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ON QUASI-METRIZATION OF BITOPOLOGICAL SPACES

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Summary

We formulate the sufficient conditions under which for a bitopological space (X, τ_1, τ_2) there exist conjugate quasi-pseudo-metrics p and q inducing the given topologies τ_1 and τ_2 respectively. The main result is presented in Theorem 6; this generalizes the result of Kelly (Corollary 8).

A function $p : X \times X \to [0, \infty)$ is called quasi-pseudo-metric on a set X if $p(x, x) = 0$ for $x \in X$, and $p(x, y) \leq p(x, z) + p(z, y)$ for $x, y, z \in X$. If p satisfies the condition $p(x, y) = 0$ if and only if $x = y$, then p is said to be quasi-metric [3].

The function q given by $q(x, y) = p(y, x)$ is also quasi-pseudo-metric; so we say that p and q are conjugate. Let us put

$$K(x, r, p) = \{y \in X : p(x, y) < r\},$$
$$K(x, r, q) = \{y \in X : q(x, y) < r\}.$$

Then the families $\{K(x, r, p) : x \in X; r > 0\}$ and $\{K(x, r, q) : x \in X, r > 0\}$ form bases of some topologies $\mathcal{P}$ and $\mathcal{Q}$ on X. For any set $A \subset X$ we use the symbols $\mathcal{P}$-cl(A) and $\mathcal{Q}$-cl(A) to denote the $\mathcal{P}$-closure or $\mathcal{Q}$-closure of A respectively. Moreover let us put $p(x, A) = \inf\{p(x, a) : a \in A\}$, $q(x, A) = \inf\{q(x, a) : a \in A\}$. Then we have $\mathcal{P}$-cl$(A) = \{x \in X : p(x, A) = 0\}$ and $\mathcal{Q}$-cl$(A) = \{x \in X : q(x, A) = 0\}$, [3].

Now let (X, τ_1, τ_2) be a bitopological space, i.e. the set X with two given topologies.

LEMMA (1). *Let (X, τ_1, τ_2) be a bitopological space, p any quasi-pseudo-metric on X and $A \subset X$. If $p(x, \cdot) : X \to [0, \infty)$ is a τ_1-upper semicontinuous function for every $x \in X$, then the function $p(\cdot, A) : X \to [0, \infty)$ is τ_1-lower semicontinuous. If $p(\cdot, y) : X \to [0, \infty)$ is τ_2-upper semicontinuous for every $y \in X$, then $q(\cdot, A) : X \to [0, \infty)$ is τ_2-lower semicontinuous, where $q(x, y) = p(y, x)$.*

The proof of this Lemma is elementary, so it is omitted.

In a bitopological space (X, τ_1, τ_2), the topology τ_1 is said to be regular with respect to τ_j if for every τ_i-open set $U \subset X$ and every $x \in U$ there exists $U_1 \in \tau_i$ such that $x \in U_1 \subset \tau_j - \text{cl}(U_1) \subset U$. A bitopological space (X, τ_1, τ_2) is said to be pairwise regular if τ_i is regular with respect to τ_j for $i, j \in \{1, 2\}, i \neq j$. [3]. (X, τ_1, τ_2) is called pairwise normal if for any τ_1-closed set A and τ_2-closed set B with $A \cap B = \emptyset$ there exist disjoint sets $U \in \tau_1$, $V \in \tau_2$ such that $A \subset V$ and $B \subset U$, [3].

LEMMA (2). [3, Th. 2.7]. *Let (X, τ_1, τ_2) be a pairwise normal bitopological space. Then for any τ_2-closed set A and τ_1-closed set B with $A \cap B = \emptyset$ there* *exists a function* $g : X \to [0,1]$ *which is* τ_1*-upper semicontinuous and* τ_2*-lower semicontinuous such that* $g(x) = 0$ *for* $x \in A$ *and* $g(x) = 1$ *for* $x \in B$.

LEMMA (3). [2, Lemma 2.3]. *A bitopological space* (X, τ_1, τ_2) *is pairwise normal if and only if for every* τ_1*-closed set M and* τ_j*-open set W such that* $M \subset W$, $i, j \in \{1,2\}$, $i \neq j$, *there exists a sequence* $\{W_n : n \geq 1\}$ *of* τ_j*-open sets such that* $M \subset \bigcup_{n=1}^{\infty} W_n$ and $\tau_1 - \text{cl}(W_n) \subset W$ *for* $n \geq 1$.

