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AN INNER PRODUCT FOR A BANACH *-ALGEBRA

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Introduction

Several methods have been used to introduce inner products on Banach spaces and Banach algebras and study these structures through Hilbert space properties. One such method is the famous Gel'fand-Naimark-Segal (GNS) construction (see for instance, [3]). Cohen [2] defined an inner product on a commutative semi-simple Banach algebra using the Gel'fand transformations and a probability measure on the maximal ideal space. He established several results pertaining to the completeness of the resultant inner product space, the closure property of ideals and the decomposition of the Banach algebra. For instance, let A be a commutative semi-simple Banach algebra and A_2 the resultant inner product space, then it is shown in [2] that A_2 is a topological algebra. Moreover, if A is an A^*-algebra then any maximal ideal, M, is A_2-closed if $M^\perp \neq \{0\}$. In this case A can be decomposed as: $A = M \oplus M^\perp$.

In this paper we develop another technique to define an inner product on a commutative Banach algebra A with an essential involution, by means of a probability measure μ on the set S of all states of A. We show that the resultant inner product space A_μ is a topological algebra (not necessarily a normed algebra) and discuss the completeness of A_μ. Moreover, we prove that every maximal ideal M with $M^\perp \neq \{0\}$, is A_μ-closed and get a decomposition of $A = M \oplus M^\perp$. We present these results in Sections 1 and 2. Denote by $P(S)$ the set of all probability measures on S. Then it is well-known that $P(S)$ is convex and weak *-compact. In Section 3, we define an "intrinsic" norm on A by taking the supremum of all such norms (derived from the inner products) over $P(S)$. It is shown that the resultant normed space, denoted by A_s, is a normed algebra satisfying the C^*-condition as well. Thus we conclude that every commutative Banach algebra with an essential involution has an auxiliary norm which turns it into an A^*-algebra.

1. An inner product

Recall some basic definitions. Let $A = (A, \|\cdot\|)$ be a complex commutative unital Banach algebra with an involution $x \mapsto x^*$. A linear functional f defined on A is called positive if $f(xx^*) \geq 0$ for all $x \in A$. A state is a positive linear functional f with $\|f\| = 1$. An involution in A is called essential if for every non-zero element $x \in A$ there exists a positive linear functional f_0 such that $f_0(xx^*) \neq 0$. It can be shown that in a Banach algebra with an essential involution, for every non-zero element x there exists a positive linear functional f such that $f(x) \neq 0$ [4, p. 61]. We remark that a commutative Banach algebra with an essential involution is an algebra without radical [4, p. 63] and hence the involution is automatically continuous [6, p. 276]. For more details about an essential involution we refer to [4].

We asssume throughout this paper that the involution in a commutative Banach algebra A is essential (without mentioning it explicitly). Denote by S the set of all states on A. Then S is a weak*-compact convex subset of the dual of A [3, p. 115]. Let μ be a probability measure which satisfies the condition of being positive on non-empty open sets. That such measures exist follows from the following result.

LEMMA (1.1). *Let A be separable. Then there exists a probability measure μ on S which is positive on non-empty open sets.*

For instance, if A is separable, then S being weak *-compact is metrizable and hence separable [6, p. 68]. Let $\{f_n\}$ be a countable dense subset of S and put $\mu = \sum_{n=1}^{\infty} a_n \delta_{f_n}$, where $\sum_n a_n = 1,\ a_n > 0,\ a_n \in R$ and δ_f is the Dirac measure at f. Then μ is a probability measure which is positive on the non-empty open sets.

1.2. *Definition*. For each x, y in A, define

$$\langle x, y\rangle = \int_S f(xy^*)d\mu(f), \quad f \in S.$$

We now show that
$\langle x, y\rangle$ defines an inner product on A.

