Boletín de la Sociedad Matemática Mexicana Vol. 33 No. 2, 1988

SOME RESULTS CONCERNING THE TIGHTNESS OF CHAIN-NET SPACES*

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In this paper, no separation axioms are assumed unless specifically mentioned. $|A|$ denotes the cardinality of a set A. If m is a cardinal (which is assumed to be an initial ordinal) then m^+ will denote the cardinal successor of m, m^{+n} will denote the n^{th} cardinal successor of m and $m^{+\omega}$ the ω^{th} successor cardinal of m. The cofinality of m will be abbreviated to $\text{cof}(m)$. Recall that a cardinal m is said to be regular if $m = \text{cof}(m)$.

We use $[A]^X$ (or simply $[A]$ if no confusion is possible) to denote the closure of A in the topological space X of which A is a subset. If m is a cardinal then $[A]_m$ will denote the m-closure of A (that is to say, $[A]_m = \cup\{[B] : B \subset A$ and $|B| \leq m\}$). Following [6], $t(X)$ will denote the tightness of a topological space X. The cardinal invariant T (called T-tightness in [3] for want of a better name) was introduced in [7]: Let X be a topological space; $T(X)$ is the smallest cardinal number m with the property that for any increasing sequence $\{F_\alpha : \alpha \in \rho\}$ of closed subsets of X such that $\text{cof}(\rho) > m$ the set $\cup\{F_\alpha : \alpha \in \rho\}$ is closed. Another variation of tightness, the set tightness t_s (see [7]) was introduced in [2] with the rather inappropriate name of quasi-character. Again let X be a topological space; $t_s(X)$ is the smallest cardinal m such that for any $A \subset X$ and $p \in [A]\backslash A$, there is a family $\mathcal{B}$ of subsets of A with $|\mathcal{B}| \leq m$, such that $p \in [\cup\mathcal{B}]$ but $p \notin \cup\{[B] : B \in \mathcal{B}\}$.

It is clear from the above definitions that $T(X) \leq t(X)$ and $t_s(X) \leq t(X)$ and it was shown in [7] that in fact, $t_s(X) \leq T(X)$.

For completeness we include the definition of a chain-net space. For any cardinal m, an m-sequence or chain-net of length m in a topological space X is a net in X directed by the ordinal m. The space X is said to be a chain-net space or (in the notation of [2]) a pseudo-radial space if whenever a subset A of X fails to be closed, there is a chain-net in A which converges to a point outside of A. It is clear that if X is a chain-net space, then the closure of any subset A of X can be obtained by iterating the chain-net closure of A (that is, adding to A the limits of all chain-nets in A). The minimum length of chain-nets needed to do this globally is called the chain-net character $\sigma_c(X)$ of the chain-net space X (see [5]).

If Y is a T_1 topological space and $x \in Y$ then a local ψ-base at x is a family $\mathcal{V}$ of open sets such that $\cap\mathcal{V} = \{x\}$. The local pseudocharacter at x is defined by

$$\psi(x, Y) = \min(|\mathcal{V}| : \mathcal{V} \text{ is a local } \psi\text{-base at } x).$$

We recall the following result.

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* AMS SUBJECT CLASSIFICATION (1980): Primary 54A25

LEMMA (1) ([2; Proposition 2.4]). *Let X be a T_1 chain-net space with $t_s(X) \leq \lambda$. If a subset A of X is λ-closed and for each $x \in X$, $\psi(x, A \cup \{x\}) \leq \lambda^+$, then A is closed.*

The following closely resembles [2: Proposition 2.7] but there, X is assumed to be T_2 and the Generalized Continuum Hypothesis is required for its proof.

LEMMA (2). *Let X be a T_1 chain-net space and $t_s(x) \leq \lambda$. If A is a subset of X and $|A| \leq \lambda^+$, then $[A]_\lambda = [A]$.*

Proof. If $|A| \leq \lambda$ then the result is obvious. Thus we assume that $|A| = \lambda^+$ and well order A as $\{x_\alpha : \alpha \in \lambda^+\}$. It is then easily seen that $[A]_\lambda = \cup\{[\{x_\alpha : \alpha < \beta\}] : \beta < \lambda^+\}$. Let $B = [A]_\lambda$; then B is λ-closed and for any $x \in X\backslash B$, putting $Y = B \cup \{x\}$, we have that

$$\{x\} = \cap\{(Y\backslash[\{x_\alpha : \alpha < \beta\}]) : \beta < \lambda^+\}$$

and so $\psi(x, B \cup \{x\}) \leq \lambda^+$. Thus by Lemma 1, B is closed and the result is clear.

