Boletín de la Sociedad Matemática Mexicana Vol. 33 No. 2, 1988

RECURSIVE NONPARAMETRIC ESTIMATION OF NONSTATIONARY MARKOV PROCESSES*

By ONÉSIMO HERNÁNDEZ-LERMA†, SERGIO O. ESPARZA AND BENJAMIN S. DURAN

1. Introduction

In this paper we consider the problem of estimating the transition probability density of a discrete-time, d-dimensional, nonstationary Markov process. To solve this problem we use recursive estimators of the type introduced by Wolverton and Wagner (1969), and Yamato (1971), and show that, under appropriate hypotheses, the estimators are uniformly consistent in mean square, strongly pointwise consistent, and strongly consistent in L_1 norm.

The nonparametric estimation of *stationary* Markov processes has been studied by many authors, e.g., Ioannides and Roussas (1987), Nguyen (1984), and Yakowitz (1985); for earlier references, see Prakasa Rao (1983), Chapter 6. The nonstationary case, however, has not received as much attention, but we can mention two papers related to our work: Gillert and Wartenberg (1984) show the weak consistency of the usual (nonrecursive) Parzen-Rosenblatt density estimators, whereas Hernández-Lerma and Doukhan (1989) combine an empirical distribution process with regression-like estimators to obtain consistent estimates for the transition law of a class of Markov *control* processes.

In fact, the motivation for considering nonstationary Markov processes is our interest in adaptive Markov control systems with *unknown* transition law, as in the last cited paper [cf. also Gordienko (1985), Hernández-Lerma (1989), Hernández-Lerma and Marcus (1987), and references therein]. The evolution of these control systems depends on the control actions applied at each time t, which in turn depend on the current estimates of the unknown transition law. The resulting process is then, in general, nonstationary.

The paper is organized as follows. In Section 2 we state some basic assumptions and give some preliminary results. Among these assumptions there is an ergodicity condition that guarantees the existence of a unique invariant probability measure μ for the Markov process, which is shown to be absolutely continuous with density, say γ. In Section 3 we introduce the recursive Wolverton-Wagner (WW) estimates $\hat{\gamma}_t(x)$ of the invariant density $\gamma(x)$, and give conditions for their consistency (Theorem 3.1). Next, in Section 4, we consider the joint density $f(x,y) = q(y|x)\gamma(x)$, where $q(y|x)$ denotes the (unknown) transition probability density. To estimate $f(x,y)$ we use again WW

* Bibliographical search for this work was partially supported by NASA Grant No. NAG-203.

† The research of this author was supported in part by the Consejo Nacional de Ciencia y Tecnología (CONACYT) under Grant PCEXCNA-050156, and it was performed while the author was a visiting faculty at Texas Tech University, during the academic year 1987/88.

AMS 1980 Subject Classification. Primary 62G05; Secondary 62M05.

estimates, denoted by $\hat{f}_t(x, y)$, and obtain the corresponding consistency result (Theorem 4.1). Estimates for $q(y|x)$ are then defined in Section 5, in the usual, obvious way, by

$$\hat{q}_t(y|x) := \hat{f}_t(x, y)/\hat{\gamma}_t(x),$$

and their consistency (Theorem 5.1) follows from the results in Sections 3 and 4. The proofs of the main results are all collected in Section 6, and we conclude in Section 7 with some general remarks on the relation of our work to Markov control (or decision) processes.

For sequences of independent and identically distributed (i.i.d.) random vectors, the WW density estimates have been studied by several authors, including Menon *et al.* (1984), Prasad (1985), and Yamato (1971). For additional references, see, e.g., Devroye and Györfi (1985), and Prakasa Rao (1983).

Notation and terminology. $\mathcal{B}^d$ denotes the Borel sigma-algebra of $\mathbf{R}^d$, and $B(\mathbf{R}^d)$ stands for the Banach space of real-valued, measurable, bounded functions on $\mathbf{R}^d$, with the supremum norm $\| u \| := \sup_x |u(x)|$. If V is a finite signed measure on $\mathcal{B}^d$, $\| V \|$ denotes its variations norm. In particular, recall that if P and Q are probability measures on $\mathbf{R}^d$ with densities p and q, respectively, then $P - Q$ is a finite signed measure with variation norm

$$(1.1) \qquad \| P - Q \| = 2 \sup_B |P(B) - Q(B)| = \int |p(x) - q(x)| dx,$$

where the sup is over all the Borel sets $B \in \mathcal{B}^d$; see, e.g., Devroye and Györfi (1985), p.2. The right-hand side (r.h.s.) of (1.1) is called the L_1-norm of $P - Q$ (or of $p - q$). Integrals without limits represent integrals over all of $\mathbf{R}^d$, and we also use the convention $0/0 := 0$.

