Boletín de la Sociedad Matemática Mexicana Vol. 34, 1989

THE UNIFORM PROPERTIES OF FOX'S SPREADS

BY JOHN H. V. HUNT†

1. Introduction

In [12] R H Fox introduced spreads. After defining a complete spread, he showed how to associate to each spread (f, X, Z) a complete spread (g, Y, Z) - called a completion of (f, X, Z) - which is uniquely determined up to certain topological properties. Spreads are topological entities, and all Fox's definitions and proofs occur in the topological category.

The purpose of this paper is to prove the results listed in §4 of [14].(1) We assume that the base space Z of a spread (f, X, Z) is a topologically complete Hausdorff space. We assume that Z carries a complete uniformity $\mathfrak{W}$, and we show how to endow the antecedent space X of the spread with a uniformity $\mathfrak{U}$ in a natural way. This uniformity is compatible with the topology on X. We first show that (f, X, Z) is a complete spread if and only if $(X, \mathfrak{U})$ is a complete space. Then we show that every completion of a spread is obtainable by means of uniform constructions in the following way. Namely, let (f, X, Z) be a spread and let $(Y, \mathfrak{V})$ be a completion of $(X, \mathfrak{U})$. Then there is a dense unimorphic embedding $j : (X, \mathfrak{U}) \to (Y, \mathfrak{V})$. Let $g : (Y, \mathfrak{V}) \to (Z, \mathfrak{W})$ be the unique uniformly continuous extension of the uniformly continuous function $f \circ j^{-1} . (j(X), \mathfrak{V}|j(X)) \to (Z, \mathfrak{W})$. Then (g, Y, Z) is a completion of (f, X, Z).(2)

We describe Fox's spreads in §2, and we describe the completion of a uniform space in terms of minimal Cauchy filters in §3. We give a basic construction in §4. Applied to spreads, this yields in §5 a connection between topology and uniformity which enables us to prove the results enunciated above. The notation that we use is in some instances slightly different, but more explicit, than that used in §3, 4 of [14].

The author became interested in Fox's paper through discussions with Francisco González-Acuña (see also [8], [9]), and would like to thank both

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† This research was partially supported by Grant No. PCCBNAL 790007 of the Programa Nacional de Ciencias Básicas de CONACYT at the Centro de Investigación y de Estudios Avanzados del IPN, México.

(1) This is a résumé of the results proved in preprint [6] of the bibliography of [14]. This preprint that is no longer appearing in its original form. The proofs of the results listed in §4 of [14] are now appearing in the present paper, and those listed in §5, 6 of [14] are appearing in [15].

(2) These results were presented in preliminary form (with the base space of a spread a paracompact Hausdorff space) at the Ohio University Topology Conference, Athens, Oh., 15-17 March, 1979 (Abstracts, p.20). They were also presented, together with the results now proved in [15], at the Symposium on Algebraic Topology in Honor of José Adem, Oaxtepec, Mor., México, 10-16 August, 1981 (see [14]), at the special session on "Knot Theory and Manifolds" at the summer meeting of the Canadian Mathematical Society, Vancouver, 2-4 June, 1983 (Program, p.11; also preface of [21]), and at the International Conference in Categorical Topology, University of Toledo, Toledo, Oh., 1-5 August, 1983 (Abstracts, p.15; also pp. x, xii of [1]). The author would like to thank Dale Rolfsen for the invitiation to participate in the special session in Vancouver and for providing support to do so, and Ed Tymchatyn and Herschel Bentley for providing support to attend the conference in Toledo.

him and Adalberto García-Máynez for their interest in this work and that in [15]. He would also like to thank Professor A H Stone for pointing out Michael's papers [17], [18] to him (in connection with the latter see also [10]), and Guillaume Brümmer for some of the references and remarks in footnote (5) below.

2. Fox's Spreads

In this section we summarize §1 – 3 of [12]. We introduce some of our own terminology and our notation for Fox's construction of the completion of a spread (c.f., §2 of [14]).