THEOREM (1.3). *$\langle x, y\rangle$ is an inner product on A.*

Proof. It is easy to see that $\langle x, y\rangle$ is linear in x, conjugate linear in y and $\langle x, x\rangle \geq 0$. To complete the proof, all we need to show is non-degeneracy, i.e. $\langle x, x\rangle = 0$ implies that $x = 0$. Now $\langle x, x\rangle = 0$ means that $\int f(xx^*)d\mu(f) = 0$ which implies that $f(xx^*) = 0$ for almost every $f \in S$. If $x \neq 0$ then there exists a state $f \in S$ such that $f(xx^*) \neq 0$. Therefore, the set $\{f \in S : f(xx^*) \neq 0\}$ is non-empty. This is also an open set because the mapping $f \to f(xx^*)$ is continuous by definition of the weak *-topology of S. Since μ is positive on non-empty open sets, hence it follows that the integral $\int f(xx^*)d\mu(f) \neq 0$. This contradiction proves the theorem. □

1.4. *Remark*. If A is a symmetric algebra without radical then the involution is always essential [4, pp. 64–65].

We shall denote the norm derived from the inner product by $\|\cdot\|_\mu$ and the resultant inner product space by A_μ.

The following Corollary follows immediately.

COROLLARY (1.5). *For each x and y in A*

(1) $\langle x, y\rangle = \langle y^*, x^*\rangle$;
(2) $\|x\|_\mu = \|x^*\|_\mu$;
(3) *if x and y are self-adjoint, then $\langle x, y\rangle$ is real.*

THEOREM (1.6). *A_μ is a topological algebra.*

Proof. It is enough to show that multiplication is separately continuous. If we fix y in A, then for all $x \in A$

$$\|x \cdot y\|_\mu^2 = \int_S f(xyy^*x^*)d\mu(f) \leq r(yy^*) \int_S f(xx^*)d\mu(f)$$
$$\leq \alpha\|x\|_\mu^2 \cdot \|y\|^2,$$

where α is some positive constant and $r(y)$ denotes the spectral radius of y. Here we have used the fact that $f(xyy^*x^*) \leq r(yy^*)f(xx^*)$ [1]. This shows that multiplication is separatey continuous. □

In a topological algebra the closure of an ideal is also an ideal and each maximal ideal is always A-closed. We now want to show that a maximal ideal is also A_μ-closed.

We first prove the following

LEMMA (1.7). *Let L be any non-zero ideal in A. Then the orthogonal complement, $L^\perp$, of L is also an ideal and $L^\perp \neq A$.*

Proof. Let $a \in L$, $b \in L^\perp$ and $x \in A$. Then

$$\langle a, bx\rangle = \int_S f(ax^*b^*)d\mu(f) = \langle ax^*, b\rangle = 0$$

since $ax^* \in L$. This implies that $bx \in L^\perp$. Hence $L^\perp$ is an ideal.

To prove that $L^\perp \neq A$, let us assume the contrary. Consider any non-zero a in L and suppose that $e \in L^\perp$. Then $\langle aa^*, e\rangle = 0$ since $aa^* \in L$. But $\langle aa^*, e\rangle = \langle a, a\rangle = 0$ implies that $a = 0$, a contradiction. This completes the proof. □

The following result gives a decomposition of A.

THEOREM (1.8). *If M is a maximal ideal of A such that $M^\perp \neq \{0\}$, then M is A_μ-closed. Moreover, A can be decomposed as: $A = M \oplus M^\perp$.*

Proof. Let $M^\perp \neq \{0\}$, then by Lemma 1.7, $M^{\perp\perp}$ is a proper ideal which includes M. This implies that $M = M^{\perp\perp}$, and hence M is A_μ-closed.

Let $B = M + M^\perp$. Then B is also an ideal, since M and $M^\perp$ are so. As $M^\perp \neq \{0\}$, M is properly contained in B. Hence by the maximality of M we get that $B = A$. □

2. Completeness of A_μ

In this section we present some results about the completeness of A_μ.

