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INTERCHANGE OF LARGE TIME AND SCALING LIMITS IN STABLE DAWSON-WATANABE PROCESSES: A PROBABILISTIC PROOF

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1. Introduction and formulation of the result

Consider a population of individuals in R^d, each of which carries unit mass, evolving in time as follows: Initially, the individuals' positions form a Poisson process with intensity measure ρ. Each particle performs symmetric stable motion with exponent $\alpha \in (0,2]$ for a random lifetime which is exponentially distributed with paramter V. At the end of this lifetime it branches into a random number N of particles, all of them obeying (independently) the dynamics just described, starting at the parent particle's final position. The random offspring number N is assumed to have moment generating function $Es^N = s + \frac{1}{2}(1-s)^{1+\beta}$, $\beta \in (0,1]$. Mathematically, this gives rise to a stochastic process $X_t^{\rho,V}$ taking its values in the counting measures on $\mathbf{R}^d$.

For a fixed constant γ, and $n = 1, 2, \ldots$, we consider the rescalings $X_t^n := \frac{1}{n} X_t^{n\lambda, n^\beta \gamma}$ of the process $X_t^{\lambda,\gamma}$ (where λ denotes Lebesgue measure on $\mathbf{R}^d$). This means that each particle carries mass $1/n$, the lifetime parameter is $n^\beta \gamma$, and the initial particles positions form a Poisson process with intensity $n\lambda$. Note that due to criticality of the branching and homogeneity of the motion there holds $EX_t^n = \lambda$, where $E\xi$ denotes the expectation of a random measure ξ.

The following facts are known (for (1.1) and (1.4) see [GW], (1.2) see [MRC], (1.3) and (1.5) see [GRCW] Theorem 1):

(1.1) For $t \to \infty$, X_t^n converges in distribution towards a random measure X_∞^n.

(1.2) For $n \to \infty$, X_t^n converges in distribution towards a random measure X_t, and $EX_t = \lambda$.

(1.3) For $t \to \infty$, X_t converges in distribution towards a random measure X_∞.

(1.4) $EX_\infty^n = \lambda$ if $d > \alpha/\beta$, and $EX_\infty^n = o$ if $d \le \alpha/\beta$.

(1.5) $EX_\infty = \lambda$ if $d > \alpha/\beta$, and $EX_\infty = o$ if $d \le \alpha/\beta$.

(where o denotes zero measure on $\mathbf{R}^d$.)

The measure-valued process (X_t) is called stable Dawson-Watanabe process; it has first been introduced and studied by Watanabe [W] and Dawson [D]. By examining the Laplace transforms of X_t^n one can prove in a rather straightforward way the following

THEOREM (1.6). ([GRCW], Thm. 2) *The large time and scaling limits (1.1) and (1.2) interchange, i.e.* X_∞^n *converges for* $n \to \infty$ *in distribution towards* X_∞.

In this paper we give a probabilistic proof of the preceding theorem which relies on a convergence theorem for infinitely divisible random measures due to Kallenberg (stated as Lemma (2.3) below) and on a representation of the canonical Palm distributions of X_t^n obtained in [GW] (see Remark 3 below).

2. Some tools from the theory of infinitely divisible random measures

LEMMA (2.1). ([K], p. 45) *Let ξ be an infinitely divisible random measure on $\mathbf{R}^d$, with distribution P. Then there exists a uniquely determined $\nu_P \in M :=$ set of locally finite measures on $\mathbf{R}^d$, and a uniquely determined measure U_P on M having the properties $U_P(\{o\}) = 0$ and $\int U_P(d\rho)(1 - e^{\langle \rho, g\rangle}) < \infty$ for all $g \in F_c :=$ set of continuous nonnegative functions on $\mathbf{R}^d$ with compact support, such that*

$$(2.2) \qquad Ee^{-\langle \xi, g\rangle} = e^{-\langle \nu_P, g\rangle} e^{-\int U_P(d\rho)(1-e^{-\langle \rho, g\rangle})} \qquad (f \in F_c).$$