We say that (f, X, Z) is a *spread* if $f : X \to Z$ is a function from a T_1-space X into a T_1-space Z such that the components of the inverse images of the open sets in Z form a basis for X. We call Z the *base space of* (f, X, Z) and X the *antecedent space of* (f, X, Z). The anteof a spread is necessarily locally connected.(3)

Let (f, X, Z) be a spread. We say that a function χ is a *spread-point in* (f, X, Z) if it is a function defined on the filter base of open neighpoint $z \in Z$ and, for each open neighis a component of $f^{-1}(W)$, and

(2.1) for each pair of open neighbourhoods W, W' of z such that $W \subset W', \chi(W) \subset \chi(W')$.

We denote the image of the function χ by $\operatorname{Im}\chi$. Thus (2.1) is equivalent to each of the following:

(2.2) $\operatorname{Im}\chi$ is a filter base,

(2.3) $\operatorname{Im}\chi$ has the finite intersection property.

Notice that $\bigcap \operatorname{Im}\chi$ either is empty or consists of a single point (see also p. 629 of [17]).

We say that a spread (f, X, Z) is *complete* if $\bigcap \operatorname{Im}\chi \neq \emptyset$, for each spread-point χ in (f, X, Z). It is easily verified that (f, X, Z) is complete if and only if $(\operatorname{Im}\chi)^+$ - the filter generated on X by $\operatorname{Im}\chi$ - is the neighbourhood filter of some point in X (such a point is necessarily unique and lies in $f^{-1}(z)$).

We say that a complete spread (g, Y, Z) is a *completion of* a spread (f, X, Z) if there is a dense embedding $j : X \to Y$ such that $j(X)$ is locally connected in Y (i.e., Y has a base of open sets whose intersections with $j(X)$ are connected - see also footnote (5) of [14]). Every spread (f, X, Z) has a completion, which is unique in the following sense : if (g_i, Y_i, Z) is a completion of (f, X, Z) and $j_i : X \to Y_i$ is a dense embedding such that $j_i(X)$ is locally connected in Y_i, for $i = 1, 2$, then there is a unique homeomorphism $J : Y_1 \to Y_2$ such that $j_2 = J \circ j_1, g_1 = g_2 \circ J$ (see §2 of [14] for the correction of an oversight in Fox's proof of the uniqueness theorem; the same oversight occurs in [17]).

We give a brief account of Fox's construction of the completion of a spread (f, X, Z). We denote by X_s the collection of all spread-points in (f, X, Z). In

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(3) We have not required Z to be locally connected, as Fox does, as it is not needed in proving the results in this article (see also [7]). In [17] Michael generalized the notion of a spread (f, X, Z) to the case in which neither X nor Z is necessarily locally connected.

order to topologize X_s we write

$$U|W = \{\chi \in X_s : U = \chi(W),$$

for each open set W in Z and each component U of $f^{-1}(W)$. The collection of all $U|W$'s forms a basis for a T_1-topology on X_s, which we assume X_s to carry. We define a function $f_s : X_s \to Z$ as follows: $f_s(X)$ is the unique point $z \in Z$ such that the domain of χ is the filter base of open neighbourhoods of z, for each $\chi \in X_s$. If W is an open set in Z and $\{U_\beta\}_\beta$ is the collection of all components of $f^{-1}(W)$, then $\{U_\beta|W\}_\beta$ is the collection of all components of $f_s^{-1}(W)$ and

$$f_s^{-1}(W) = \bigcup_\beta (U_\beta|W).$$

Thus (f_s, X_s, Z) is a spread.

Let ψ be a spread-point in (f_s, X_s, Z), and let z be the unique point in Z such that the domain of ψ is the filter base of open neighbourhoods of z. For each open neighbourhood W of z, let $\chi(W)$ be the component of $f^{-1}(W)$ such that $\psi(W) = \chi(W)|W$. Then χ is a spread-point in (f, X, Z) and $\bigcap \mathrm{Im}\psi = \{\chi\}$. Thus (f_s, X_s, Z) is a complete spread.

We define a function $j_s : X \to X_s$ as follows: $j_s(x)$ is the spread-point in (f, X, Z) that associates with each open neighbourhood W of $f(x)$ the component of $f^{-1}(W)$ containing x, for each $x \in X$. Then $f = f_s \circ j_s$. Also

$$j_s(U) = (U|W) \cap j_s(X),$$

where W is an open set in Z and U is a component of $f^{-1}(W)$. Thus $j_s : X \to X_s$ is a dense embedding and $j_s(X)$ is locally connected in X_s. Consequently (f_s, X_s, Z) is a completion of (f, X, Z).