2. Preliminaries

Let $\{x_t, t = 0, 1, \ldots\}$ be an $\mathbf{R}^d$-valued Markov process with transition density $q(y|x)$, and arbitrary initial distribution μ_0. Thus for every $t = 1, 2, \ldots$, the distribution μ_t of x_t is given recursively by

$$(2.1) \qquad \mu_t(B) = \int Q(B|x)\mu_{t-1}(dx), \qquad \text{for } B \in \mathcal{B}^d,$$

where $Q(B|x)$ denotes the transition probability measure, i.e.,

$$(2.2) \qquad Q(B|x) := \int_B q(y|x) dy, \qquad \text{for } x \in \mathbf{R}^d, \text{ and } B \in \mathcal{B}^d.$$

We assume throughout that the following ergodicity condition holds [cf. Dobrushin (1956), Iosifescu (1972), Ueno (1957)].

ASSUMPTION 2.1. *There exists a positive number* $\alpha < 1$ *such that*

$$\| Q(\cdot|x) - Q(\cdot|y) \| \leq 2\alpha \qquad \text{for all } x \text{ and } y \text{ in } \mathbf{R}^d.$$

Sufficient conditions for Assumption 2.1 are given, e.g., by Georgin (1978), Hernández-Lerma and Cavazos-Cadena (1988), or Hernández-Lerma (1989, Section III.3); for autoregressive models $x_{t+1} = f(x_t) + \xi_t$, where $\{\xi_t\}$ is a sequence of i.i.d. random vectors, see Doukhan and Ghindès (1983), or Hernández-Lerma and Doukhan (1989). On the other hand, Assumption 2.1 implies that, independent of the initial distribution μ_0, the Markov process has a unique invariant distribution μ, i.e.,

$$\mu(B) = \int Q(B|x)\mu(dx) \qquad \text{for all } B \in \mathcal{B}^d, \tag{2.3}$$

and moreover, for any $t \geq 0$,

$$\| \mu_t - \mu \| \leq 2\alpha^t, \tag{2.4}$$

or equivalently,

$$\sup_x \| Q^t(\cdot|x) - \mu \| \leq 2\alpha^t,$$

where $Q^t(\cdot|x)$ denotes the t-step transition probability, given $x_0 = x$.

To estimate μ, we first assume the following.

ASSUMPTION 2.2. (a) *The initial distribution* μ_0 *is absolutely continuous with a bounded density* γ_0.

(b) *There is a constant* $\overline{q}$ *and a function* $g \in B(\mathbf{R}^d)$ *such that* $q(y|x) \leq \overline{q}$, *and* $|q(y|x) - q(y'|x)| \leq |g(x)||y - y'|$ *for all* x, y, *and* y' *in* $\mathbf{R}^d$.

Assumption 2.2 (a), together with (2.1), (2.2), and an induction argument, implies that μ_t is absolutely continuous with a density γ_t given recursively by

$$\gamma_t(y) = \int q(y|x)\gamma_{t-1}(x)dx \qquad \text{for all } y \in \mathbf{R}^d \text{ and } t \geq 1, \tag{2.5}$$

and this in turn, by (2.4), yields that μ is absolutely continuous, so that there exists a nonnegative measurable function γ such that

$$\mu(B) = \int_B \gamma(x)dx \qquad \text{for } B \in \mathcal{B}^d.$$

Thus, from (2.2) and (2.3), $\gamma(y) = \int q(y|x)\gamma(x)dx$ for almost all $y \in \mathbf{R}^d$, whereas using (1.1), we can write the inequality (2.4) as

(2.6) $$\int |\gamma_t(x) - \gamma(x)| dx \leq 2\alpha^t.$$

On the other hand, Assumption 2.2 (b), combined with (2.5), implies that

(2.7) $$\| \gamma_t \| \leq c_0 \qquad \text{for all } t \geq 0,$$

where $c_0 := \max\{\bar{q}, \| \gamma_0 \|\}$, and

(2.8) $$|\gamma_t(y) - \gamma_t(y')| \leq \| g \| \, |y - y'|$$

for all y and y' in $\mathbf{R}^d$, and $t \geq 1$. Similarly,

$$|\gamma(y) - \gamma(y')| \leq \| g \| \, |y - y'|,$$

so that γ is a Lipschitz bounded function (note that $\| \gamma \| \leq \bar{q} \leq c_0$), and from (2.5)

$$|\gamma_t(y) - \gamma(y)| \leq \int q(y|x) |\gamma_{t-1}(x) - \gamma(x)| dx \leq 2\bar{q}\alpha^{t-1}.$$

Therefore, we conclude:

PROPOSITION 2.1. $\| \gamma_t - \gamma \| = \sup_x |\gamma_t(x) - \gamma(x)| \to 0$ *as* $t \to \infty$.