The following result may be easily obtained by repeated application of the preceding lemma.

COROLLARY. Let X be a T_1 chain-net space and $t_s(x) \leq \lambda$. If $A \subset X$ and $|A| \leq \lambda^{+n}$ for some non-negative integer n, then $[A]_\lambda = [A]$.

The atomic tightness $a(p, A)$ of a limit point p of a set A in a space X is defined by:

$$a(p, A) = \min\{|B| : B \subset A\backslash\{p\} \text{ and } p \in [B]\}.$$

We will need the following simple results:

LEMMA (3) (7; Lemma 1]). *If $p \in [A]$ then* $\operatorname{cof}(a(p, A)) \leq T(X)$.

LEMMA (4) ([7; Corollary 1]). *If $t(X)$ is a successor cardinal then $t(X) = T(X)$.*

The next lemma generalizes [7; Corollary 2].

LEMMA (5). *Let λ be a cardinal. If in a topological space X every μ-closed set is closed for all $\mu < \lambda^{+\omega}$, then either $T(X) < \lambda$ or $t(X) = T(X)$.*

Proof: We will assume that $T(X) \geq \lambda$. Let $A \subset X$ and define

$$D = \cup\{[B] : B \subset A \text{ and } |B| \leq \lambda^{+\omega}\}.$$

We proceed to show that D is μ-closed for all $\mu < \lambda^{+\omega}$ and hence is closed. To this end, suppose that $F \subset D$ and $|F| < \lambda^{+\omega}$. There is then a family $\mathcal{B}$ of subsets of A such that $|\mathcal{B}| < \lambda^{+\omega}$ and for each $B \in \mathcal{B}$, $|B| \leq \lambda^{+\omega}$ and such that $F \subset \cup\{[B] : B \in \mathcal{B}\}$. Thus $|\cup\mathcal{B}| \leq \lambda^{+\omega}$ and so $[F] \subset D$. If $t(X) < \lambda^{+\omega}$

then since $\lambda \leq T(X) \leq t(X)$ it follows either that $t(X) = \lambda$, in which case $t(X) = T(X)$, or that $t(X) = \lambda^{+m}$ for some positive integer m. The result now follows from Lemma 4. If on the other hand it happens that $t(X) = \lambda^{+\omega}$, then let us assume that $T(X) = \lambda^{+n}$ for some non-negative integer n. Thus for any subset A of X and $p \in [A]$, since $\mathrm{cof}(a(p, A)) \leq \lambda^{+n}$ and $a(p, A) \leq t(X) = \lambda^{+\omega}$, it follows that $a(p, A) \leq \lambda^{+n}$ or $a(p, A) = \lambda^{+\omega}$. Since $t(X) = \sup\{a(p, A) : A \subset X$ and $p \in [A]\}$ it follows that there is some $A \subset X$ and some $p \in [A]$ such that $a(p, A) = \lambda^{+\omega}$. Fix this p and A and let

$$F = \cup\{[B] : B \subset A \text{ and } |B| \leq \lambda^{+n}\}.$$

F is not closed since $p \in [F]\backslash F$ and so in order to obtain a contradiction, we need show only that F is μ-closed for each $\mu < \lambda^{+\omega}$. However, if $H \subset F$ and $|H| < \lambda^{+\omega}$ then for any $q \in [H]$, $a(q, H) < \lambda^{+n}$ by the argument given in the previous paragraph. Hence $q \in F$ and the result is proved.

LEMMA (6). *Let X be a topological space. If $t(X) = \rho$ is a regular cardinal and furthermore, there is some subset A of X and some $p \in X$ such that $p \in [A]$ but $p \notin [B]$ for all subsets B of A with $|B| < \rho$ (that is to say, ρ is attained), then $t(x) = T(X)$.*

Proof: Let A and p be as in the hypothesis of the theorem. Then there exists some subset C of A such $|C| = \rho$ and $p \in [C]$. Well order C as $\{x_\alpha : \alpha < \rho\}$. Then $\cup\{\{x_\alpha : \alpha < \beta\} : \beta < \rho\}$ is not closed and so

$$t(X) = \rho = \mathrm{cof}(\rho) \leq T(X).$$

The inverse inequality is always true as noted earlier.