We call (f_s, X_s, Z) the *canonical completion of* (f, X, Z), and $j_s : X \to X_s$ the *canonical embedding of X in X_s*.

The following result for arbitrary topological spaces X, Y, Z is a consequence of the lemma in §3 of [12].

LEMMA (2.3). *Let $j : X \to Y$ be a dense embedding such that $j(X)$ is locally connected in Y, and let $f : X \to Z$, $g : Y \to Z$ be continuous functions such that $f = g \circ j$. Then, for each open set W in Z, the correspondence between the collection of all components of $g^{-1}(W)$ and the collection of all components of $j(f^{-1}(W))$ defined by*

$$V \mapsto V \cap j(X)$$

is a bijection.

3. Minimal Cauchy filters and completions

We assume that uniform spaces are defined in terms of uniformities of coverings, as in [19]. (Notice that in this case the uniform topology of a uniform space is necessarily Hausdorff.) We assume familiarity with the elements of uniform spaces and - with the exception of the completion of a uniform space in terms of minimal Cauchy filters described below - we refer to [19] for notation, definitions and standard results.

The completion of a uniform space is constructed in terms of equivalence classes of Cauchy filters in [19]. Unfortunately minimal Cauchy filters, which we wish to use for this purpose, are not defined in [19] (references are [2], [3], [20], in each of which, however, uniformities of relations are used). Thus we describe minimal Cauchy filters and the construction of the completion of a uniform space in terms of them.

Let $(X, \mathfrak{U})$ be a uniform space. Two Cauchy filters ξ, η in $(X, \mathfrak{U})$ are equivalent if $\xi \cap \eta$ is also a Cauchy filter in $(X, \mathfrak{U})$. The intersection of an equivalence class of Cauchy filters in $(X, \mathfrak{U})$ is also a Cauchy filter in $(X, \mathfrak{U})$, which, being contained in no other Cauchy filter in $(X, \mathfrak{U})$, is called a minimal Cauchy filter in $(X, \mathfrak{U})$. It is well-that, for a base $\{\mathcal{U}_\lambda\}_\lambda$ of $\mathfrak{U}$,

(3.1) a Cauchy filter ξ in $(X, \mathfrak{U})$ is a minimal Cauchy filter in $(X, \mathfrak{U})$ if and only if, for each $A \in \xi$, there is some $\mathcal{U} \in \mathfrak{U}$ and some $U \in \mathcal{U}$ such that $\mathrm{St}(U, \mathfrak{U}) \subset A$

(c.f., proposition 5, p. 183 of [2]; lemma 2, p.139 of [3]; proposition 1, p. 182 of [20]). Consequently

(3.2) if ξ is a minimal Cauchy filter in $(X, \mathfrak{U})$, then $\xi \cap \bigcup_\lambda \mathcal{U}_\lambda$ is a base for ξ.

We denote by $\widehat{X}$ the collection of all minimal Cauchy filters in $(X, \mathfrak{U})$, and for each $A \subset X, \mathcal{U} \in \mathfrak{U}$, we write

$$\widehat{A} = \{\xi \in \widehat{X} : A \in \xi\},$$
$$\widehat{\mathcal{U}} = \{\widehat{U} : U \in \mathcal{U}\}.$$

Then $\{\widehat{\mathcal{U}} : \mathcal{U} \in \mathfrak{U}\}$ is a uniformity base on $\widehat{X}$, and we denote by $\widehat{\mathfrak{U}}$ the uniformity that it generates on $\widehat{X}$. The uniform space $(\widehat{X}, \widehat{\mathfrak{U}})$ is complete, and the function

$$\widehat{\jmath} : (X, \mathfrak{U}) \to (\widehat{X}, \widehat{\mathfrak{U}}), \quad x \mapsto \xi_x,$$

where ξ_x is the neighbourhood filter of x in the uniform topology of $(X, \mathfrak{U})$,is a dense unimorphic embedding. We call it the *canonical unimorphic embedding of* $(X, \mathfrak{U})$ *in* $(\widehat{X}, \widehat{\mathfrak{U}})$, and we call $(\widehat{X}, \widehat{\mathfrak{U}})$ the *canonical completion of* $(X, \mathfrak{U})$.(4)

4. A basic construction

Let X be a topological space with topology $\mathscr{R}$.

