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THE UNION PROBLEM AND DOMAINS OF HOLOMORPHY

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Let X be a complex space and $\Omega_1 \subset \Omega_2 \subset \cdots$ a sequence of open Stein subsets in X. Let $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$. The question is, is Ω Stein? If Ω is a Stein manifold, that is the case. For an arbitrary manifold X, J. E. Fornaess in [3], has given an example of a sequence of increasing Stein subsets in a manifold whose union is not Stein. If X has singularities, nothing is known about this. In this paper I show that if X is a normal Stein space and Ω is relatively compact in X, then Ω is a domain of holomorphy. In [10] I gave the anouncement of this result with the additional condition that $\partial\Omega \backslash S$ is dense in $\partial\Omega$, where S is the singular set of X. I give in the first part the proof of this result and next, I show that the additional condition is not necessary.

For the basic notions and properties of Stein spaces we refer to [6] and [8].

THEOREM (1). *Let X be a normal Stein space, $\Omega_1 \subset \Omega_2 \subset \cdots$ a sequence of open Stein subsets of X and let $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$, be irreducible. Let S be the singular set of X. If $\Omega \subset\subset X$ and $\partial\Omega \backslash S$ is dense in $\partial\Omega$, then Ω is a domain of holomorphy.*

Proof. The proof is based in the techniques of Fornaess-Narasimhan utilized in [5]. We will show for any $p \in (\partial\Omega \backslash S)$ and any sequence $\{z_n\}$ of points of Ω, $z_n \to p$, there is a subsequence $\{z_{n_i}\}$ and a holomorphic function F in Ω, which diverges in $\{z_{n_i}\}$. We may assume that X is a pure n-dimensional and connected space. Andreotti and Narasimhan in [1] show that a pure n-dimensional Stein Space may be realized as a branched domain over C^n in such a way that: we can find holomorphic mappings $\Phi_r : X \to C^n$, $r = 1, 2, \ldots l$ such that if Z_r^l is the branch locus of Φ_r then

i) Φ_r has discrete fibers

ii) Φ_r is a local isomorphism in $\Omega \backslash Z_r^l$

iii) $S = \bigcap_{r=1}^{l} Z_r^l$

Moreover, we can find holomorphic functions $f_1, f_2, \ldots, f_l$ on X such that if we set $Z_r = \{f_r = 0\}$, then $Z_r^l \subset Z_r$ and $\cap Z_r = S$.

Let d_r be the boundary distance of the unramified domain

$$\Phi_r : \Omega \backslash Z_r \to C^n$$

Since $X \backslash Z_r$ is Stein without singularities and each $\Omega_j \backslash Z_r$ is Stein, then $\Omega \backslash Z_r$ is an increasing sequence of open Stein subsets of $X \backslash Z_r$, therefore is Stein [2], $-\log d_r$ is plurisubharmonic in $\Omega \backslash Z_r$ and the function

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$$\varphi_r(x) = \begin{cases} \max(0, -\log d_r + M_r \log|f_r(x)|) & \text{on } \Omega\backslash Z_r \\ 0 \quad \text{on } Z_r \end{cases}$$

is plurisubharmonic on Ω, where M_r, is a large enough constant, see [1].

Let $p \in (\partial\Omega\backslash S)$, and choose r such that $p \notin Z_r$. We can find $h_1, h_2 \ldots, h_m$ holomorphic functions on X such that

$$\{p\} = \bigcap_{i=1}^{m} \{h_i = 0\}.$$

Write $\Phi_r(x) - \Phi_r(p) = \{\Phi_r^s(x) - \Phi_r^s(p)\}_{s=1}^n$, where $\Phi_r^1, \ldots, \Phi_r^n$ are the coordinate functions of the map Φ_r. Since the function

$$x \to \Phi_r^s(x) - \Phi_r^s(p)$$

vanishes in p, by the Nullstellensatz, there exists a neighborhood $V(p)$ of p and a positive integer N_s, such that

$$V(p) \cap Z_r = \emptyset \text{ and } (\Phi_r^s(x) - \Phi_r^s(p))^{N_s} = \sum_{i=1}^{m} \alpha_{s_i}(x) h_i(x)$$

for some $\alpha_{s_i}, \ldots, \alpha_{s_m}$ holomophic functions on $V(p)$. Take $V' \subset\subset V(p)$ a neighborhood of p and let $T_s := \sup_{x \in V'} \{\sum_{i=1}^m |\alpha_{s_i}(x)|^2\}$.