An alternative (less direct) way to obtain Proposition 2.1 is by using the converse to Scheffé's Theorem given by Boos (1985). Assumptions 2.1 and 2.2 are supposed to hold throughout the following.

3. Estimation of the invariant density

For each $n = 0, 1, \ldots$, let u_n be the function defined by

$$u_n(x) := b_n^{-d} u(x/b_n), \qquad \text{for } x \in \mathbf{R}^d,$$

where u is a given probability density function (p.d.f.), sometimes referred to as the *kernel* function, and $\{b_n\}$ is a sequence of positive numbers. The recursive *Wolverton-Wagner* (WW) *estimate* $\hat{\gamma}_t$ of the density γ is then defined as

$$\hat{\gamma}_t(x) := t^{-1} \sum_{n=0}^{t-1} u_n(x_n - x), \qquad \text{for } x \in \mathbf{R}^d, \text{and } t = 1, 2, \ldots.$$

Observe that these are indeed recursive estimators, since

$$(3.1) \qquad (t+1)\hat{\gamma}_{t+1}(x) = t\hat{\gamma}_t(x) + u_t(x_t - x),$$

and that, on the other hand,

$$\hat{\mu}_t(B) := \int_B \hat{\gamma}_t(x)dx, \qquad \text{for } B \in \mathcal{B}^d,$$

defines an estimate for the invariant probability measure μ.

We assume throughout that the kernel u is *bounded*, $\| u \| < \infty$, and also $\bar{u} := \int |x| u(x) dx < \infty$. With respect to the sequence $\{b_t\}$, we will assume that it is *nonincreasing*, and it satisfies some of the following conditions.

Condition 3.1. (a) $b_t \to 0$. (b) $tb_t^d \to \infty$. (c) $\sum_t t^{-3/2} b_t^{-d} < \infty$.

A sequence satisfying these conditions is, for example, $b_t := t^{-r}$, with $0 < r < 1/2d$.

To state our first consistency result, let us define the *bias* function

$$B_t(x) := E\hat{\gamma}_t(x) - \gamma(x),$$

and the *mean square error* function

$$(3.2) \qquad M_t(x) := E[\hat{\gamma}_t(x) - \gamma(x)]^2 = \mathrm{Var}[\hat{\gamma}_t(x)] + B_t(x)^2,$$

where $\mathrm{Var}(\cdot)$ denotes the variance.

THEOREM 3.1. *Suppose that Condition* 3.1(a) *holds. Then, as* $t \to \infty$,
(a) $\sup_x |B_t(x)| \to 0$.
If in addition, Condition 3.1(b) *holds, then*
(b) $\sup_x \mathrm{Var}[\hat{\gamma}_t(x)] \le ct^{-1} b_t^{-d} \to 0$, *where* $c := 2 \| u \| c_0/(1-\alpha)$, *and*
(c) $\sup_x M_t(x) \to 0$.
Suppose also that Condition 3.1(c) *holds. Then*
(d) $\hat{\gamma}_t(x) \to \gamma(x)$ *almost surely* (a.s.) *for all* $x \in \mathbf{R}^d$, *and*
(e) $\| \hat{\mu}_t - \mu \| = \int |\hat{\gamma}_t(x) - \gamma(x)| dx \to 0$ a.s.

Notice that under Conditions 3.1(a) and 3.1(b), equation (3.2) yields that parts (a) and (c) in Theorem 3.1 are *equivalent*. This has been proved by Prasad (1985) for i.i.d. sequences $\{x_t\}$. He also gives conditions under which, for i.i.d. sequences, Theorem 3.1(c) is equivalent to *uniform* strong consistency, but we have been unable (as yet) to extend Prasad's result to Markov sequences.

The proof of Theorem 3.1 is given in Section 6. Following the outline sketched in the Introduction, we now turn to the estimation of the joint density $f(x,y) = q(y|x)\gamma(x)$.