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(4) An account of uniform spaces in terms of uniformities of coverings and the construction of the completion of a uniform space in terms of minimal Cauchy filters is given in [13]. This is the same as here.

For a collection $\mathcal{A}$ of subsets of X, we denote by $c(\mathcal{A}, R)$ the collection of all components of all sets in $\mathcal{A}$, and for a family $\mathfrak{A}$ of collections of subsets of X, we write $c(\mathfrak{A},\mathcal{R}) = \{c(\mathcal{A},\mathcal{R}) : \mathcal{A} \in \mathfrak{A}\}$.

Clearly, if $\mathcal{T}, \mathcal{T}'$ are equivalent topology bases on X, then so are $c(\mathcal{T},\mathcal{R})$, $c(\mathcal{T}',\mathcal{R})$; and, if $\mathfrak{U}$, $\mathfrak{U}'$ are equivalent uniformity bases on X, then so are $c(\mathfrak{U},\mathcal{R})$, $c(\mathfrak{U}',\mathcal{R})$.

Now let $\mathcal{T}$ be a topology base on X, and let $\mathfrak{U}$ be a uniformity base on X. We denote by $[\mathcal{T}]$ the topology generated on X by $\mathcal{T}$, and by $[\mathfrak{U}]$ the uniformity generated on X by $\mathfrak{U}$. Then the following result is easily proved.

PROPOSITION (4.1). *If* $[\mathcal{T}]$ *is the uniform topology of* $(X, [\mathfrak{U}])$, *then* $[c(\mathcal{T},\mathcal{R})]$ *is the uniform topology of* $(X, [c(\mathfrak{U},\mathcal{R}]))$.

We call $[c(\mathcal{T},\mathcal{R})]$ the *$\mathcal{R}$-component topology of* $[\mathcal{T}]$ *on* X, and $[c(\mathfrak{U},\mathcal{R})]$ the *$\mathcal{R}$-component uniformity of* $[\mathfrak{U}]$ *on* X.

5. The uniform properties of a spread over a topologically complete Hausdorff space

In this section (f, X, Z) denotes a spread in which Z is a topologically complete Hausdorff space (i.e., Z carries some complete uniformity compatible with its topology).(5)

Thus both X, Z are Hausdorff spaces, and we denote their topologies by $\mathcal{R}, \mathcal{T}$, respectively. We denote by $f^{-1}(\mathcal{T})$ the collection of all inverse images of all sets in $\mathcal{T}$; thus $f^{-1}(\mathcal{T})$ is the initial topology induced on X by $f : X \to Z$. That (f, X, Z) is a spread is then by definition the statement that the $\mathcal{R}$-component topology of $f^{-1}(\mathcal{T})$ coincides with $\mathcal{R}$, i.e.,

$$\mathcal{R} = [c(f^{-1}(\mathcal{T}),\mathcal{R})].$$

We assume that the space Z carries a complete uniformity $\mathfrak{W}$ which is compatible with its topology $\mathcal{T}$. We write $f^{-1}(\mathfrak{W}) = \{f^{-1}(\mathcal{W}) : \mathcal{W} \in \mathfrak{W}\}$, where $f^{-1}(\mathcal{W})$ is the collection of inverse images of all sets in $\mathcal{W}$, for each $\mathcal{W} \in \mathfrak{W}$. Then $f^{-1}(\mathfrak{W})$ is a uniformity base on X and $[f^{-1}(\mathfrak{W})]$ is the initial uniformity on X induced by $f : X \to Z$. We write

$$\mathfrak{U} = [c(f^{-1}(\mathfrak{W}),\mathcal{R})];$$

i.e., $\mathfrak{U}$ is the $\mathcal{R}$-component uniformity of $[f^{-1}(\mathfrak{W})]$ on X. Since the initial uniformity $[f^{-1}(\mathfrak{W})]$ on X is compatible with the initial topology $f^{-1}(\mathcal{T})$ on X (see pp. 176, 177 of [2]), $\mathfrak{U}$ is compatible with $\mathcal{R}$ by (4.1).