LEMMA (4). *Let* (X, τ_1, τ_2) *be a pairwise regular bitopological space. If* τ_1 *has a* τ_i*-open base which is* σ*-locally finite in* (X, τ_j) *for* $i, j \in \{1,2\}$, $i \neq j$, *then* (X, τ_1, τ_2) *is pairwise normal.*

Proof. Let M be a τ_i-closed set, $W \in \tau_j$ and let $M \subset W$. By $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n \subset \tau_j$ we denote a base of τ_j such that $\mathcal{B}_n$ is τ_i-locally finite for $n \geq 1$. For every point $x \in M$ we choose $n = n(x) \geq 1$ and $U(x) \in \mathcal{B}_{n(x)}$ such that $x \in U(x) \subset \tau_1 - \text{cl}(U(x)) \subset W$. Let us put $W_n = \cup\{U(x) : n(x) = n\}$. Then we have $W_n \in \tau_j$, $\tau_i - \text{cl}(W_n) = \cup\{\tau_i - \text{cl}(U(x)) : n(x) = n\}$, $M \subset \bigcup_{n=1}^{\infty} W_n$ and $\tau_i - \text{cl}(W_n) \subset W$ for $n \geq 1$. Thus from Lemma 3, (X, τ_1, τ_2) is pairwise normal.

LEMMA (5). *Assume that* (X, τ_1, τ_2) *is a bitopological space and* $\{p_n : n \geq 1\}$ *is a sequence of quasi-pseudo-metrics on X which all are bounded by* 1. *Let*

$$p(x,y) = \sum_{n=1}^{\infty} 2^{-n} p_n(x,y)$$

$$q_n(x,y) = p_n(y,x) \text{ and } q(x,y) = p(y,x).$$

If the following two conditions (1), (2) *are satisfied*

(1) $p_n(x, \cdot) : X \to [0,1]$ *is* τ_1*-upper semicontinuous and* τ_2*-lower semicontinuous for* $x \in X$, $n \geq 1$;

(2) *for each* τ_1*-clossed set* $A \subset X$ *and a point* $x \notin A$ *there exists* $n \geq 1$ *such that* $p_n(x, A) > 0$;

then the topology $\mathcal{P}$ *induced by the quasi-pseudo-metric p coincides with* τ_1.

If the following two conditions (3), (4) *are satisfied*

(3) $p_n(\cdot, y) : X \to [0,1]$ *is* τ_1*-lower semicontinuous and* τ_2*-upper semicontinuous for* $y \in X$, $n \geq 1$;

(4) *for each* τ_2*-closed set* $B \subset X$ *and a point* $y \notin B$ *there exists* $n \geq 1$ *such that* $q_n(y, B) > 0$;

then the topology $\mathcal{Q}$ *induced by q coincides with* τ_2.

Moreover, if conditions (1), (2) (or (3), (4)) *hold and both* τ_1 *and* τ_2 *are* T_1*-topologies, then p and q are quasi-metrics inducing* τ_1 (resp. τ_2).

Proof. It is evident that p and q are quasi-pseudo-metrics. Moreover, if (1) (or (3)) is satisfied, then $p(x, \cdot); X \to [0,1]$ is τ_1-upper semicontinuous τ_2-lower semicontinuous for $x \in X$ (resp. $p(\cdot, y) : X \to [0,1]$ is τ_1-lower semicontinuous τ_2-upper semicontinuous).

Assume that (1) and (2) hold and let $x \in \tau_1-\text{cl}(A)$. Then for some $n \geq 1$ we have $p_n(x, \tau_1 - \text{cl}(A)) = r > 0$. Hence

$$p(x, A) \geq p(x, \tau_1 - \text{cl}(A)) \geq 2^{-n} p_n(x, \tau_1 - \text{cl}(A)) = 2^{-n}\ r > 0.$$

So we have shown that the condition $p(x, A) = 0$ implies $x \in \tau_1-\text{cl}(A)$. According to Lemma 1 the function $p(\cdot, A) : X \to [0, 1]$ is τ_1-lower semicontinuous. Let us put $\mathcal{T} = \{(a, 1] : 0 \leq a < 1\} \cup \{\emptyset, [0, 1]\}$ and $f(x) = p(x, A)$; then $f : (X, \tau_1) \to ([0, 1], \mathcal{T},)$ is a continuous function. For a point $x \in \tau_1-\text{cl}(A)$ we have $f(x) \in f(\tau_1 - \text{cl}(A)) \subset \mathcal{T} - \text{cl}(f(A))$. Because $f(A) = \{p(z, A) : z \in A\} = \{0\}$ and $\mathcal{T}-\text{cl}(\{0\}) = \cap\{[0, \epsilon] : 0 < \epsilon \leq 1\} = \{0\}$ we obtain $f(x) = 0$. Thus from the condition $x \in \tau_1 - \text{cl}(A)$ it follows $p(x, A) = 0$. Consequently $x \in \tau_1 - \text{cl}(A)$ if and only if $p(x, A) = 0$. It means that the topology $\mathcal{P}$ induced by p coincides with τ_1.