If $T = \max_s T_s$, $N \geq 2 \max_s N_s$ and $x \in V'$, let s_x be the integer such that

$$|\Phi_r^{s_x}(x) - \Phi_r^{s_x}(p)| = \max_s \{|\Phi_r^s(x) - \Phi_r^s(p)|\},$$

then

$$\begin{aligned} \| \Phi_r(x) - \Phi_r(p) \|^N &\leq (\sum_{s=1}^{n} |\Phi_r^s(x) - \Phi_r^s(p)|)^N \leq n^N (\max_s \{|\Phi_r^s(x) - \Phi_r^s(p)|\})^N \\ &\leq n^N |\Phi_r^{s_x}(x) - \Phi_r^{s_x}(p)|^{2N_s} \leq n^N (\sum_{i=1}^{m} |\alpha_{s_i}(x)|^2)(\sum_{i=1}^{m} |h_i(x)|^2) \\ &\leq n^N T (\sum_{i=1}^{m} |h_i(x)|^2) \end{aligned}$$

Since $-\log d_r(x) + M_r \log|f_r(x)| \leq \varphi_r(x)$, it follows that there exists a constant $C_0 > 0$, such that, if $x \in V' \cap \Omega$, then $d_r(x) \geq C_0 \exp(-\varphi_r(x))$, so

$$C_0^N \exp(-N\varphi_r(x)) \leq d_r(x)^N \leq \| \Phi_r(x) - \Phi_r(p) \|^N \leq n^N T (\sum_{i=1}^{m} |h_i(x)|^2)$$

Since $h_1, h_2, \dots, h_m$ are continuous in Ω, there exist constants $C_1, C_2 > 0$ such that

$$\sum_{i=1}^{m} |h_i|^2 \geq C_1 \text{ in } \Omega\backslash V' \text{ and } \sum_{i=1}^{m} |h_i|^2 \leq C_2 \text{ in } \Omega$$

Let $C_3 = \min\left(\frac{C_0^N}{n^N T}, C_1\right)$ therefore if $x \in \Omega$,

$$C_3 \exp\left(-N\varphi_r(x)\right) \leq \sum_{i=1}^{m} |h_i(x)|^2 \leq C_2 \exp\left(\varphi_r(x)\right)$$

Consider now the following results.

LEMMA. *Let $\Pi : D \to C^n$ a finite bounded unramified domain of holomorphy over C^n. Let $\varphi \geq 0$, p.s.h. function on D. Let $h_1, h_2, \dots, h_m$ be holomorphic functions on D and α, β, A, B positive constants such that*

$$\alpha \exp\left(-A\varphi(x)\right) \leq \sum_{i=1}^{m} |h_i(x)|^2 \leq \beta \exp\left(B\varphi(x)\right).$$

Then, there exist holomorphic functions $g_1, g_2, \dots, g_m$ on D satisfying

$$\sum_{i=1}^{m} g_i h_i = 1 \text{ and } \sum_{i=1}^{m} \int_D |g_i(x)|^2 \exp\left(-C\varphi(x)\right) dv < 0$$

where C is a constant depending on A, B, n, m.