We can represent the connection between uniformity and topology on a spread diagrammatically as follows:

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(5) This is the common usage of "topologically complete" nowadays (problem 6L of [16]; chap. 2 of [6]; p.370 of [13]). It is confusing to call a completely regular T_1-space which is a G_δ in its Stone-Čech compactification "topologically complete" ([4], [19], [22]). Such a space is more properly called "Čech-complete" (9.2.e of [5]; 3.9 of [11] - notice, however, that a topologically complete space is called "Dieudonné complete" in 8.5.13 of [11]).

$$\begin{array}{ccc}
\text{UNIFORMITY} & & \text{TOPOLOGY} \\
(X,\mathfrak{U}) & \dashrightarrow & (X,\mathcal{R}) \\
\Big\downarrow{id} & & {id}\Big\downarrow \\
(X,[f^{-1}(\mathfrak{W})]) & \dashrightarrow & (X,f^{-1}(\mathcal{T})) \\
\Big\downarrow{f} & & {f}\Big\downarrow \\
(Z,\mathfrak{W}) & \dashrightarrow & (Z,\mathcal{T})
\end{array}$$

The broken arrows indicate that the topological spaces on the right carry the uniform topologies of the corresponding uniform spaces on the left. The functions are continuous on the right, and uniformly continuous on the left.

We denote by $\{\mathcal{W}_\lambda\}_\lambda$ the collection of all open coverings of Z in $\mathfrak{W}$ (it is a uniformity base for $\mathfrak{W}$), and we write

$$\mathcal{U}_\lambda = c(f^{-1}(\mathcal{W}_\lambda),\mathcal{R}),$$

for each λ. Then $\{\mathcal{U}_\lambda\}_\lambda$ is a collection of open coverings of X and, as a uniformity base on X, it generates $\mathfrak{U}$.

PROPOSITION (5.1). *The correspondence between the collection of all spread-points in (f,X,Z) and the collection of all minimal Cauchy filters in $(X,\mathfrak{U})$ defined by*

$$\chi \mapsto (Im\chi)^+$$

is a bijection.

Proof. Let χ be a spread-point in (f,X,Z), and let z be the unique point in Z such that the domain of χ is the filter base of open neighbourhoods of z. Then, for each $\mathcal{W}_\lambda$, there is some $W \in \mathcal{W}_\lambda$ such that $z \in W$. Consequently $\chi(W) \in \mathcal{U}_\lambda$, and so $\mathrm{Im}\chi \cap \mathcal{U}_\lambda \neq \emptyset$. Since $\mathrm{Im}\chi$ is a filter base (by 2.2), this shows that $(\mathrm{Im}\chi)^+$ is a Cauchy filter in $(X,\mathfrak{U})$.

In order to see that $(\mathrm{Im}\chi)^+$ is a minimal Cauchy filter in $(X, \mathfrak{U})$, let $U = \chi(W)$, where W is an open neighbourhood of z. Since the neighbourfilter in $(X, \mathfrak{U})$, there is by (3.1) some $\mathcal{W}_\lambda$ and some open neighbourhood W' of z such that $W' \in \mathcal{W}_\lambda$ and $\mathrm{St}(W', \mathcal{W}_\lambda) \subset W$. Let $U' = \chi(W')$. Then $U' \in \mathcal{U}_\lambda$ and $U' \subset U$ by (2.1). We claim that $\mathrm{St}(U', \mathcal{U}_\lambda) \subset U$. Suppose that $U' \cap U'' \neq \emptyset$, where U'' is a component of $f^{-1}(W'')$ and $W'' \in \mathcal{W}_\lambda$. Then $W' \cap W'' \neq \emptyset$, and consequently $W'' \subset W$. Thus U'' is contained in some component of $f^{-1}(W)$. Since $U'' \cap U \neq \emptyset$, it follows that $U'' \subset U$. Thus $\mathrm{St}(U', \mathcal{U}_\lambda) \subset U$. This shows that $(\mathrm{Im}\chi)^+$ is a minimal Cauchy filter in $(X, \mathfrak{U})$ by (3.1).