THEOREM (2.1). *Let X be a Banach space with respect to the norms $\|\cdot\|_1$ and $\|\cdot\|_2$. If there exists a constant k such that $\|x\|_1 \leq k\|x\|_2$ for all $x \in X$, then the two norms are equivalent.*

Proof. The proof follows from Banach's bounded inverse theorem, see for instance [5, p. 77]. □

THEOREM (2.2). *A_μ is a Hilbert space if and only if both the norms $\|\cdot\|$ and $\|\cdot\|_\mu$ are equivalent.*

Proof. Obviously if the two norms are equivalent, then A_μ is complete since A is complete with respect to $\|\cdot\|$.

Conversely, assume that A_μ is a Hilbert space. Since $\|x\|_\mu \le \beta\|x\|$ for some constant β, then it follows from Theorem 2.1 that the norms are equivalent. □

Let Δ be the set of all non-zero complex multiplicative homomorphisms of A equipped with its Gel'fand topology. The set Δ is usually called the maximal ideal space of A.

2.3. *Definition*. For each $x \in A$, define the spectral radius norm of x to be $\|x\|_r = \sup\{|\hat{x}(h)| : h \in \Delta\}$, where $\hat{x}$ is the Gel'fand transform associated with x. Denote by $A_r = (A, \|\cdot\|_r)$. A_r is a normed algebra.

Note that for all $x \in A$, $\|x\|_\mu \le \alpha_1\|x\|_r \le \alpha_2\|x\|$, for some positive constants α_1, α_2. As a consequence of Theorem 2.2 we get that, if A_μ is a Hilbert space then all the three norms are equivalent and hence A_r is complete. Also A_r is isometric to the subalgebra of Gel'fand transformations of $C(\Delta)$ via the mapping $x \mapsto \hat{x}$. Thus we have:

COROLLARY (2.4). *Let X_μ be a Hilbert space. Then the subalgebra of Gel'fand transformations of $C(\Delta)$ is closed.*

THEOREM (2.5). *Let A be an A^*-algebra. Then A_μ is a Hilbert space if and only if A is finite dimensional.*

Proof. The proof of Theorem 2.5 mainly follows the arguments similar to that of Cohen [2]. □

3. The intrinsic norm and some of its properties

Consider the set $P(S)$ of all probability measures on S. It is well-known that $P(S)$ is convex and weak *-compact.

3.1. *Definition*. For each $x \in A$ define

$$\|x\|_s = \sup_{\mu \in P(S)} \|x\|_\mu.$$

Obviously, the supremum is always attained and $\|x\|_s$ satisfies all the axioms of a norm. Denote by $A_s = (A, \|\cdot\|_s)$.

THEOREM (3.2). *A_s is a topological algebra.*

Proof. The proof is immediate from Theorem 1.6. □

We shall need the following

LEMMA (3.3). *For each $x \in A$ there exists a measure $\mu_0 \in P(S)$ which is an extreme point of the convex set $P(S)$ such that*

$$\|x\|_s^2 = \int_S f(xx^*)\,d\mu_0(f).$$

Proof. Let us fix x in A and consider the function $\psi_x : P(S) \to R$ given by $\psi_x(\mu) = \|x\|_\mu^2$. Then $\psi_x(\alpha\mu_1 + \beta\mu_2) = \alpha\psi_x(\mu_1) + \beta\psi_x(\mu_2)$, $\forall \mu_1, \mu_2 \in P(S)$ and for α, β non-negative reals such that $\alpha + \beta = 1$. We need to show that there exists μ_0 an extreme point of $P(S)$ such that