Proof. See [9, Theor. 1, Cor. 1], where this result is proved by Skoda for pseudoconvex domains in C^n, but the proof remains valid for finite bounded unramified domains over C^n. □

Applying this result we have that there exist holomorphic functions $g_1, g_2, \dots, g_m$ on $\Omega\backslash Z_r$ and a constant C^* such that

$$\sum_{i=1}^{m} g_i h_i = 1 \text{ on } \Omega\backslash Z_r \text{ and } \int_{\Omega\backslash Z_r} |g_i(x)|^2 \exp\left(-C^*\varphi_r(x)\right) d\lambda < \infty$$

LEMMA. (Fornaess-Narasimhan [5]). *Let D be a normal complex space and $\Phi : D \to \Delta^n$ a representation of D as a finite covering map on Δ^n. Let f be holomorphic on D such that Φ is unramified out of $Z = \{f = 0\}$. Let N be the sheet number of Φ. Then for any holomorphic function g on $D\backslash Z$ such that*

$$\int_{D\backslash Z} |g|^2 dv < \infty.$$

The function $f^{N-1}g$ extends to a holomorphic function on D. □

Appliying these results, we have $\sum_{i=1}^{m} g_i h_i = 1$ on $\Omega\backslash Z_r$ and $F_i = f_r^* g_i$ is holomorphic on Ω. Since $f_r(p) \neq 0$, if $x_n \to p$, for some $i = 1, \dots, n$, we have $|F_i(x)| \to \infty$. Since $\partial\Omega\backslash S$ is dense in $\partial\Omega$ the Theorem follows. □

THEOREM (2). *Let X be an irreductible Stein space of pure dimension n and A an analytic set which is at least two-codimensional in X. Then it is not possible to express the difference space* (Y, O_Y), *where* $Y = X\backslash A$ *and* $O_Y := O_{X|Y}$, *as an increasing and nested union of open Stein subsets of X.*

Proof. Suppose the result is not true. Then there exists X and A as above, and an increasing and nested sequence of open Stein subsets $\{\Omega_j\}_{j\in N}$ of X, such that $X\backslash A = \bigcup_{j=1}^{\infty} \Omega_j$.

For any $x \in X$, we know there exists a neighborhood basis $\{D_m\}$ of X and a proper and light map $\Pi : D_m \to \Delta^n(o, r_m)$, such that D_m is an analytic polyhedron and the triple $(D_m, \Pi, \Delta^n(o, r_m))$ is a finite branched analytic covering.

Since the level sets of Π are compact and Π is light, these level sets are zero-dimensional. Then, $\dim_x \Pi = \dim_x X -$ dim $L_x(\Pi) = n$, where $L_X(\Pi)$ denotes the level set of Π in X. By the proper mapping theorem, we have, $\Pi(D_m) = \Delta^n(o, r_m)$.

Let $c_0 \in A$. take the following restrictions:

a) Choose a neighborhood U' of c_0 small enough so that the branching order of Π in any neighborhood of c_0 contained in U' is constant.

b) Since $\Pi^{-1}(\Pi(A \cap U'))|U'$ is the union of the restriction of A to U' with an analytic set A_0, take $z_0 \in (A \cap A_0^c)$ and choose a neighborhood U_0 of z_0 in such a way that $U_0 \cap A_0$ is empty and U_0 is Stein.

Since A is at least two=dimensional in X, $U_0\backslash A$ is not Stein. Now Π is an holomorphic and finite mapping between two spaces of the same dimension, therefore $\Delta^n(y_0, r)\backslash\Pi(A)$ is not Stein either. Then we can find a Hartogs figure

$$H \subset\subset (\Delta^n(y_0, r)\backslash\Pi(A))$$

such that its Hartogs completion H^* is contained in $\Delta^n(y_0, r)$, but H^* is not contained in $\Delta^n(y_0, r)\backslash\Pi(A)$.

From the above we obtain $\Pi^{-1}(\overline{H}) \cap (U_0 \cap A) = \emptyset$, so there exists an integer N, such that $\Pi^{-1}(\overline{H}) \subset (U_0 \cap \Omega_N)$. $U_0 \cap \Omega_N$ is Stein, so we can find an Oka-Weil domain W such that $\Pi^{-1}(\overline{H}) \subset W \subset\subset U_0 \cap \Omega_N$,[8].

Notice that Π is a local homeomorphism in $U_0\backslash\Pi(A^*)$, where A^* is a negligible set formed by the images of the singular set of X and the regular points of U_0 where $\text{rank}_x\Pi \neq n$.