In order to see that the correspondence is surjective, let ξ be a minimal Cauchy filter in $(X, \mathfrak{U})$. Since $f : (X, \mathfrak{U}) \to (Z, \mathfrak{W})$ is uniformly continuous, $(f(\xi))^+$-the filter generated on Z by $f(\xi)$ (the filter base consisting of the images of all the sets in ξ) - is a Cauchy filter in the complete space $(Z,$, and as such it converges to a point $z \in Z$. We define a spread-point χ in (f, X, Z) such that $\xi = (\mathrm{Im}\chi)^+$ as follows. Let W be an open neighbourhood of z. Since $W \in (f(\xi))^+$, there is an element $U' \in \xi \cap \bigcup_\lambda \mathcal{U}_\lambda$ such that $W \supset f(U')$, by (3.2). Since U' is connected, it follows that U' is contained in a component U of $f^{-1}(W)$. Since ξ is a filter, U is the unique component of $f^{-1}(W)$ in ξ. We define $\chi(W) = U$. Since $\mathrm{Im}\chi \subset \xi$, $\mathrm{Im}\chi$ has the finite intersection property. Thus χ is a spread-point in (f, X, Z) by (2.3). Consequently $(\mathrm{Im}\chi)^+$ is a minimal Cauchy filter in $(X, \mathfrak{U})$, as we have seen in the first part of this proof. Thus $\xi = (\mathrm{Im}\chi)^+$.

In order to see that the correspondence is injective, let χ, χ' be spread-points in (f, X, Z) and suppose that $\chi \neq \chi'$. Let the domains of χ, χ' be the filter bases of open neighbourhoods of the points $z, z' \in Z$, respectively. If $z \neq z'$, then there are disjoint open neighbourhoods W, W' of z, z' respectively. Thus $\chi(W) \cap \chi'(W') = \emptyset$, which implies that $\chi(W) \notin (\mathrm{Im}\chi')^+$, because $(\mathrm{Im}\chi')^+$ is a filter ; i.e., $(\mathrm{Im}\chi)^+ \neq (\mathrm{Im}\chi')^+$. If $z = z'$, then there is an open neighbourhood W of this point such that $\chi(W), \chi'(W)$ are different components of $f^{-1}(W)$. Thus $\chi(W) \cap \chi'(W) = \emptyset$, which as before implies that $\chi(W) \notin (\mathrm{Im}\chi')^+$; i.e., again $(\mathrm{Im}\chi)^+ \neq (\mathrm{Im}\chi')^+$. Q.E.D.

COROLLARY (5.2). *(f, X, Z) is a complete spread if and only if $(X, \mathfrak{U})$ is a complete space.*

Proof. If ξ is a minimal Cauchy filter in $(X, \mathfrak{U})$ and $x \in X$, then ξ converges to x if and only if $\bigcap \xi = \{x\}$. Thus the result follows immediately from (5.1). Q.E.D.

Theorem (5.3). *Let (g, Y, Z) be a completion of (f, X, Z) and let $j : X \to Y$ be a dense embedding such that $j(X)$ is locally connected in Y and $f = g \circ j$. Let us denote the topology on Y by $\mathcal{S}$ and write*

$$\mathfrak{V} = [c(g^{-1}(\mathfrak{W}), \mathcal{S})];$$

i.e., $\mathfrak{V}$ is the $\mathcal{S}$-component uniformity of $[g^{-1}(\mathfrak{W})]$. Then

(i) *$(Y, \mathfrak{V})$ is a complete space,*

(ii) *$j : (X, \mathfrak{U}) \to (Y, \mathfrak{V})$ is a dense unimorphic embedding,* *so that* $(Y, \mathfrak{V})$ *is a completion of* $(X, \mathfrak{U})$.

Proof. Since (i) has been shown in (5.2), we show (ii).
Let us write

$$\mathcal{V}_\lambda = c(g^{-1}(\mathcal{W}_\lambda), \mathcal{S}),$$

for each λ. Then $\mathcal{V}_\lambda$ is an open covering of Y and it follows from (2.3) that

$$j(\mathcal{U}_\lambda) = \mathcal{V}_\lambda | j(X),$$

for each λ. This shows that $j : (X, \mathfrak{U}) \to (j(X), \mathfrak{V}|j(X))$ is a unimorphism, because $\{\mathcal{V}_\lambda\}_\lambda$ is a base for $\mathfrak{V}$. Thus $j : (X, \mathfrak{U}) \to (Y, \mathfrak{V})$ is a dense unimorphic embedding. Q.E.D.

