classes (see also the generalization in [1]). The setting and spirit of this paper is similar to that of [3].

1. Generalized monoidal transforms and the residual circuit

Let G be any topological group admitting a classifying space BG. Our setting is that of simplicial complexes and since the notion of a fundamental class is required, we will restrict attention to oriented n-circuits. Indeed, we keep in mind those n-circuits which are n (complex) dimensional algebraic varieties (possibly singular) with their natural orientation arising from their complex structures and their appropriate triangulations as simplicial complexes. General references are [2] and [6].

Let M and N be oriented n-circuits and $[M] \in H_n(M, \mathbf{Z}) \cong \mathbf{Z}$ and $[N] \in H_n(M, \mathbf{Z}) \cong \mathbf{Z}$ their respective fundamental classes. Let $K \subset M$ and $L \subset N$ be subcomplexes of M and N respectively (K and L are not necessarily circuits of lower dimension). We consider maps of the following type:
Let $\psi : (M, K) \to (N, L)$ be a simplicial map of pairs such that

i) $\psi^{-1}(L) = K$

ii) $\psi : M - K \to N - L$ is a homeomorphism

By abuse of notation we shall use the same notation to denote a restriction where the context is understood. Such maps will be called *generalized monoidal transformations* (or *gmt* for short). When L is a point, K is seen to play the role of the exceptional divisor in the classical "blowing up" process (see below).

Let $\mathcal{U}$ be a closed regular neighborhood of K obtained via a barycentric subdivision and let $\mathcal{V} = \psi(\mathcal{U})$. Then $\psi : \mathcal{U} - K \to \mathcal{V} - L$ is a homeomorphism.

PROPOSITION (1.1). *The* **Z**-*cohomology of a generalized monoidal transformation satisfies*

$$H^\star(M) \cong \psi^\star H^\star(N) \oplus H^\star(K)/\psi^\star H^\star(L).$$

Proof: Let $\mathcal{V}^\star = \mathcal{V} - L$, $\mathcal{U}^\star = \mathcal{U} - K$, $N^\star = N - L$ and $M^\star = M - K$. We apply the Mayer-Victoris sequence (with $\mathbf{Z}$ coefficients), noting that $N = N^\star \cup \mathcal{V}$, $M = M^\star \cup \mathcal{U}$, and that we have contraction mappings $H^\star(\mathcal{V}) \xrightarrow{\cong} H\star(L)$ and $H\star(\mathcal{U}) \xrightarrow{\cong} H\star(K)$.

The sequence gives

$$\begin{array}{ccccccc} H^{i-1}(\mathcal{V}^\star) & \longrightarrow & H^i(M) & \longrightarrow & H^i(N^\star)\oplus H^i(K) & \longrightarrow & H^i(\mathcal{V}^\star) \\ \| & & \big\downarrow \psi^* & & & & \\ H^{i-1}(\mathcal{V}^\star) & \longrightarrow & H^i(N) & \longrightarrow & H^i(N\star)\oplus H^i(L) & \longrightarrow & H^i(\mathcal{V}^\star) \end{array}$$

The result follows from the injectivity of $\psi^\star$. □

Now let $W = W(\psi)$ be the n-circuit obtained as follows: let $\mathcal{U}\vee\mathcal{V}$ be disjoint union and identify $x \in \partial\mathcal{U}$ with $\psi(x) \in \partial\mathcal{V}$.

Definition (1.2). $W = \mathcal{U} \vee \mathcal{V}/x \sim \psi(x)$ is the *residual n-circuit* of ψ.

W is well defined and is an n-circuit since ψ is simplicial and it is a homeomorphis between $\partial\mathcal{U}$ and $\partial\mathcal{V}$. Also, since any two regular neighborhoods are isotopic, it follows that any such neighborhoods give rise to homeomorphic W's.