Notation. Let ξ, P, ν_P and U_P be as in Lemma 1. For all measurable $B \subseteq \mathbf{R}^d$ and $F \subseteq M$ one puts

$$C_{\tilde{P}}(B \times F) := \nu_P(B) 1_F(o) + \int \rho(B) 1_F(\rho) U_P(d\rho)$$

Note that the first marginal of$C_{\tilde{P}}$ is

$$C_{\tilde{P}}(B \times M) = \nu_P(B) + \int \rho(B) U_P(d\rho) = E\xi(B),$$

i.e. the intensity measure of ξ. In case $E\xi$ is locally finite, let $(\tilde{P}_b)_{b \in \mathbf{R}^d}$ be a regular desintegration of $C_{\tilde{P}}$ with respect to its first marginal $E\xi =: \Lambda_P$, and let for each $b \in \mathbf{R}^d$ $\tilde{\xi}_b$ be a random measure having distribution $\tilde{P}_b$. We will call $\tilde{\xi}_b$ a *canonical Palm random measure at b*. For each $f \in F_c$ such that $\langle \Lambda_P, f\rangle > 0$, let $\tilde{\xi}_f$ be the random measure which arises from $\tilde{\xi}_b$ when the point b is chosen at random with probability distribution $\frac{1}{\langle \Lambda_P, f\rangle} f(b) \Lambda_P(db)$; note that $\tilde{\xi}_f$ has distribution $\tilde{P}_f := \frac{1}{\langle \Lambda_P, f\rangle} \int \tilde{P}_b(\cdot) f(b) \Lambda_P(db)$.

We will call $\tilde{\xi}_f$ a *canonical Palm random measure* of ξ, *randomized by f*.

LEMMA (2.3). ([K], Lemma 10.8) *Let $\xi, \xi_1, \xi_2, \ldots$ be infinitely divisible random measures having locally finite intensity measures. Then any two of the following statements implies the third*:

1) $\xi_k \to \xi$ in distribution,
2) $E\langle \xi_k, f\rangle \to E\langle \xi, f\rangle$ for all $f \in F_c$,
3) $(\tilde{\xi}_k)_f \to \tilde{\xi}_f$ in distribution for all $f \in F_c$ with $\langle \rho, f\rangle > 0$.

LEMMA (2.4). *Let $\xi, \xi_1, \xi_2, \ldots$ be i.i.d. infinitely divisible random populations and put, for some $n \in \mathbf{N}, \eta := \frac{1}{n}(\xi_1 + \ldots \xi_n)$. Then a version of $\tilde{\eta}_b$ is given by $\frac{1}{n}\tilde{\xi}_b$.*

Proof. Denote the distribution of ξ by P and that of η by Q. One checks easily that $U_Q = nU_P(\frac{\rho}{n} \in (\cdot))$.

Since $\nu_P = \nu_Q = o$, there results

$$C_{\tilde{Q}}(B \times F) = C_{\tilde{P}}(B \times \{\rho | n\rho \in F\}) \quad \text{for all measurable } B \subseteq \mathbf{R}^d, F \subseteq M,$$

and hence $\tilde{Q}_b = \tilde{P}_b(\frac{\rho}{n} \in (\cdot))$. □

3. Proof of the theorem

Let us state at once that in the case $d \leq \alpha/\beta$ the assertion of the theorem is immediate from (1.4) and (1.5); therefore we assume in the rest of the paper that $d > \alpha/\beta$.