Applying these same arguments we can prove that under assumptions (3), (4) the topology $\mathcal{Q}$ induced by q is equal to τ_2.

Now let (1), (2) be satisfied and let both τ_1, τ_2 be T_1-topologies. For any points $x, y \in X$, $x \neq y$, we have $x \notin \tau_1 - \text{cl}(\{y\}) = \{y\}$; so for some $n \geq 1$ it holds $p_n(x, y) = p_n(x, \{y\}) > 0$. Consequently $p(x, y) > 0$, and that finishes the proof.

THEOREM (6). *Let (X, τ_1, τ_2) be a pairwise regular bitopological space such that τ_i has a τ_i-open base which is σ-locally finite in (X, τ_j) for $i, j \in \{1, 2\}$ $i \neq j$. Then there exist conjugate quasi-pseudo-metrics p,q inducing the given topologies τ_1, τ_2. Moreover, if both τ_1, τ_2 are T_1-topologies, then p, q are quasi-metrics.*

Proof. We use a method analogous to that in the proof of the Nagata - Smirnov metrization theorem [1, p. 351]. Let $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n \subset \tau_1$ and $\mathcal{B}' = \bigcup_{n=1}^{\infty} \mathcal{B}'_n \subset \tau_2$ be bases of τ_1 and τ_2 such that $\mathcal{B}'_n$ is τ_1-locally finite and $\mathcal{B}_n$ is τ_2-locally finite for $n \geq 1$. Let $\mathcal{B}_n = \{U_s : s \in S_n\}$. For each pair (m, n) of natural numbers and each $s \in S_n$ we denote $V_s = \cup\{U' \in \mathcal{B}_m : \tau_2-\text{cl}(U') \subset U_s\}$. Since $\mathcal{B}_m$ is τ_2-locally finite we obtain $\tau_2 - \text{cl}(V_s) \subset U_s$. From Lemma 4 (X, τ_1, τ_2) is pairwise normal; so according to Lemma 2 there exists a function $f_s : X \to [0, 1]$ which is τ_1-lower semicontinuous τ_2-upper semicontinuous and such that $f_s(x) = 0$ for $x \in X\setminus U_s$ and $f_s(x) = 1$ for $x \in \tau_2-\text{cl}(V_s)$. For every point $x \in X$ we can choose a τ_2-neighbourhood $U(x)$ and a finite set $S(x) \subset S_n$ such that $U_s \cap U(x) = \emptyset$ for $s \in S_n\setminus S(x)$. The family $\{U(x) \times U(y) : x, y, \in X\}$ is then a $\tau_2 \times \tau_2$-open cover of $X \times X$. For every member of this cover we define a function $g_{xy} : U(x) \times U(y) \to [0, \infty)$ assuming

$$g_{xy}(x', x'') = \sum_{s \in S(x) \cup S(y)} \max\{0, f_s(x') - f_s(x'')\}.$$

If $s' \in S_n\setminus(S(x) \cup S(y))$, then we have $U(x) \cup U(y) \subset X\setminus U_{s'}$, so $f_{s'}(U(x) \cup U(y)) = \{0\}$. Therefore the above formula can be written in the from

$$g_{xy}(x', x'') = \sum_{s \in S_n} \max\{0, f_s(x') - f_s(x'')\}.$$

For each $x' \in U(x)$ the function $g_{xy}(x', \cdot) : U(y) \to [0, \infty)$ is τ_1-upper semicontinuous τ_2-lower semicontinuous and for $x'' \in U(y)$ the function $g_{xy}(\cdot, x'') : U(x) \to [0, \infty)$ is τ_1-lower semicontinuous τ_2-upper semicontinuous. The family $\{g_{xy} : x, y \in X\}$ consists of compatible functions, so it implies that their common extension $g_{mn} : X \times X \to [0, \infty)$ is such that $g_{mn}(x, \cdot) : X \to [0, \infty)$ is τ_1-upper semicontinuous τ_2-lower semicontinuous for $x \in X$ and $g_{mn}(\cdot, y) : X \to [0, \infty)$ is τ_1-lower semicontinuous τ_2-upper semicontinuous. Now let $\rho_{mn} : X \times X \to [0, 1]$ be the function defined by letting $\rho_{mn}(x, y) = \min\{1, g_{mn}(x, y)\}$.