To exemplify, let $\pi : \tilde{\mathbf{C}}^n \to \mathbf{C}^n$ be the blow up of $\mathbf{C}^n$ at the origin. Then $\tilde{\mathbf{C}}^n$ is a complex manifold and π is an analytic map. Furthermore, $\pi : \tilde{\mathbf{C}}^n - \pi^{-1}\{0\} \to \mathbf{C}^n - \{0\}$ is a biholomorphism and $\pi^{-1}\{0\} \cong \mathbf{C}P^{n-1} \overset{\text{def}}{=} D$, the exceptional divisor. It is clear that π is a *gmt*: $\psi = \pi$, $M = \tilde{\mathbf{C}}^n$, $N = \mathbf{C}^n$, $k = D$, $L = \{0\}$. The corresponding residual n-circuit W is homeomorphic to $\mathbf{C}P^n$.

Let $\mathcal{U}$ = closed unit ball in $\mathbf{C}^n$ and $\mathcal{U} = \pi^{-1}(\mathcal{V})$. Then $\mathcal{U}$ is a closed tubular neighborhood of the exceptional divisor D and its interior is complex analytically equivalent to the unit disc bundle of the Hopf bundle $\eta : E(\eta) \to \mathbf{C}P^{n-1}$ and $\partial\mathcal{V} \cong S^{2n-1}$. Since W is obtained by attaching a $2n$-closed disc along the boundary, we see that $W \cong \mathbf{C}P^n$.

We proceed to describe the fibered version of the above blow up, or the monoidal transformation (*mt*) following [4]: Let X be a non-singular compact algebraic variety (or more generally, a compact complex manifold) with $\dim_{\mathbf{C}} X = n$. Let $Y \subset X$ be a non-singular subvariety, $\operatorname{codim}_{\mathbf{C}} Y = q$. Then the *mt* of X along Y is given by:

i) $\pi : X' \to X$ is an analytic map of complex manifolds,

ii) $\pi^{-1}(Y) \overset{\text{def}}{=} Y'$ is a non-singular subvariety of X'.

iii) $\pi : Y' \to Y$ is a complex locally trivial fibration with fiber $\mathbf{C}P^{q-1}$, and is complex analytically equivalent to the associated projectivization of the normal bundle of Y in X.

iv) $\pi : X' - Y' \to X - Y$ is a biholomorphism.

Let $\mathcal{V}$ be a closed tubular neighborhood of $Y \subset X$ and let $\mathcal{U} = \pi^{-1}(\mathcal{V})$. Then $W = \mathcal{U} \vee \mathcal{V}/x \in \partial\mathcal{U} \sim \pi(x) \in \partial\mathcal{V}$ can be given the structure of a compact algebraic variety which is a $\mathbf{C}P^q$ bundle over Y such that there exists

a) a complex structure on W for which $\overline{\pi} : W \to Y$ is complex analytic

b) $\overline{\pi}$ is a locally-trivial fibration with fibre $\mathbf{C}P^q$.

2. Classifying maps and cobordism

The structural group for (generally continuous) locally trivial fibrations with fibre CP^m will be the complex Lie group of automorphisms of CP^m i.e. the complex Lie group of projective transformations:

$$PGL(m, \mathbf{C}) \cong GL(m+1, \mathbf{C})/\mathbf{C}^\star.$$

Let $H \subset \mathbf{C}P^m$ be a hyperplane and $G \subset PGL(m, \mathbf{C})$ be the subgroup stabilizing H. We have $G \cong GL(m, \mathbf{C})$, since we can always regard H as the hyperplane at infinity. Let $j : G \hookrightarrow PGL(m, \mathbf{C})$ and $j^\star : BG \to BPGL(m, \mathbf{C})$ be the induced map between classifying spaces. Let $f : Y \to BG = BGL(q, \mathbf{C})$ be the classifying map of the normal bundle of Y in X. Now we add to a) and b) above the property:
c) the classifying map of the $\mathbf{C}P^q$ fibration $\overline{\pi} : W \to Y$ is $\overline{f} : Y \to BPGL(m, \mathbf{C})$ with

$$\overline{f} = j^\star f.$$

All of the above conditions are obtained by blowing up at each normal fibre and forming the residual circuit. In this case, W happens to be isomorphic to the $\mathbf{C}P^q$ bundle obtained from the normal bundle of the submanifold $Y \subset X$ by ading to each fibre a hyperplane at infinity.