$$M \stackrel{\text{def}}{=}: \sup_{\mu \in P(S)} \psi_x(\mu) = \psi_x(\mu_0).$$

Let $P_0 = \psi_x^{-1}(\{M\}) \subset P(S)$. Then P_0 is a non-empty, convex and weak *-compact subset of $P(S)$. By Krein-Milman theorem (see for instance [6]), there exists an extreme point, μ_0, of P_0. We claim that μ_0 is also an extreme point of all of $P(S)$. If this were not the case, then there exist μ_1 and μ_2 in $P(S)$, $\mu_1, \mu_2 \notin P_0$; and α, $\beta > 0$ with $\alpha + \beta = 1$ such that $\mu_0 = \alpha\mu_1 + \beta\mu_2$. By convexity of ψ_x, $\psi_x(\mu_0) = \alpha\psi_x(\mu_1) + \beta\psi_x(\mu_2)$, we immediately get that either $\psi_x(\mu_1) > M$ or $\psi_x(\mu_2) > M$. This contradicts the hypothesis. Hence μ_0 is an extreme point of $P(S)$. Thus we have shown that $\|x\|_s^2$ is attained and moreover, on an extreme point of $P(S)$. Therefore, $\|x\|_s^2 = \int f(xx^*)d\mu_0(f)$, which proves the lemma. □

Since the extreme points of $P(S)$ correspond to the Dirac measures we have $\mu_0 = \delta_{f_x}$ for some $f_x \in S$. This implies that $\|x\|_s^2 = \delta_{f_x}(\varphi) = f_x(xx^*)$ (f_x depending only on x and it may not be unique), where the function $\varphi : S \to R$ is given by $\varphi(f) = f(xx^*)$. Thus we have

COROLLARY (3.4). *For each $x \in A$*

$$\|x\|_s^2 = \sup_{\|f\|=1} f(xx^*).$$

Let us again fix x in A and consider the function $\varphi(f) = f(xx^*)$. Then φ is a convex function.

Proceeding as in Lemma 3.3, one proves that there exists a state f_x which is an extreme point of S and

$$\|x\|_s^2 = f_x(xx^*).$$

Then f_x must be multiplicative (see [6, p. 286]). □

Summarizing all this we have

THEOREM (3.5). *Let A be a commutative Banach algebra with an essential involution. Then for each $x \in A$ there exists a state f_x, which is an extreme point of S, such that*

$$\|x\|_s^2 = f_x(xx^*).$$

Now, we prove the following

LEMMA (3.6). *A_s is a normed algebra and moreover, $\|xx^*\| = \|x\|_s^2$.*

Proof. Let $x, y \in A$. By Theorem 3.5, there exists $f_{x,y}$ such that $\|xy\|_s^2 = f_{x,y}(xyy^*x^*)$. Note that $f_{x,y}$ is multiplicative. Therefore, it follows that

$$\begin{aligned}\|xy\|_s^2 &= f_{x,y}(xyy^*x^*) = f_{x,y}(xx^*)f_{x,y}(yy^*)\\ &\leq \sup_{\|f\|=1} f(xx^*)] \sup_{\|f\|=1} f(yy^*)]\\ &= \|x\|_s^2 \cdot \|y\|_s^2,\end{aligned}$$

using Corollary 3.4.

On the other hand, a further application of Theorem 3.5 and Corollary 3.4 yields

$$\begin{aligned}\|xx^*\|_s^2 &= f_{x,x^*}(xx^*xx^*) = f_{x,x^*}(xx^*)f_{x,x^*}(xx^*)\\ &= [f_{x,x^*}(xx^*)]^2 = \sup_{\|f\|=1} [f(xx^*)]^2 = \|x\|_s^2,\end{aligned}$$

i.e., $\|xx^*\|_s = \|x\|_s^2$. Here we have used the fact that the suprema are attained at multiplicative positive functionals. This completes the proof. □

3.7. *Remark*. The norm introduced so far seems to depend upon the original norm of A. However, for a commutative unital Banach *-algebra, one has

$$\|x\|^2 = \sup_{f\in M} f(xx^*),$$

where $M = \{f \in S : f \text{ is multiplicative}\}$. Therefore, the norm is in fact intrinsic.

Using Lemma 3.6 we get our main result.

THEOREM (3.8). *Let A be a commutative Banach algebra with an essential involution. Then there exists an auxiliary norm $\|\cdot\|_s$ on A such that A becomes an A^*-algebra.*

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