LEMMA (3.1). *For each $n = 1, 2, \ldots$, and each $t \in [0, \infty]$, a family of canonical Palm random measures of X_t^n is given by $\frac{1}{n}(\widetilde{X_t^{\lambda, n^\beta\gamma}})_b, \quad b \in \mathbf{R}^d$.*

Proof. This is immediate from Lemma (2.4), since $X_t^{n\lambda, n^\beta\gamma}$ equals in distribution the sum of n independent copies of $X_t^{\lambda, n^\beta\gamma}$. □

Let $N_S^{x,V}$ denote a random population of individuals which arises after time s from one initial individual at site $x \in \mathbf{R}^d$ by the branching dynamics described in the introduction (with lifetime parameter V).

Remark (3.2). For each $n = 1, 2, \ldots$ and each $t \in [0, \infty]$, a family of canonical Palm random populations of $X_t^{\lambda, n^\beta\gamma}$ is provided by [GW], Theorem 2.3 and Lemma 5.1, namely by

$$\delta_b + \int_{[0,t)} (\sum_{i=1}^{Z_s} N_{s,i}^{a_s, n^\beta\gamma}) \mu_{n^\beta}(ds)$$

where μ_{n^β} is a random Poisson configuration on $[0, \infty)$ with intensity $n^\beta\gamma$, (a_s) is a random path of the basic process (i.e. symmetric stable motion with exponent α) starting in b, Z_s, $s > 0$, are random numbers with $P[Z_s = k] = (k+1)p_{k+1}$, $k = 1, 2, \ldots$, (p_k) are the weights of the offspring distribution, $N_{s,i}^{x, n^\beta\gamma}$, $i = 1, 2, \ldots$, $x \in \mathbf{R}^d$, has the same distribution as $N_s^{x, n^\beta\gamma}$, and all these random objects are independent.

Combining Lemma (3.1) and Remark (3.2), we arrive at

PROPOSITION (3.3). *Consider an arbitrary but fixed $f \in F_c$ with $\langle \lambda, f \rangle > 0$. For each $n = 1, 2, \ldots$ and $t \in [0, \infty)$, a canonical Palm random measure of X_t^n, randomized by f, is given by*

$$Y_t^n := \frac{1}{n}\delta_b + \frac{1}{n} \int_{[0,t)} (\sum_{i=1}^{Z_s} N_{s,i}^{a_s, n^\beta\gamma}) \mu_{n^\beta}(ds) \tag{3.4}$$

where b (which is the starting point of (a_s)) is randomly distributed according to the probability measure $(1/\langle\lambda, f\rangle)f(x)\lambda(dx)$.

For the rest of the paper we fix a function $f \in F_c$ such that $\langle\lambda, f\rangle > 0$.

Remark (3.5).

a) It follows from Lemma (2.3) together with (1.2) that, for any $t \in [0,\infty)$, $(\widetilde{X_t^n})_f = Y_t^n$ converges, for $n \to \infty$, in distribution towards the random measure $(\widetilde{X_t})_t =: Y_t$.
b) On the other hand, it follows from Lemma (2.3) together with (1.3) and (1.5) that $(\widetilde{X_t})_f = Y_t$ converges, for $t \to \infty$, in distribution towards the random measure $(\widetilde{X_\infty})_f =: Y_\infty$.
c) It is clear from (3.4) that for all $n = 1, 2, \ldots$ and all bounded $B \subseteq \mathbf{R}^d$ there holds

$$Y_t^n(B) \underset{t\to\infty}{\nearrow} Y_\infty^n(B). \tag{3.6}$$

Remark (3.7). The assertion of the theorem (in the case $d > \alpha/\beta$) now follows immediately from Proposition (3.8) below together with (1.5) and Lemma (2.3). Once again note that also in the case $d \leq \alpha/\beta$ the theorem holds true, since then all large time limits vanish due to (1.4) and (1.5).