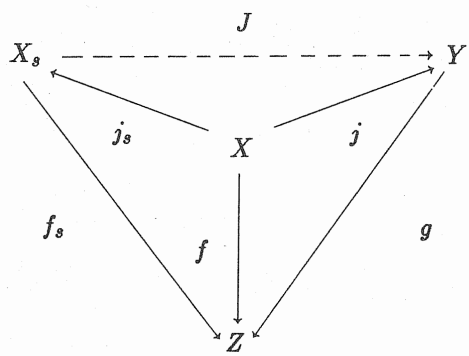
THEOREM (5.4). *Let* $(Y, \mathfrak{V})$ *be a completion of* $(X, \mathfrak{U})$, *and let* $j : (X, \mathfrak{U}) \to (Y, \mathfrak{V})$ *be a dense unimorphic embedding. Then* $f \circ j^{-1} : (j(X), \ \mathfrak{V}|j(X)) \to (Z, \mathfrak{W})$ *is uniformly continuous and as such admits a unique uniformly continuous extension* $g : (Y, \mathfrak{V}) \to (Z, \mathfrak{W})$, *and* (g, Y, Z) *is a completion of* (f, X, Z).
Moreover, if $\mathcal{S}$ *denotes the uniform topology of* $(Y, \mathfrak{V})$, *then*

$$\mathfrak{V} = [c(g^{-1}(\mathfrak{W}), \mathcal{S})];$$

i.e., $\mathfrak{V}$ *is the* $\mathcal{S}$*-component uniformity of* $[g^{-1}(\mathfrak{W})]$.

Proof. Since $j^{-1} : (j(X), \mathfrak{V}|j(X)) \to (X, \mathfrak{U}), f : (X, \mathfrak{U}) \to (Z, \mathfrak{W})$ are uniformly continuous, $f \circ j^{-1} : (j(X), \ \mathfrak{V}|j(X)) \to (Z, \mathfrak{W})$ is uniformly continuous. Thus it is standard that it admits a unique uniformly continuous extension $g : (Y, \mathfrak{V}) \to (Z, \mathfrak{W})$, because $j(X)$ is dense in Y and $(Z, \mathfrak{W})$ is complete.

We make use of the canonical completion (f_s, X_s, Z) of (f, X, Z) and the canonical embedding $j_s : X \to X_s$ in order to prove that (g, Y, Z) is a completion of (f, X, Z). Let us denote by $\mathfrak{U}_s$ the spread uniformity of (f_s, X_s, Z) on X_s. Since $(X_s, \mathfrak{U}_s)$ is a complete space and $j_s : (X, \mathfrak{U}) \to (X_s, \mathfrak{U}_s)$ is a dense unimorphic embedding by (5.3)(i), (ii), it is standard that there is a unique unimorphism $J : (X_s, \mathfrak{U}_s) \to (Y, \mathfrak{V})$ such that $j = J \circ j_s$. Hence $j(X)$ is locally connected in Y, because $j_s(X)$ is locally connected in X_s and $J : X_s \to Y$ is a homeomorphism. Also it follows from the commutativity of the inner triangles in the diagram



that $f_s|j_s(X) = g \circ J|j_s(X)$. Consequently $f_s = g \circ J$, since $j_s(X)$ is dense in X_s and Z is a Hausdorff space; i.e., the whole diagram is commutative. From this it follows that (g, Y, Z) is a complete spread, because (f_s, X_s, Z) is a complete spread and $J : X_s \to Y$ is a homeomorphism. This shows that (g, Y, Z) is a completion of (f, X, Z).

In order to prove the last part of the theorem, let us write

$$\mathfrak{V}' = [c(g^{-1}(\mathfrak{W}), \mathscr{S})].$$

Since $j : (X, \mathfrak{U}) \to (Y, \mathfrak{V}')$ is a unimorphic embedding by (5.3) (ii), it follows from (5.4)(ii) that the identity function $id : (j(X), \mathfrak{V}|j(X)) \to (j(X), \mathfrak{V}'|j(X))$ is a unimorphism ; i.e., $\mathfrak{V}|j(X) = \mathfrak{V}'|j(X)$. Since $j(X)$ is dense in Y, it is standard that it follows from this that $\mathfrak{V} = \mathfrak{V}'$. Q.E.D.