4. Estimation of the joint density

In this section we extend Theorem 3.1 to the $2d$-dimensional joint process $z_t := (x_t, x_{t+1}), t = 0, 1, \ldots$. To do this, we begin by noting that $\{z_t\}$ is a Markov process, and z_t has density

$$(4.1) \qquad f_t(x, y) = q(y|x)\gamma_t(x), \qquad t \geq 0,$$

which converges in L_1-norm to $f(x, y) = q(y|x)\gamma(x)$; in fact, from (2.6),

$$\int\int |f_t(x, y) - f(x, y)| dy dx = \int |\gamma_t(x) - \gamma(x)| dx \leq 2\alpha^t.$$

Moreover, both $f(x, y)$ and $f_t(x, y)$ are uniformly bounded, namely,

$$(4.2) \qquad |f(x, y)| \leq \bar{q} \parallel \gamma \parallel \leq c_0^2, \text{ and } |f_t(x, y)| \leq \bar{q} c_0 \leq c_0^2,$$

and since $|f_t(x, y) - f(x, y)| \leq \bar{q} \parallel \gamma_t - \gamma \parallel$, Proposition 2.1 yields

$$(4.3) \qquad \parallel f_t - f \parallel = \sup_{x,y} |f_t(x, y) - f(x, y)| \to 0 \text{ as } t \to \infty.$$

To obtain the analogue of the Lipschitz condition (2.8), we strengthen Assumption 2.2(b) as follows.

Assumption 4.1. There is a function $h \in B(\mathbf{R}^d)$ such that, for all x, x', and y in $\mathbf{R}^d$,

$$|q(y|x) - q(y|x')| \leq |h(y)||x - x'|.$$

With this additional assumption, we have:

PROPOSITION 4.1. *$f_t(x, y)$ satisfies the uniform Lipschitz condition*

$$|f_t(x, y) - f_t(x', y')| \leq c_1 \cdot max\{|x - x'|, |y - y'|\}$$

for all $2d$-vectors (x, y) and (x', y'), where $c_1 := c_0(2 \parallel g \parallel + \parallel h \parallel)$.

To define an estimate $\hat{f}_t(x, y)$ of the joint density $f(x, y)$, let us consider again the sequence $\{b_n\}$ and the functions $u(x)$ and $u_n(x)$ introduced in Section 3, and let

$$u_n^*(x, y) := u_n(x)u_n(y) = b_n^{-2d} u(x/b_n)u(y/b_n), \qquad \text{for } (x, y) \in \mathbf{R}^{2d}.$$

Then we define

$$\hat{f}_t(x, y) := t^{-1} \sum_{n=0}^{t-1} u_n^*(x_n - x, x_{n+1} - y) \tag{4.4}$$
$$= t^{-1} \sum_{n=0}^{t-1} u_n(x_n - x) u_n(x_{n+1} - y).$$

To state the corresponding consistency result, we introduce the *bias* function

$$B_t^*(x, y) := E\hat{f}_t(x, y) - f(x, y),$$

and the *mean square error* function

$$M_t^*(x, y) := E[\hat{f}_t(x, y) - f(x, y)]^2 = \mathrm{Var}[\hat{f}_t(x, y)] + B_t^*(x, y)^2.$$

Let us also consider (cf. Conditions 3.1(b) and (c)):

Condition 4.1. (a) $tb_t^{2d} \to \infty$. (b) $\sum t^{-3/2} b_t^{-2d} < \infty$.

THEOREM 4.1. *Suppose that Condition* 3.1(a) *holds,* i.e., $b_t \to 0$. *Then, as* $t \to \infty$,
(a) $\sup_{x,y} |B_t^*(x, y)| \to 0$.
If in addition, Condition 4.1(a) *holds, then*
(b) $\sup_{x,y} \mathrm{Var}[\hat{f}_t(x, y)] < ct^{-1} b_t^{-2d} \to 0$, *for some constant c, and*
(c) $\sup_{x,y} M_t^*(x, y) \to 0$.
Suppose also that Condition 4.1(b) *holds. Then*
(d) $\hat{f}_t(x, y) \to f(x, y)$ a.s. *for all* $(x, y) \in \mathbf{R}^{2d}$, *and*
(e) $\int\int |\hat{f}_t(x, y) - f(x, y)| dx dy \to 0$ a.s.

The proof is given in Section 6.

5. Estimation of the transition density

Having the estimates $\hat{\gamma}_t(x)$ and $\hat{f}_t(x, y)$ of $\gamma(x)$ and $f(x, y)$, respectively, we can now define an estimate $\hat{q}_t(y|x)$ for the transition density $q(y|x) = f(x, y)/\gamma(x)$ in the obvious way:

$$\hat{q}_t(y|x) := \hat{f}_t(x, y)/\hat{\gamma}_t(x), \qquad \text{for } (x, y) \in \mathbf{R}^{2d} \text{ and } t \geq 1.$$

Then the consistency Theorems 3.1 and 4.1 yield the following.