PROPOSITION (3.8). *Y_∞^n converges, for $t \to \infty$, in distribution towards Y_∞.*

Proof. In view of Remark (3.5), we are faced with the following diagram of convergences:

$$\begin{array}{ccc} Y_t^n & \xrightarrow[t\to\infty]{} & Y_\infty^n \\ n\to\infty \ \Big\downarrow & & \\ Y_t & \xrightarrow[t\to\infty]{} & Y_\infty^n \end{array}$$

We claim that also $Y_\infty^n \to Y_\infty$ holds true. In Lemma (3.11) below we will show:

(3.9) $\forall$ bounded $B \subseteq \mathbf{R}^d \forall\varepsilon > 0 \exists t > 0 \forall n = 1, 2, \ldots : P[Y_\infty^n(B) - Y_t^n(B) \geq \varepsilon] \leq \varepsilon$

(In this sense, convergence in t is uniformly in n.)

Now consider, for any $g \in F_c$ with $g < 1$, the Laplace transforms $Ee^{-\langle Y_\infty^n, g\rangle}$; we claim that they converge towards $Ee^{-\langle Y_t, g\rangle}$. To this end we rewrite

$$\begin{aligned} |Ee^{-\langle Y_\infty^n, g\rangle} - Ee^{-\langle Y_t^n, g\rangle}| &= E[e^{-\langle Y_t^n, g\rangle}(1 - e^{-\langle Y_\infty^n - Y_t^n, g\rangle})] \\ &\leq E[1 - e^{-\langle Y_\infty^n - Y_t^n, g\rangle}] \leq E[\langle Y_\infty^n - Y_t^n, g\rangle \wedge 1] \\ &\leq \varepsilon + P[\langle Y_\infty^n - Y_t^n, g\rangle \geq \varepsilon] \quad \text{for all } \varepsilon > 0. \end{aligned}$$

Now take according to (3.9) for any fixed $\varepsilon > 0$ the time t large enough such that

$$P[\langle Y_\infty^n - Y_t^n, g\rangle \geq \varepsilon] \leq \varepsilon.$$

Hence results for this t:

$$|Ee^{-\langle Y_\infty^n, g\rangle} - Ee^{-\langle Y_t^n, g\rangle}| \leq 2\varepsilon \quad (n = 1, 2, \ldots) \tag{3.10}$$

Since $\langle Y_s^n, g\rangle$ increases, for $s \to \infty$, towards $\langle Y_\infty^n, g\rangle$ (see Remark (3.5)c), we can in view of Remark (3.5)b) choose t so large that besides (3.10) also

$$|Ee^{-\langle Y_\infty^\infty, g\rangle} - Ee^{-\langle Y_t^\infty, g\rangle}| < \varepsilon \tag{3.11}$$

holds true. Now we can apply the triangle inequality:

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} |Ee^{-\langle Y_\infty, g\rangle} - Ee^{-\langle Y_\infty^n, g\rangle}| \\
&\leq |Ee^{-\langle Y_\infty, g\rangle} - Ee^{-\langle Y_t, g\rangle}| \\
&+ \overline{\lim_{n\to\infty}} |Ee^{-\langle Y_t, g\rangle} - Ee^{-\langle Y_t^n, g\rangle}| \\
&+ \overline{\lim_{n\to\infty}} |Ee^{-\langle Y_t^n, g\rangle} - Ee^{-\langle Y_\infty^n, g\rangle}| \leq 3\varepsilon.
\end{aligned}$$

(Note that we applied (3.10) and (3.11) to estimate the first and third summand, respectively, and Remark (3.5)a) to guarantee that the second summand vanishes). Since $\varepsilon > 0$ was arbitrary, this yields the assertion. □

We are thus left with the hard core in the proof of the theorem, namely

LEMMA (3.12). *The convergence* $Y_t^n \xrightarrow[t\to\infty]{} Y_\infty^n$ *is uniform in the sense of* (3.9).

Proof. 1. Without loss of generality we assume that $B \subseteq \mathbf{R}^d$ is a ball centered around the origin. Let $\varepsilon > 0$ be fixed.