One readily sees that $\{\rho_{mn} : m, n \geq 1\}$ is a sequence of quasi-pseudo-metrics on X which are bounded by 1 and satisfy the condition (1) in Lemma 5.

Now let A be a non-empty τ_1-closed set and let $x \notin A$. We can choose $m, n \geq 1$ and $s \in S_n$, $s' \in S_m$ such that $x \in U_{s'} \subset \tau_2 - \mathrm{cl}(U_{s'}) \subset U_s$ and $A \subset X \backslash U_s$. The function f_s defined for the pair (m, n) satisfies the conditions $f_s(x) = 1$ and $f_s(a) = 0$ for $a \in A$. It implies $\rho_{mn}(x, a) = 1$ for $a \in A$ and consequently $\rho_{mn}(x, A) = 1$. Thus we have shown that $\{\rho_{mn} : m, n \geq 1\}$ satisfies also (2) in Lemma 5. According to this Lemma quasi-pseudo-metrics ρ_{mn} generate the quasi-pseudo-metric p_1 on X such that $p_1(x, \cdot) : X \to [0, \infty)$ is τ_1-upper semicontinuous τ_2-lower semicontinuous for $x \in X$ and $p_1(\cdot, y) : X \to [0, \infty)$ is τ_1-lower semicontinuous τ_2-upper semicontinuous. The topology $\mathcal{P}_1$ induced by p_1 coincides with τ_1. Denoting by q_1 the conjugate quasi-pseudo-metric we have that the function $q_1(x.,) : X \to [0, \infty)$ is τ_2-upper semicontinuous for every $x \in X$. Therefore the sets $K(x, r, q_1)$ are τ_2-open, what implies $\mathcal{Q}_1 \subset \tau_2$, where $\mathcal{Q}_1$ is the topology induced by q_1. Now, beginning from the base $\mathcal{B}'$ and using the same arguments we construct the quasi-pseudo-metric q_2 such that $q_2(x, \cdot)$ is τ_1-lower semicontinuous τ_2-upper semicontinuous for $x \in X$ and $q_2(\cdot, y)$ is τ_1-upper semicontinuous τ_2-lower semicontinuous. Let $p_2(x, y) = q_2(y, x)$. Then the pair (p_2, q_2) of conjugate quasi-pseudo-metrics induces topologies $\mathcal{P}_2, \mathcal{Q}_2$ such that $\mathcal{P}_2 \subset \tau_1$ and $\mathcal{Q}_2 = \tau_2$. Finally we define functions p, q by $p = p_1 + p_2$ and $q = q_1 + q_2$. Obviously p, q are conjugate quasi-pseudo-metrics. Let $\mathcal{P}, \mathcal{Q}$ be the topologies induced by p and q. The evident inclusions $K(x, r, p) \subset K(x, r, p_1)$ and $K(x, r, q) \subset K(x, r, q_2)$ for $x \in X$, $r > 0$ imply $\tau_1 \subset \mathcal{P}$ and $\tau_2 \subset \mathcal{Q}$. Because $p(x, \cdot)$ is a τ_1-upper semicontinuous function and $q(x, \cdot)$ is τ_2-upper semicontinuous, the sets $K(x, r, p)$ are τ_1-open and the sets $K(x, r, q)$ are τ_2-open for $x \in X$, $r > 0$. From this it follows $\mathcal{P} = \tau_1$ and $\mathcal{Q} = \tau_2$.

If τ_1 and τ_2 are T_1-topologies, then by virtue of Lemma 5 the constructed functions p_1 and q_2 are quasi-metrics. Hence also p and q are quasi-metrics.

COROLLARY (7). *Let* (X, τ_1, τ_2) *be a pairwise regular bitopological space such that* τ_1 *has a* τ_i*-open base which is* σ*-discrete in* $(X\tau_j)$ *for* $i \neq j$, $i, j \in \{1, 2\}$. *Then there exist conjugate quasi-pseudo-metrics inducing the given topologies.*

COROLLARY (8). [3, Th. 2.8]. *If* (X, τ_1, τ_2) *is a pairwise regular bitopological space such that both* τ_1 *and* τ_2 *are second countable, then there exist conjugate quasi-pseudo-metrics inducing the given topologies.*

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REFERENCES

[1] R. Engelking, General topology, Monografie Matematyczne, Polish Scientific Publishers, Warszawa 1977.

[2] J. Ewert, Multivalued maps and bitopological spaces, Slupsk 1985 (Polish)

[3] J. C. Kelly: Bitopological spaces, Proc. London Math. Soc. **13** (1963), 71-89.