Definition (2.1). Let $\pi : (X', Y') \to (X, Y)$ be a monoidal transformation of X along Y. Let W, and $\overline{\pi} : W \to Y$ be described as above. Then W is called the *residual manifold* and $\overline{\pi}$ the *canonical fibration*.

REMARKS

1. We see that W and π only depend on the normal bundle of $Y \subset X$. Note that W has a natural orientation.
2. Then term "residual" seems a natural one to adopt since the results to be proven are related to de Rham theory (via differential forms representing characteristic classes.)

Following the dictum that some of the best ideas are amongst the simplest to comprenhed, we shall state and prove the following simple (but useful):

PROPOSITION (2.2). *Let (π, X', X, Y) be a monoidal transformation of X along Y, with residual manifold W. Then with respect to the natural orientations of X', X and W, $X + W$ is (oriented) cobordant to X'.*

Proof: Let $V_1 = X' \times [0,1]$, $V_2 = X \times [0,1]$ and $\mathcal{V}^0$ an open tubular neighborhood of $Y \subset X$ with $\mathcal{V}^0 = \pi(\mathcal{U}^0)$. We consider $V_1 \vee V_2/\sim$ where the identification is

$$(x, 0) \in (X \times \{0\}) - (\mathcal{V}^0 \times \{0\}) \sim (\pi(x), 1) \in (X' \times \{1\}) - (\mathcal{U}^0 \times \{1\}).$$

Let $S = V_1 \vee V_2/\sim$. The on smoothing the corners created by the identification, S is an oriented manifold whose oriented boundary is diffeomorphic to $X + W - X'$. □

Note that there are canonical (smooth) embeddings of each of X, X' and W in S.

COROLLARY (2.3). *Let $\tau[M]$ denote the index (or signature) of a closed oriented manifold. Then*

$$\tau(X) + \tau(W) = \tau(X').$$

The corollary essentially follows from the fact that any additive cobordism invariant i satisfies $i(X) + i(W) = i(X')$.

3. The main result

The above construction of S motivates the following

Definition (3.1). Let us call S the *canonical cobordism* of the monoidal transformation (π, X', X, Y). Now as a manifold S is relatively simple to describe. Effectively, if G is any topological group admitting a classifying space BG, let $g : X \to BG$ and $g' : X' \to BG$ be the classifying maps of two principal G-bundles over X and X' respectively. Let $n = \dim_{\mathbf{C}} X = \dim_{\mathbf{C}} X'$.

Suppose that

$$g'|_{X'-\mathcal{U}} : X' - \mathcal{U} \to BG$$

is homotopic to

$$g \circ (\pi|_{X'-\mathcal{U}^0}) : X' - \mathcal{U}^0 \to BG$$

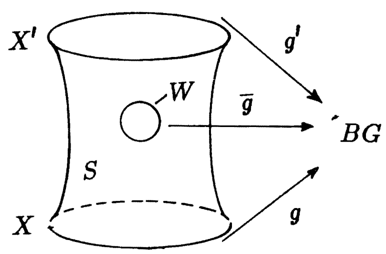
where $\mathcal{U}^0$ is an open tubular neighborhood of $Y' \subset X$. Then we have