We intend to show:

$$\exists t > 0 \ \forall n = 1, 2, \ldots : P[\frac{1}{n} \int_{[t,\infty)} \sum_{i=1}^{Z_s} N_{s,i}^{a_s, n^\beta \gamma}(B) \mu_{n^\beta}(ds) > \varepsilon] < \varepsilon. \tag{3.13}$$

Since $\xi_{s,i}^n := N_{s,i}^{0, n^\beta \gamma}(B)$ obeys, by symmetry of the motion,

$$P[\xi_{s,i}^n > r] \geq P[N_{s,i}^{x, n^\beta \gamma}(B) > r] \quad \text{for all } x \in \mathbf{R}^d, \ r \geq 0,$$

it suffices to show:

$$\exists t > 0, \ \forall n = 1, 2, \ldots : P[\frac{1}{n} \int_{[t,\infty)} (\sum_{i=1}^{Z_s} \xi^n_{s,i}) \mu_{n\beta}(ds) > \varepsilon] < \varepsilon \tag{3.14}$$

Denoting by (T_s) the semigroup of the basic process, we have by criticality of the branching and by the scaling property of the stable motion:

$$E\xi^n_{s,i} = T_s 1_B(0) = T_1 1_{s^{-1/\alpha}B}(0) \leq c_B s^{-d/\alpha} \quad \text{for all } s \geq 1, \tag{3.15}$$

where the constant c_B depends only on B.

2. To illustrate what is going on, we first consider the case $\beta = 1$. In this case, the offspring distribution simply is given by $p_0 = p_2 = \frac{1}{2}$, i.e. the branching is binary. Hence $Z_s \equiv 1$, and using (3.15) we get for all $n = 1, 2, \ldots$

$$E\frac{1}{n} \int_{[t,\infty)} \xi^n_{s,i} \mu_n(ds) \leq \frac{1}{n} c_B \int_{[t,\infty)} s^{-\frac{d}{\alpha}} n ds = c_B \frac{\alpha}{d - \alpha} t^{-\frac{d}{\alpha}+1}$$

Hence, by Markov's inequality,

$$P[\frac{1}{n} \int_{[t,\infty)} \xi_s \mu_n(ds) > \varepsilon] \leq \frac{1}{\varepsilon} c_B \frac{\alpha}{d - \alpha} t^{-\frac{d}{\alpha}+1} \tag{3.16}$$

In order to guarantee (3.14) it thus suffices to choose t large enough so that the right hand side of (3.16) is smaller than ε.

This completes the proof in the case $\beta = 1$.

3. In the case $0 < \beta < 1$, a straightforward argument like that in step 2 fails, since then the random numbers Z_s are not integrable any more. Their distribution $q_k := P[Z_s = k] = (k + 1)p_{k+1}$, however, obeys a power law of the following form:

There exist positive constants c_1, c_2 such that

$$c_1 k^{-(\beta+1)} \leq q_k \leq c_2 k^{-(\beta+1)} \quad (k = 1, 2, \ldots) \tag{3.17}$$

(3.18) can be checked, e.g., by expanding the moment generating function of (q_k), which is $1 - \frac{1+\beta}{2}(1 - s)^\beta$, into a binomial series.

Now we turn to the proof of (3.14) in the case $0 < \beta < 1$:

In view of the estimate (3.15), it is reasonable to divide the support of the random Poisson configuration $\mu_{n\beta}$ into two parts, namely:

those points s for which $Z_s > s^{d/\alpha}$, forming $\mu^1_{n^\beta}$,

and those points s for which $Z_s \leq s^{d/\alpha}$, forming $\mu^2_{n^\beta}$.