The main result of the paper generalizes parts a) and c) of [7; Corollary 5]. Part b) has been generalized in [4]

THEOREM. *Let X be a chain-net space. Then $t(X) = t_s(X) = T(X)$ provided either of the following conditions hold*:

1) *$t(X)$ is a successor cardinal or a regular cardinal which is attained.*

2) *$T(X) \geq \lambda$ and X does not contain any convergent chain-net whose length is a regular cardinal greater than $\lambda^{+\omega}$.*

Proof: 1) This follows immediately from [6; Proposition 4], Lemma 4 and Lemma 6.

2) The result will follow from Lemma 5 if we can show that for each $\mu < \lambda^{+\omega}$ every μ-closed subset of X is closed. However, if a subset A of X is not closed then there is a chain-net in A of regular cardinality ρ which converges to a point x outside of A. Clearly $\rho < \lambda^{+\omega}$, but then $x \in A$.

In [1], the question was posed as to whether or not a chain-net space with countable tightness has to be sequential; that is to say, whether its chain-net character (σ_c) has to be $\aleph_0$. It was shown in [5] that under CH this is false and in [8] it was further shown to be false in ZFC. However, the examples constructed in these papers all have $\sigma_c = \aleph_1$, and the question arises as to whether $\sigma_c \leq t^+$ in the class of Hausdorff spaces. Another example in [5] showed that in the class of T_1 spaces with unique chain limits (the so-called T_c spaces), the chain-net character of a chain-net space with countable tightness can be $2^{\aleph_0}$. The rest of this article is dedicated to the construction under MA+$\rceil$CH of a T_2 chain-net space with countable tightness whose chain-net character is $2^{\aleph_0}$. The construction uses a combination of the techniques employed in [5] and [8].

Let μ be the first ordinal of cardinality $2^{\aleph_0}$ and let $\{C_\alpha : \alpha < \mu\}$ be a family of nowhere dense subsets of the Cantor set whose union is not meager (as a subset of the Cantor set). MA implies that $\cup\{C_\alpha : \alpha < \xi\}$ is meager for each $\xi < \mu$. We can clearly assume that the C_α are non-empty and disjoint.

Since $\cup\{C_\alpha : \alpha < \xi\}$ is meager there is a countable family $\mathcal{J}_\xi$ of closed nowhere dense subsets of the Cantor set such that $\cup\{C_\alpha : \alpha < \xi\} \subset \cup\mathcal{J}_\xi$. Since $\cup\{C_\alpha : \alpha < \mu\}$ is not meager, it follows that for each $\xi < \mu$ there exists a minimal $\beta(\xi) \geq \xi$ such that $C_{\beta(\xi)} - \cup\mathcal{J}_\xi \neq \emptyset$. For each $\xi < \mu$ choose $x_{\beta(\xi)} \in C_{\beta(\xi)} - \cup\mathcal{J}_\xi$ and let $X = \{x_{\beta(\xi)} : \xi < \mu\}$. It is clear that $|X| = 2^{\aleph_0}$ since μ is regular and the C_α are disjoint.

Suppose now that A is an uncountable subset of X of regular cardinality $\kappa < 2^{\aleph_0}$. It follows that for some minimal $\lambda < \mu$, $\{x_{\beta(\xi)} : \xi < \lambda\} \supset A$. But $\{x_{\beta(\xi)} : \xi < \lambda\} \subset \cup\{C_{\beta(\xi)} : \xi < \lambda\} \subset \cup\mathcal{J}_\eta$, where $\eta = \sup\{\beta(\xi) : \xi < \lambda\}$. Hence $A \subset \cup\mathcal{J}_\eta$ and since κ is a regular cardinal it follows that $A \cap T$ is of cardinality κ for some $T \in \mathcal{J}_\eta$. Then $[(A\cap T)]^X = [(A\cap T)]^{\mathbf{R}} \cap X \subset T \cap X$ since T is closed. But $T \cap X \subset (\cup\mathcal{J}_\eta) \cap X \subset \{x_{\beta(\xi)} : \xi < \lambda\}$. Hence $|T \cap X| < 2^{\aleph_0}$. Thus $A \cap T$ is of cardinality κ and its closure has cardinality less than $2^{\aleph_0}$. We denote by $*$ this property of the space X.