THEOREM 5.1. *Let* $x \in \mathbf{R}^d$ *be such that* $\gamma(x) > 0$. *If the assumptions of Theorems* 3.1(d) *and* 4.1(c) *hold, then, as* $t \to \infty$,
(a) $\sup_y E|\hat{q}_t(y|x) - q(y|x)|^2 \to 0$.
If also Condition 4.1(b) *holds, then*
(b) $\hat{q}_t(y|x) \to q(y|x)$ a.s. *for all* $y \in \mathbf{R}^d$, *and*
(c) $\| \hat{Q}_t(\cdot|x) - Q(\cdot|x) \| = \int |\hat{q}_t(y|x) - q(y|x)| dy \to 0$ a.s., *where*

$$\hat{Q}_t(B|x) := \int_B \hat{q}_t(y|x)dy, \qquad \textit{for } B \in \mathcal{B}^d,$$

is an estimate of the transition probability measure $Q(B|x)$ *in* (2.2).

6. Proofs

Let us first state some properties of the functions $u_n(x) := b_n^{-d}u(x/b_n)$.

LEMMA 6.1. *For every* $x \in \mathbf{R}^d$ *and* $n \geq 0$,
(a) $u_n(\cdot)$ *is a* p.d.f. *on* $\mathbf{R}^d$.
(b) $Eu_n(x_n - x) \leq b_n^{-d} \parallel u \parallel$.
(c) $Eu_n(x_n - x) \leq c_0$, *where* c_0 *is the constant in* (2.7).
(d) $\mathrm{Var}[u_n(x_n - x)] \leq Eu_n^2(x_n - x) \leq b_n^{-d} \parallel u \parallel c_0$.
(e) *The covariance function* $\Gamma(n,m) := \mathrm{cov}[u_n(x_n - x), u_m(x_m - x)]$ *satisfies*

$$|\Gamma(n,m)| \leq \parallel u \parallel b_m^{-d}\alpha^{m-n}Eu_n(x_n - x) \leq \parallel u \parallel c_0 b_m^{-d}\alpha^{m-n}$$

for all $0 \leq n \leq m$, *where* $\alpha < 1$ *is the number in Assumption* 2.1.

Proof. (a) and (b) are obvious, by the definition of u_n. Part (c) follows from (2.7) and the equality

$$\begin{aligned} Eu_n(x_n - x) &= \int u_n(y - x)\gamma_n(y)dy \\ &= \int \gamma_n(b_n y + x)u(y)dy. \end{aligned} \tag{6.1}$$

Writing $u_n^2 = u_n \cdot u_n$, (d) follows from the definition of u_n and part (c), and finally, (e) is a special case of Ueno's (1957) Lemma 3. □

Proof of Theorem 3.1. (a) By definition,

$$B_t(x) := E\hat{\gamma}_t(x) - \gamma(x) = t^{-1}\sum_{n=0}^{t-1} E[u_n(x_n - x) - \gamma(x)], \tag{6.2}$$

and by (6.1),

$$|Eu_n(x_n - x) - \gamma(x)| \leq \int |\gamma_n(b_n y + x) - \gamma(x)|u(y)dy.$$

Now inside the absolute value on the r.h.s., add and substract $\gamma_n(x)$, and then use the Lipschitz property (2.8), to obtain

$$\begin{aligned} |Eu_n(x_n - x) - \gamma(x)| &\leq \parallel g \parallel b_n \int |y|u(y)dy + \parallel \gamma_n - \gamma \parallel \\ &= \parallel g \parallel \bar{u}b_n + \parallel \gamma_n - \gamma \parallel . \end{aligned}$$

Thus, part (a) follows from (6.2), Proposition 2.1, and Condition 3.1(a).

(b) In terms of the covariance $\Gamma(n,m)$ in Lemma 6.1(e), and noting that

$$(6.3) \qquad \sum_{n,m} \Gamma(n,m) \leq \| u \| c_0 b_t^{-d} \sum_{n,m} \alpha^{|m-n|} \leq \| u \| c_0 b_t^{-d} 2t/(1-\alpha),$$

where the sum is over $n, m = 0, 1, \ldots, t-1$, we obtain

$$\mathrm{Var}[\hat{\gamma}_t(x)] = t^{-2} \sum_{n,m} \Gamma(n,m) \leq c t^{-1} b_t^{-d},$$

where $c = 2 \| u \| c_0/(1-\alpha)$. This proves (b).

c) This part follows from (a) and (b), together with (3.2).