$\mu^1_n\beta$ can be considered a "random thinning" of μ_{n^β}, and hence has the same distribution as a random Poisson configuration Φ_1 with intensity measure $h(s)n^\beta\lambda(ds)$, where $h(s) := P[Z_s > s^{d/\alpha}]$. Denoting by Φ_2 a random Poisson configuration with intensity measure $(1-h(s))n^\beta\lambda(ds)$ (independent of Φ_1), we note that $\Phi_1 + \Phi_2$ equals μ_{n^β} in distribution, and moreover:

$$\frac{1}{n}\int_{[t,\infty)}(\sum_{i=1}^{Z_s}\xi^n_{s,i})\mu_{n^\beta}(ds) \quad \text{is equal in distribution to}$$

$$\frac{1}{n}\int_{[t,\infty)}(\sum_{i=1}^{\bar{Z}_s}\xi^n_{s,i})\Phi_1(ds) + \frac{1}{n}\int_{[t,\infty)}(\sum_{i=1}^{\underline{Z}_s}\xi^n_{s,i})\Phi_2(ds) =: G^n_t + H^n_t, \tag{3.18}$$

where $\bar{Z}_s$ is assumed to have distribution $P[Z_s \in (\cdot)|Z_s > s^{d/\alpha}]$, and $\underline{Z}_s$ is assumed to have distribution $P[Z_s \in (\cdot)|Z_s < s^{d/\alpha}]$, the random variables $\underline{Z}_s$, $\bar{Z}_s$ all being independent.

4. In order to estimate the summand H^n_t, first note that by (3.17) one has for all $s \geq 1$:

$$\begin{aligned} E[\underline{Z}_s] = E[Z_s|Z_s \leq s^{d/\alpha}] &\leq \frac{1}{P[Z_s \leq s^{d/\alpha}]}\sum_{k=1}^{s^{d/\alpha}} kc_2k^{-(\beta+1)} \\ &\leq \frac{1}{c_1}\sum_{k=1}^{s^{d/\alpha}} c_2k^{-\beta} \leq c_3 s^{\frac{d}{\alpha}(1-\beta)} \end{aligned}$$

with a suitable constant $c_3 > 0$ (independent of s).

Hence results by Wald's identity (similar as in step 2):

$$EH^n_t \leq \frac{1}{n}\int_{[t,\infty)} c_2E[\underline{Z}_s]\cdot E[\xi^n_{s,i}]n^\beta ds \leq c_B\cdot c_3\int_{[t,\infty)} s^{-\frac{d\beta}{\alpha}}ds,$$

which leads, by Markov's inequality to:

$$\exists T > 0 \ \forall t \geq T \ \forall n = 1,2,\dots: \quad P[H^n_t > \varepsilon] < \varepsilon. \tag{3.19}$$

5. We now turn to estimate the first summand G^n_t in (3.18). By (3.17) there holds for all $s \geq 1$

$$h(s) = P[Z_s > s^{d/\alpha}] \le c_4 s^{-\frac{d\beta}{\alpha}} \tag{3.20}$$

for a suitable constant c_4 (independent of s), hence Φ_1 is, on the interval $[1, \infty)$, "stochastically thinner" than a random Poisson configuration $\mu_n^* \beta$ with intensity measure $c_4 s^{-\frac{d\beta}{\alpha}} \lambda(ds)$. Writing, for abbreviation,

$$\eta_s^n := \sum_{i=1}^{\bar{Z}_s} \xi_{s,i}^n,$$

we thus observe, for all $t \ge 1$ and $n = 1, 2, \ldots$

$$P[G_t^n \ge \varepsilon] \le P[\int_{[t,\infty)} \eta_s^n \mu_{n\beta}^*(ds) \ge \varepsilon]. \tag{3.21}$$

6. In view of (3.18), (3.19) and (3.21), the proof will be complete if we succeed to show:

$$\exists t > 0 \ \forall n = 1, 2, \ldots : P[\int_{[t,\infty)} \eta_s^n \mu_{n\beta}^*(ds) \ge \varepsilon] \le \varepsilon. \tag{3.22}$$

Indeed, then we have, using step 1, for this t and all $n = 1, 2, \ldots$:

$$P[Y_\infty^n(B) - Y_t^n(B) \ge 2\varepsilon] \le P[G_t^n + H_t^n \ge 2\varepsilon] \le P[G_t^n \ge \varepsilon] + P[H_t^n \ge \varepsilon] \le 2\varepsilon$$

(note that t in (3.22) can be taken, without loss of generality, larger than T figuring in (3.19)).