Clearly $|X| = 2^{\aleph_0}$ and it is easy to see that there are $2^{\aleph_0}$ closed sets in X with cardinality $2^{\aleph_0}$. (X must have $2^{\aleph_0}$ points of complete accumulation.) Hence there are $2^{\aleph_0}$ countable sets in X whose closures have cardinality $2^{\aleph_0}$. We enumerate these as $\{N_\alpha : \alpha < \mu\}$ in such a way that each set occurs $2^{\aleph_0}$ times in the enumeration. For each $\alpha < \mu$ we choose $y_\alpha \in [N_\alpha]\backslash N$ and a sequence $S_\alpha \subset N_\alpha$ convergent to y_α in such a way that $y_\alpha \neq y_\beta$ if $\alpha \neq \beta$.

Let $Z = \{y_\alpha : \alpha < \mu\}$. Z clearly has property $*$ in the relative metric topology ρ that Z inherits from X.

Let $Z_\alpha = \{y_\beta : \beta < \alpha\}$ and define topologies τ_α on Z_α as follows:

τ_1 in the discrete topology on Z_1.

Having defined a locally countable, first countable topology on Z_β for each $\beta < \alpha$, we define a locally countable, first countable topology τ_α on Z_α in such a way that for each $\beta < \alpha$, (Z_β, τ_β) is an open subspace of (Z_α, τ_α) and $\rho|Z_\alpha \subset \tau_\alpha$.

1) If α is a limit ordinal then $\tau_\alpha = \cup\{\tau_\beta : \beta < \alpha\}$.
2) If $\alpha = \gamma + 1$ and $S_\alpha \cap Z_\alpha$ is finite then y_γ is isolated in Z_α.
3) If $S_\alpha \cap Z_\alpha$ is infinite then we choose a nested local base of ρ-clopen sets at

y_γ, $\{B_n : n \in \omega\}$ in such a way that $(B_n \backslash B_{n+1}) \cap (S_\alpha \cap Z_\alpha) \neq \emptyset$. Choose $z_n \in (B_n \backslash B_{n+1}) \cap (S_\alpha \cap Z_\alpha)$ and countable τ_γ-clopen, ρ-closed sets U_n so that $z_n \in U_n \subset B \backslash B_{n+1}$. Neighbourhoods of y_γ will then be of the form

$$\{y_\gamma\} \cup \bigcup \{U_n : n \geq K\} : (k \in w).$$

It is clear that if each U_n is ρ-closed and τ_γ-clopen then each of these sets is ρ-closed since y_γ is the only point of accumulation of the family $\{U_n : n \in w\}$. Furthermore, $\tau_\alpha \supset \rho$ and hence the neighbourhoods are τ_α-closed.

Let $\tau = \bigcup_{\alpha<\mu} \tau_\alpha$.

It is clear that τ is a topology on Z which refines ρ and hence has property $*$. Now define $Y = Z \cup \{p\}$ (where $p \notin Z$) and where neighbourhoods of p are of the form $U \cup \{p\}$ where $Z \backslash U$ is closed in Z and of cardinality less than $2^{\aleph_0}$.

We claim that:

1) Y is chain-net.
Suppose $p \in [A]^Y$ for some $A \subset Z$ (closures are taken in the topology τ). Then $[A]^Z$ must have cardinality $2^{\aleph_0}$. $[A]^Z$ may be obtained from A by sequences and then a chain net of length $2^{\aleph_0}$ in $[A]^Z$ converges to p. We note that no smaller chain can converge to p since Z has property $*$. Thus we have shown that $\sigma_c(Y) = 2^{\aleph_0}$.

2) Y has countable tightness. Again let $p \in [A]^Y$ (some $A \subset Z$). $[A]^Z$ must have cardinality $2^{\aleph_0}$. Hence in the metric topology ρ, A has a countable dense set $N \subset A$. N is enumerated $2^{\aleph_0}$ times in the enumeration N_α, and hence N has $2^{\aleph_0}$ accumulation points in the topology τ ($N \subset Z_\alpha$ for some $\alpha < \mu$ and then for all $\gamma > \alpha$ such that $N_\gamma = N$, $|S_y \cap Z_y| = \aleph_0$ and hence $y_\gamma \in [N]^Z$). Thus $[N]^Z$ has cardinality $2^{\aleph_0}$ and so $p \in [N]^Y$.

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This work was completed while the second author was a visiting professor at the University of São Paulo. He wishes to express his gratitude for support provided by the Consejo Nacional de Ciencia y Tecnología (Mexico) and the University of São Paulo.

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