It is easily seen that the bijection

$$I : X_s \to \widehat{X},$$
$$\chi \mapsto (\mathrm{Im}\chi)^+,$$

given in (5.1), satisfies $\widehat{j} = I \circ j_s$. In fact, since $\xi_x \supset \mathrm{Im}\chi_x$ by definition of χ_x (see §2.3 for notation), it follows that $\xi_x = (\mathrm{Im}\chi_x)^+$, because $(\mathrm{Im}\chi_x)^+$ is a minimal Cauchy filter in $(X, \mathfrak{U})$ by (5.1). That is, $\widehat{j}(x) = I \circ j_s(x)$, for each $x \in X$.

Since $\widehat{X}$ is a Hausdorff space, it follows from this that I coincides with the unique unimorphism $J : (X_s, \mathfrak{U}_s) \to (\widehat{X}, \widehat{\mathfrak{U}})$ such that $\widehat{j} = J \circ j_s$, the existence of which is established in the proof of (5.4). We conclude by proving directly that

$$I : (X_s, \mathfrak{U}_s) \to (\hat{X}, \hat{\mathfrak{U}})$$

is a unimorphism in order to pose a question.

Let us denote by $\mathcal{S}_s$ the topology on X_s, and let us write $\mathcal{U}_\lambda^s = c(f_s^{-1}(\mathcal{W}_\lambda), \mathcal{S}_s)$ for each λ. Thus $\{\mathcal{U}_\lambda^s\}$ is a collection of open coverings of X_s which as a uniformity base on X_s generates $\mathfrak{U}_s$. We show that

(i) $I(\mathcal{U}_\lambda^s) < \hat{\mathcal{U}}_\lambda$,

(ii) $I^{-1}(\hat{\mathcal{U}}_\lambda) < (\mathcal{U}_\lambda^s)^*$,

for each λ.

Let $U|W \in \mathcal{U}_\lambda^s$, where U is a component of $f^{-1}(W)$ and $W \in \mathcal{W}_\lambda$. Then $\hat{U} \in \hat{\mathcal{U}}_\lambda$, and we show that $I(U|W) \subset \hat{U}$. Let $\chi \in U|W$. Then $U = \chi(W)$, and so $U \in \operatorname{Im}\chi$. Consequently $U \in (\operatorname{Im}\chi)^+$; i.e., $(\operatorname{Im}\chi)^+ \in \hat{U}$. This shows that $I(U|W) \subset \hat{U}$, which proves (i). Now let $\hat{U} \in \hat{\mathcal{U}}_\lambda$, where U is a component of $f^{-1}(W)$ and $W \in \mathcal{W}_\lambda$. Then $U|W \in \mathcal{U}_\lambda^s$, and we claim that $I^{-1}(\hat{U}) \subset \operatorname{St}(U|W, \mathcal{U}_\lambda^s)$. Thus suppose that $(\operatorname{Im}\chi)^+ \in \hat{U}$, where χ is spread-point of some point $z \in Z$. Some $W' \in \mathcal{W}_\lambda$ contains z. Let $U' = \chi(W')$, where U' is a component of $f^{-1}(W')$. Then $U, U' \in (\operatorname{Im}\chi)^+$. Thus $U \cap U' \neq \emptyset$, which implies that $(U|W) \cap (U'|W') \neq \emptyset$. Since $\chi \in U'|W'$ and $U'|W' \in \mathcal{U}_\lambda^s$, it follows that $\chi \in \operatorname{St}(U|W, \mathcal{U}_\lambda^s)$. This shows that $I^{-1}(\hat{U}) \subset \operatorname{St}(U|W, \mathcal{U}_\lambda^s)$, which proves (ii).

The relations (i), (ii) imply respectively that I, I^{-1} are uniformly continuous.

QUESTION (5.5). *Under what circumstances is*

$$I(\mathcal{U}_\lambda^s) = \hat{\mathcal{U}}_\lambda?$$

6. Conclusion

In [15] we show how, when the category of spreads is restricted to those in which the base space is a topologically complete Hausdorff space, the theory of spreads can be developed using familiar definitions and theorems from the theory of uniform spaces (see §5, 6 of [14]). By virtue of the results in the present paper it is known that this program can be carried out.

CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS DEL IPN
MÉXICO, D. F., MÉXICO 07000

UNIVERSITY OF SASKATCHEWAN
SASKATOON, CANADA

UNIVERSITY OF THE WITWATERSRAND
JOHANNESBURG, SOUTH AFRICA

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