7. We now proceed to show (3.22). Let, for $m = 1, 2, \ldots$, t_m be such that

$$\int_{t_m}^{\infty} s^{-\frac{d\beta}{\alpha}} ds = \frac{1}{m}.$$

Let $\mu_n^{(m)}\beta$ be a random Poisson configuration with intensity measure $m \cdot n^\beta s^{-\frac{d\beta}{\alpha}} \lambda(ds)$. Obviously, $S_{m,n} := \int_{[t_m,\infty)} \eta_s^n \mu_{n\beta}^{(m)} \beta(ds)$ arises, in distribution, as a sum of m independent copies $L_{m,n}^{(1)}, \ldots, L_{m,n}^{(m)}$ of $L_{m,n} := \frac{1}{n} \int_{[t_m,\infty)} \eta_s^n \mu_{n\beta}^*(ds)$.

Now assume the contrary of (3.22) , which would imply the existence of a sequence $m_j \to \infty$ and a sequence n_j such that

$$P[L_{m_j,n_j} \ge \varepsilon] > \varepsilon \quad (j = 1, 2, \ldots). \tag{3.23}$$

Let $r > 0$ be a lower bound for the probability of $\frac{N\varepsilon}{2}$ successes in N cointosses with success probability ε. (Note that such a strictly positive lower bound actually exists, since by the law of large numbers there holds $P[\text{number of successes} > \frac{N\varepsilon}{2}] \xrightarrow[N\to\infty]{} 1$). From (3.23) there results

$$P[S_{m_j,n_j} \geq \varepsilon \frac{m_j \varepsilon}{2}] = P[L^{(1)}_{m_j,n_j} + \cdots + L^{(1)}_{m_j,n_j} \geq \varepsilon \frac{m_j \varepsilon}{2}] \geq r.$$

This would imply that the family of random variables $\{S_{m,n}\}_{m,n\in\mathbf{N}}$ is not tight. To guarantee (3.22) it is thus sufficient to show that $\{S_{m,n}\}_{m,n\in\mathbf{N}}$ actually *is* tight.

8. We now turn to the proof of tightness of $\{S_{m,n}\}_{m,n\in\mathbf{N}}$. In step 9 below we will show:

$$\text{(3.24)} \qquad \exists c_5 > 0 \ \forall s \geq 1 \ \forall k = 1,2,\dots \ \forall n = 1,2,\dots : P[\eta^n_s \geq k] \leq c_5 k^{-\beta}.$$

Having (3.24) at hand, it is possible to estimate the random variables η^n_s by an upper bound "in distribution". To this end, let K be large enough such that

$$\sum_{j=K+1}^{\infty} c_5 \beta j^{-(\beta+1)} \leq 1,$$

put $\pi_j := c_5\beta j^{-(\beta+1)}$ if $j > K$, $\pi_K := 1 - \sum_{j=K+1}^{\infty} \pi_j$, $\pi_j := 0$ if $j < K$, and let $(\chi_s)_{s\geq 1}$ be a family of i.i.d. random variables, each with distribution $(\pi_j)_{j=1,2,\dots}$. Since by (3.24) for all $k = 1,2,\dots,s \geq 1$ and $n = 1,2,\dots$ there holds

$$P[\chi_s \geq k] \geq P[\eta^n_s \geq k],$$

tightness of $\{S_{m,n}\}_{m,n\in\mathbf{N}}$ will follow from tightness of

$$\frac{1}{n}\int_{[t_m,\infty)} \chi_s \mu^{(m)}_{n^\beta}(ds), \quad m,n \in \mathbf{N}.$$