To prove the strong consistency in (d) we will use a Lemma by Van Ryzin (1969), stated here for ease of reference:

LEMMA 6.2. (Van Ryzin). *Let $\{Y_t\}$ and $\{Y_t'\}$ be two sequences of random variables on a probability space $(\Omega, \mathcal{G}, P)$, and let $\{\mathcal{G}_t\}$ be an increasing sequence of sub-sigma-algebras of $\mathcal{G}$. Suppose that Y_t and Y_t' are measurable with respect to $\mathcal{G}_t$, and:*
(i) $Y_t \geq 0$ a.e.,
(ii) $EY_1 < \infty$,
(iii) $E(Y_{t+1}|\mathcal{G}_t) \leq Y_t + Y_t'$ a.s., *and*
(iv) $\sum_t E|Y_t'| < \infty$.
Then Y_t converges a.s. to a finite limit.

Proof of Theorem 3.1(d). The idea (as in the proof of Van Ryzin's (1969) Theorem 1) is to use Lemma 6.2 to show that the sequence

$$(6.4) \qquad Y_t := [\hat{\gamma}_t(x) - E\hat{\gamma}_t(x)]^2,$$

where $x \in \mathbf{R}^d$ is an arbitrary fixed point, converges a.s. to a finite limit, say Y, and this, together with part (b), implies $Y = 0$ a.s. This in turn yields

$$\hat{\gamma}_t(x) - \gamma(x) = \hat{\gamma}_t(x) - E\hat{\gamma}_t(x) + E\hat{\gamma}_t(x) - \gamma(x) \to 0 \text{ a.s.,}$$

by part (a).

Let $\mathcal{G}_t$ be the sigma-algebra generated by $x_0, \ldots, x_{t-1}$, for $t = 1, 2, \ldots$. Clearly, Y_t in (6.4) is $\mathcal{G}_t$-measurable, and it satisfies (i) and (ii) in Lemma 6.2. Let us define

$$\Gamma_t := \hat{\gamma}_t(x) - E\hat{\gamma}_t(x), \quad \text{and} \quad U_t := u_t(x_t - x) - Eu_t(x_t - x).$$

Then $Y_t = \Gamma_t^2$, and the recursive relation (3.1) yields

$$\Gamma_{t+1} = \hat{\gamma}_{t+1}(x) - E\hat{\gamma}_{t+1}(x) = (t+1)^{-1}(t\Gamma_t + U_t),$$

so that

$$\begin{aligned} Y_{t+1} &= Y_t + (t+1)^{-2}[U_t^2 + 2t\Gamma_t U_t - (2t+1)Y_t] \\ &\le Y_t + (t+1)^{-2}(U_t^2 + 2t\Gamma_t U_t), \end{aligned}$$

since $Y_t \ge 0$. Thus

$$E(Y_{t+1}|\mathcal{G}_t) \le Y_t + Y_t' \text{ a.s.},$$

where

$$Y_t' := (t+1)^{-2}E(U_t^2 + 2t\Gamma_t U_t|\mathcal{G}_t).$$

To verify condition (iv) in Van Ryzin's Lemma 6.2, first note that

$$E|Y_t'| \le (t+1)^{-2}\{E(U_t^2) + 2tE|\Gamma_t U_t|\}, \tag{6.5}$$

with $E(U_t^2) = \text{Var}[u_t(x_t - x)] \le b_t^{-d} \parallel u \parallel c_0$, by Lemma 6.1(d). On the other hand, using

$$E(\Gamma_t^2) = \text{Var}[\hat{\gamma}_t(x)] \le ct^{-1}b_t^{-d},$$

and Schwartz inequality, we obtain

$$E(|\Gamma_t U_t|) \le [E(\Gamma_t^2)E(U_t^2)]^{1/2} \le Ct^{-1/2}b_t^{-d},$$

for some constant C. Thus, (6.5) becomes (for some constant C_1)

$$\begin{aligned} E|Y_t'| &\le C_1(t+1)^{-2}(1+t^{1/2})b_t^{-d} \\ &\le 2C_1 t^{-3/2}b_t^{-d}, \end{aligned}$$

and Condition 3.1(c) yields $\sum E|Y_t'| < \infty$. Therefore, by Lemma 6.2, Y_t in (6.4) converges a.s. to a finite limit, and as noted earlier, part (d) follows.

Finally, part (e) follows from (d) and the extension to Scheffé's Theorem in Glick's (1974) Corollary C, which can also be found (e.g.) in Devroye and Györfi (1985), p. 10, or Prakasa Rao (1983), p. 191. This completes the proof of Theorem 3.1. □

Proof of Theorem 4.1. This theorem follows from *exactly the same* arguments used in the proof of Theorem 3.1, translated to the $2d$-dimensional case, and we omit the details. However, just for the sake of illustration, we note that, for instance, Lemma 6.1 for $u_n(x)$ can be translated to $u_n^*(x, y)$ as follows.