THEOREM (3.2). *There exists a map $\overline{g} : S \to BG$ such that*

i) *$\overline{g}|_{\overline{X}}$ is homotopic to g and $\overline{g}|_{\overline{X'}}$ is homotopic to g'.*

ii) *Any two maps $\overline{g}_1 : S \to BG$ and $\overline{g}_2 : S \to BG$ which satisfy* i) *have the property that $[\overline{g}_1|_W] = [\overline{g}_2|_W]$ where $[\overline{g}_1|_W]$ and $[\overline{g}_2|_W] \in H_{2n}(BG, \mathbf{Z})$ are the images under $\overline{g}_*$ of the fundamental class of W.*

Before proceeding to the proof we remark that with some abuse of notation, we have identified ∂S with the disjoint union $X \vee X' \vee W$. See figure



Proof of Theorem 3.2: S is homeomorphic to the space obtained from the disjoint union

$$(X \times [0,1/3]) \vee ((X' - \mathcal{U}^0) \times [1/3,2/3]) \vee (X' \times [2/3,1])$$

with the obvious identifications:

$$(x,1/3) \sim (x',1/3), \quad x \in X \text{ and } x' \in X - \mathcal{U}^0 \text{ and}$$
$$(\pi(x),2/3) \sim (x,2/3), \quad x \in X' - \mathcal{U}^0(\pi(\mathcal{U}^0) = \mathcal{V}^0).$$

Let F be the above disjoint union of the sets $A = X \times [0,1/3]$, $B = (X - \mathcal{V}^0) \times [1/3,2/3]$ and $C = X' \times [2/3,1]$ modulo the above identifications.

Define
a) $\overline{g} : A \to BG$ by $\overline{g}(x,t) = g(x)$, $x \in X$, $t \in [0,1/3]$.
b) $\overline{g} : B \to BG$ by $\overline{g}(x,t) = h_t(x)$, $x \in X - \mathcal{V}^0$, $t \in [1/3,2/3]$ where $h_t : X' - \mathcal{U}^0 \to BG$ is a homotopy parametrized by $[1/3,2/3]$, between $g' : X' - \mathcal{U}^0 \to BG$ and $g \circ (\pi|_{X'-\mathcal{U}^0})$.
c) $\overline{g} : F \to BG$ by $\overline{g}(x,t) = g'(x)$, $x \in X'$, $t \in [2/3,1]$.

Then $\overline{g} : G \to BG$ is well defined and continuous and satisfied the above hypotheses. Evidently, F is homeomorphic to S, verifying the existence of $\overline{g}$. Now let $\overline{g}_1$ and $\overline{g}_2$ be two maps from S into BG satisfying i). Then $(\overline{g}_1)_\star[W] = (\overline{g}_2)_\star[W]$ where $[W] \in H_{2n}(W,\mathbf{Z})$ is the fundamental class. This is clear since

$$\begin{aligned}(\overline{g}_1)_\star[W] &= (\overline{g}_1)_\star[X'] - (\overline{g}_1)[X] \\ &= (g')_\star[X'] - (g)_\star[X]\end{aligned}$$

and likewise

$$(\overline{g}_2)_\star[W] = (g')_\star[X'] - (g_\star)[X]$$

(note that S determines the homology between such classes). □

REMARK

It is not true in general that $\overline{g}_1|_W$ is homotopic with $\overline{g}_2|_W$. Hence the principal G-bundles induced by these maps may not necessarily be isomorphic.

COROLLARY (3.3). *Under the hypothesis of Theorem (3.2), there exists a family of principal G-bundles over W such that if $f_1 : W \to BG$ and $f_2 : W \to BG$ are the classifying maps, then*

$$(f_1)_\star[W] = (f_2)_\star[W] \in H_{2n}(BG,\mathbf{Z}).$$

Let us denote this common element above, by $\alpha H_{2n}(BG,\mathbf{Z})$ which we name the *residual homology class* of the monoidal transformation. Clearly, we have

$$\alpha = g_\star[W] - {g'}_\star[X].$$

From Theorem (3.2) we derive the main formula of this paper.