To check tightness of the latter family, note that $\mu^{(m)}_{n^\beta}([t_m,\infty))$ is a Poisson random variable with mean n^β. It is thus enough to show that the family

$$\varsigma_n := \frac{1}{n}\sum_{j=1}^{N_n} \chi_j, \quad n = 1,2,\dots$$

is tight, where N_n is a Poisson random variable with mean n^β.

$$\text{But } \varsigma_n = \frac{(N_n)^{1/\beta}}{n} \frac{1}{(N_n)^{1/\beta}} \sum_{j=1}^{N_n} \chi_j \text{ converges, for } n \to \infty,$$

in distribution, since χ_s is in the domain of normal attraction of a totally asymmetric stable law with exponent β.

Hence $\{\varsigma_n\}_{n\in\mathbf{N}}$ is tight, and so is $\{S_{m,n}\}_{m,n\in\mathbf{N}}$.

9. It remains to close the gap in the previous step, namely to show (3.24). Note that $\eta_s^n = A_s^n \cdot D_s^n \quad (s \geq 1, n = 1, 2, \ldots)$, where

$$A_s^n := \frac{1}{\bar{Z}_s} \sum_{i=1}^{Z_s} \xi_{s,i}^n s^{d/\alpha}, \quad D_s := \bar{Z}_s s^{-d/\alpha}.$$

In order to estimate the distribution of D_s, choose in view of (3.17) a constant $c_6 > 0$ such that

$$\forall s \geq 1\ \forall x \geq 1 : P[Z_s \geq x] \geq c_6 x^{-\beta}. \tag{3.25}$$

By (3.17) and (3.25) we have for all $j = 1, 2, \ldots$

$$\begin{aligned} P[D_s \in [j, j+1)] &= P[\bar{Z}_s \in [j s^{d/\alpha}, (j+1) s^{d/\alpha})] \\ &= P[Z_s \in [j s^{d/\alpha}, (j+1) s^{d/\alpha}) | Z_s \geq s^{d/\alpha}] \\ &\leq c_2 \sum_{i=[j s^{d/\alpha}]}^{[(j+1) s^{d/\alpha}]+1} i^{-(\beta+1)} \frac{1}{c_6} s^{d\beta/\alpha} \\ &\leq \frac{c_2}{c_6} (j[s^{d/\alpha}])^{-(\beta+1)} (s^{d/\alpha} + 1) s^{d\beta/\alpha} \end{aligned}$$

$\leq c_7 (j+1)^{-(b+1)}$ for a suitable $c_7 > 0$ independent of j and s.

Hence results, for all $k = 1, 2, \ldots$

$$P[\eta_s^n \geq k] = P[A_s^n D_s \geq k] \leq \sum_{j=2}^{\infty} P[D_s \in [j-1, j)] P[A_s^n \geq \frac{k}{j} | D_s \in [j-1, j)]$$

$$\leq \sum_{j=2}^{k} c_7 j^{-(\beta+1)} \frac{j}{k} E[A_s^n | D_s \in [j-1, j)] + \sum_{j=k+1}^{\infty} c_7 j^{-(\beta+1)}. \tag{3.26}$$

Since the conditional expectation $E[A_s^n | \bar{Z}_s]$ is, due to (3.15), bounded by c_B, and since moreover

$$\sum_{j=2}^{k} j^{-\beta} \leq \int_1^k x^{-\beta} dx = \frac{1}{1-\beta} k^{1-\beta}$$

and

$$\sum_{j=k+1}^{\infty} j^{-\beta+1} \leq \int_k^{\infty} x^{-\beta+1} dx = \frac{1}{\beta} k^{-\beta},$$

the validity of (3.24) follows directly from (3.26), and the proof of the lemma is complete. □

The preceding lemma closes the gap in the proof of Proposition (3.8), which, in view of Remark (3.7), finishes the proof of the theorem.

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