LEMMA 6.3. *For every* $(x, y) \in \mathbf{R}^{2d}$ *and* $n \geq 0$,
(a) $u_n^*(\cdot, \cdot)$ *is a* p.d.f. *on* $\mathbf{R}^{2d}$.
(b) $Eu_n^*(x_n - x, x_{n+1} - y) \leq b_n^{-2d} \| u \|^2$.
(c) $Eu_n^*(x_n - x, x_{n+1} - y) \leq c_0^2$.
(d) $\mathrm{Var}[u_n^*(x_n - x, x_{n+1} - y)] \leq E[u_n^*(x_n - x, x_{n+1} - y)]^2 \leq c_0^2 \| u \|^2 b_n^{-2d}$.
(e) *The covariance* $\Gamma^*(n, m) := Cov[u_n^*(x_n - x, x_{n+1} - y), u_m^*(x_m - x, x_{m+1} - y)]$ *satisfies*

$$|\Gamma^*(n,m)| \leq c_0^2 \| u \|^2 b_m^{-2d} \alpha^{m-n-1} \quad \text{if } m \geq n+1, \quad \text{or}$$
$$\leq c_0^2 \| u \|^2 b_n^{-2d} \quad \text{if } m = n. \quad \square$$

Proof of Theorem 5.1. Let $x \in \mathbf{R}^d$ be such that $\gamma(x)$ is positive, say, $\gamma(x) \geq \varepsilon > 0$. Then by Theorem 3.1(d), $\hat{\gamma}_t(x) > \varepsilon/2$ a.s. for all t sufficiently large, and therefore,

$$\begin{aligned} &|\hat{q}_t(y|x) - q(y|x)| \\ &\leq [\gamma(x)\hat{\gamma}_t(x)]^{-1}\{\gamma(x)|\hat{f}_t(x,y) - f(x,y)| + f(x,y)|\hat{\gamma}_t(x) - \gamma(x)|\} \\ &< 2\varepsilon^{-2}\{\gamma(x)|\hat{f}_t(x,y) - f(x,y)| + f(x,y)|\hat{\gamma}_t(x) - \gamma(x)|\}. \end{aligned} \tag{6.6}$$

Thus, part (a) follows from the inequality $(a+b)^2 \leq 2(a^2 + b^2)$ and Theorems 3.1(c) and 4.1(c). Part (b) is obvious from Theorems 3.1(d) and 4.1(d), and finally, integrating in (6.6) yields

$$\int |\hat{q}_t(y|x) - q(y|x)|dy \leq 2\varepsilon^{-2}\gamma(x)\{\int |\hat{f}_t(x,y) - f(x,y)|dy + |\hat{\gamma}_t(x) - \gamma(x)|\},$$

so that part (c) follows from Theorems 3.1(d) and 4.1(d) and (e). $\square$

7. Concluding remarks

We have presented consistency results for a class of recursive density estimators for Markov processes, but as already noted in the Introduction, the motivation for considering nonstationary Markov processes is our interest in *adaptive* Markov control systems. More explicitly, concerning Theorem 5.1, our (long-term?) goal is to extend it to Markov control/decision processes with *unknown* transition law

$$Q(B|x,a) := \mathrm{Prob}(x_{t+1} \in B | x_t = x, a_t = a), \tag{7.1}$$

where x_t and a_t denote, respectively, the state variable and the control action applied at time t. If the state space is *finite*, there are several ways to estimate $Q(B|x,a)$ in (7.1); see, e.g., Kurano (1987) and his references. Similarly, if the control system is of the form

$$x_{t+1} = F(x_t, a_t, \xi_t), \qquad t = 0, 1, \ldots, \tag{7.2}$$

where F is a given function and the ξ_t are i.i.d. disturbances with common distribution μ, (7.1) becomes

$$Q(B|x, a) = \int I_B[F(x, a, z)]\mu(dz), \tag{7.3}$$

where I_B is the indicator function, and the problem of estimating Q reduces to the problem of estimating μ ; cf. Gordienko (1985), Hernández-Lerma (1989), Hernández-Lerma and Marcus (1987). In particular, if μ is absolutely continuous with density γ, it is clear that our Theorems 3.1 or 5.1 contain the "controlled" case (7.2), (7.3), because the control variables $a_t = a$ have no effect whatsoever on the disturbances ξ_t. In the general case (7.1), however, it is not clear how one should incorporate the effect of the control actions into the estimators.