COROLLARY (3.3). *Let $\beta \in H^{2m}(BG, \mathbf{Z})$ be any characteristic class. Then under the hypotheses of Theorem* (3.2), *we have*

$$< (g')^{\star}(\beta), [X'] > - < (g)^{\star}(\beta), [X] > = < \beta, \alpha >$$

where $< , >$ denotes Kronecker pairing

4. Applications

Let (π, X', X, Y) be an *mt* as above and let $g : X \to BGL(n, \mathbf{C})$, $g' : X' \to BGL(n, \mathbf{C})$ be the classifying maps associated with the respective tangent bundles TX and TX'. Then the formula of Corollary (3.3) gives a relation between characteristic numbers in the top dimension of X, X' and Y.

If we now consider a sequence of *mt*'s

$$X^{(n)} \xrightarrow{\pi^{(n)}} X^{(n-1)} \xrightarrow{\pi^{(m-1)}} \cdots \longrightarrow X^{1} \xrightarrow{\pi} X, \quad (X^{1} = X')$$

then we have a straight forward extension of the preceding ideas:

PROPOSITION (4.1). *For a sequence of monoidal transformations there exists an oriented manifold S such that its oriented boundary ∂S satisfies*

$$\partial S = X + W_1 + \cdots + W_n - X^{(n)}$$

where each residual manifold W_i, $1 \le i \le n$, has a natural complex structure and is a locally trivial complex fibration $\pi_i : Wi \to Y_i$ of complex manifolds whose fibre is a complex projective space. We have $Y_i \subset X^{(i)}$ and W_i is a $\mathbf{C}P^q$-bundle over Y_i where $q = codim_{\mathbf{C}} Y_i$, that is totally determined by the normal bundle of Y_i in $X^{(i)}$.

THEOREM (4.2). *Let G be any topological group admitting a classifying space BG. Let $g^{(n)} : X^{(n)} \to BG$ and $g : X \to BG$ be the classifyning maps of a sequence of monoidal transformations, such that $g^{(n)}$ and g determine isomorphic principal G-bundles outside of the divisor of the sequence with $\pi = \pi^{(n)} \circ \cdots \circ \pi^{(1)}$, $\pi : X^{(n)} \to X$. Then there exists $\alpha \in H_{2n}(BG, \mathbf{Z})$ such that for any $\beta \in H^{2n}(BG, \mathbf{Z})$, we have*

$$< (g^{(n)})^{\star}(\beta), \quad [X^{(n)}] > - < g^{\star}(\beta), [X] > = < \beta, \alpha > .$$

Now the previous results generalize to *gmt* as follows: In the notation of §1, let $\psi : (M, K) \to (N, L)$ be a *gmt* and let S be obtained from the disjoint union $(M \times [0,1]) \vee (N \times [0,1])$ by identifying $(x, 1) \sim (\psi(x), 0)$ for $x \in M - \mathcal{U}$. Then S is an $(n+1)$-circuit with boundary $N + W - M$. Then we may proceed almost exactly as before to establish:

THEOREM (4.3). *In the above notation let G be a topological group admitting a classifying space BG. Let $g : N \to BG$ and $g' : M \to BG$ be classifying maps* *of principal G-bundles. Assume that* $g'|_{M-\mathcal{U}}$ *is homotopic to* $(g \circ (\psi|_{M-\mathcal{U}})$. *Then*

a) *there exists a classifying map* $\overline{g} : S \to BG$ *such that*

i) $\overline{g}|_N = g$

ii) $\overline{g}|_M = g'$;

b) *any two such* $\overline{g}_1 : S \to BG$ *and* $\overline{g}_2 : S \to BG$ *satisfying* i) *have the property* $(g_1)_*[W] = (\overline{g}_2)_*[W]$

c) *there exists a well defined class* $\alpha \in H_n(BG, \mathbf{Z})$ *such that for any* $\beta \in H^{2n}(BG, \mathbf{Z})$ *we have*

$$< (g')^*(\beta), [M] > - < (g)^*(\beta), [N] >=< \beta, \alpha > .$$

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