Acknowledgment. The authors wish to thank Dr. Rolando Cavazos-Cadena for his thoughtful reading of the paper and a number of suggestions, including some needed corrections and changes in an earlier version.

Centro de Investigación del IPN
Apartado Postal 14-740
México, D. F., México 07000

Texas Tech University
Lubbock, Texas, U. S. A. 79409

References

[1] D. D. Boos, *A converse to Scheffé's theorem*. Ann. Statist. **13**(1985), 423-427.
[2] L. Devroye and L. Györfi, Nonparametric Density Estimation: The L_1 View, Wiley, New York (1985).
[3] R. L. Dobrushin, *Central limit theorem for nonstationary Markov chains*, I. Theor. Probability Appl. **1**(1956), 65-80.
[4] P. Doukhan et M. Ghindès, *Estimation de la transition de probabilité d'une chaîne de Markov Doeblin - récurrente*. Stoch. Proc. Appl. **15**(1983), 271-293.
[5] J. P. Georgin, *Contrôle de chaînes de Markov sur des espaces arbitraires*. Ann. Inst. H. Poincaré **14**, Sect. B,(1978), 255-277.
[6] H. Gillert and A. Wartenberg, *Density estimation for non-stationary Markov processes*. Math. Operationsforsch. u. Statist., ser. Statist. **15**(1984), 263-275.
[7] N. Glick, *Consistency conditions for probability estimators and integrals of density estimators*. Utilitas Math. **6**(1974), 61-74.
[8] E. I. Gordienko, *Adaptive strategies for certain classes of controlled Markov processes*. Theory Probab. Appl. **29**(1985), 504-518.
[9] O. Hernández-Lerma, Adaptive Markov Control Process. Springer-Verlag, New York (1989).
[10] ——— and R. Cavazos-Cadena. *Continuous dependence of stochastic control models on the noise distribution*. Appl. Math. Optim. **17**(1988), 79-89.
[11] ——— and P. Doukhan, *Nonparametric estimation and adaptive control of a class of discrete-time stochastic systems*. (1989), Submitted for publication.

[12] ——— AND S. I. MARCUS, *Adaptive policies for discrete-time stochastic control systems with unknown disturbance distribution*. Syst. Control Lett. **9**(1987), 307-315.

[13] D. IOANNIDES AND G. G. ROUSSAS, *Note on the uniform convergence of density estimates for mixing random variables*. Statist. Probab. Lett. **5**(1987), 279-285.

[14] M. IOSIFESCU, *On two recent papers on ergodicity in nonhomogeneous Markov chains*. Ann. Math. Statist. **43**(1972), 1732-1736.

[15] M. KURANO, *Learning algorithms for Markov decision processes*. J. Appl. Probab. **24**(1987), 270-276.

[16] M. LOEVE, Probability Theory II, 4th ed., Springer-Verlag, New York (1978).

[17] V. V. MENON, B. PRASAD AND R. S. SINGH, *Nonparametric recursive estimates of a probability density function and its derivatives*. J. Statist. Plann. Inference **9**(1984), 73-82.

[18] H. T. NGUYEN, *Recursive nonparametric estimation in stationary Markov processes*. Pub. Inst. Statist. Univ. Paris **24**(1984), 65-84.

[19] B. L. S. PRAKASA RAO, Nonparametric Functional Estimation, Academic Press, New York (1983).

[20] B. PRASAD, *Equivalence of uniform asymptotic unbiasedness, mean square and strong consistencies of recursive estimates of a density and its derivatives*. J. Statist. Plann. Inference **12**(1985), 81-86.

[21] H. L. ROYDEN, Real Analysis, 2nd. ed. Macmillan, New York (1968).

[22] T. UENO, *Some limit theorems for temporally discrete Markov processes*. J. Fac. Sci. Univ. Tokyo, Sect. 1, **7**(1957), 449-462.

[23] J. VAN RYZIN, *On strong consistency of density estimates*. Ann. Math. Statist. **40**(1969), 1765-1772.

[24] C.T. WOLVERTON AND T. J. WAGNER, Recursive estimates of probability densities. IEEE Trans. Syst. Sci. Cyber. **5**(1969) 246-247.

[25] S. YAKOWITZ, *Nonparametric density estimation, prediction, and regression for Markov sequences*. J. Amer. Statist. Assoc. **80**(1985), 215-221.

[26] H. YAMATO, *Sequential estimation of a continuous probability density function and mode*. Bull. Math. Statist. **14**(1971) 1-12.