Take $y^* \in (H^* \cap \partial\Pi(W))$. Let $\Pi^{-1}(y^*) = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$. If $y_m \to y^*$, with $\{y_m\} \subset \Pi(w)$, then, there exist corresponding sequences $\{v_m^i\} \to \alpha_i$, $i = 1, \dots, r$, with $\{v_m^i\} \subset W$. Therefore we can find a holomorphic function f in W which is not bounded in each sequence $\{v_m^i\}$.

For each $y \in (\Pi(w)\backslash A^*)$, if $\Pi^{-1}(y) = \{z_1, z_2, \dots, z_k\}$, let Δy be a neighborhood of y contained in $\Pi(W)\backslash A^*$ such that $\Pi^{-1}(\Delta y) = \bigcup_{i=1}^{k} \Delta_{s_i}$, where Δ_{s_i} is a neighborhood of z_i and

$$\Pi : \Delta_{s_i} \to \Delta y$$

is a biholomorphism.

Define $f_y^i := (f_{|\Delta_{s_i}}) \circ \Pi^{-1}$. Clearly f_y^i is holomorphic in Δy. Consider the symmetric polynomial

$$P(a_1, a_2, \dots, a_k) = a_1 \cdot a_2 \cdots a_k.$$

Then define the function P^* in $H(W)\backslash A^*$ as

$$P^* := P(f_y^1, f_y^2, \dots, f_y^k)$$

in Δy. If $\Delta y \cap \Delta y' \neq \emptyset$ and $y, y' \in (\Pi(W)\backslash A^*)$, note that, $(f_y^1, f_y^2, \dots, f_y^k)$ is just a permutation of $(f_{y'}^1, f_{y'}^2, \dots, f_{y'}^k)$, therefore P^* is well-defined in $\Pi(W)\backslash A^*) \supset (H\backslash A^*)$. Since f is continuos in W, f is locally bounded in $\Pi(W)$, so it is possible to extend P^* to H^*. But P^* is holomorphic in H and not bounded in $\{y_m\}$, a contradiction. □

COROLLARY. *Let X be a normal Stein space of pure dimension n and $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$, where each Ω_j is an open Stein subset of X and such that $\Omega_j \supset \Omega_{j+1}$, $j \in N$. Then the singular set of X is nowhere dense in the boundary of Ω.*

Proof. Denote S the singular set of X. We assert $\partial\Omega\backslash S$ is dense in $\partial\Omega$. Otherwise, there would be $p \in S \cap \partial\Omega$ and a Stein neighborhood U_p of p, such that

$$U_p \bigcap (\partial\Omega\backslash S) = \emptyset$$

i.e. S would be dense in $\partial\Omega$ in some neighborhood of p. Since S is closed in $\partial\Omega$, it follows that

$$S_{|U_p} = \partial\Omega_{|U_p}$$

Now, S is at least two-codimensional in X, so $S \cap \partial\Omega$ does not separate U_p in two open disjoint subsets of X. It follows $U_p\backslash S \subset \Omega$, therefore

$$U_p\backslash S = \bigcup_{j=1}^{\infty} (\Omega_j \cap U_p)$$

But each $\Omega_j \cap U_p$ is Stein, so the above equality contradicts the Theorem. □

From this Corollary and Theorem (1), it follows immediately:

THEOREM (3). *Let X be a normal Stein space, $\Omega_1 \subset \Omega_2 \subset \cdots$ a nested sequence of open Stein subsets of X and let $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$ be irreducible. If $\Omega \subset\subset X$, then Ω is a domain of holomorphy.*

Definition. Let X, Y be two complex spaces of the same dimension. We say that the holomorphic mapping $\Pi : X \to Y$ defines a ramified covering of Y, if for every point $y \in Y$ there exists a neighborhood U such that $\Pi^{-1}(U)$ is the disjoint union of complex spaces $W_1, W_2, \ldots$ for each of which every induced map $\Pi_\nu|_{W_\nu} \to U$ is finite.

P. Le Barz in [11] proved the next important result.

THEOREM (4). (Le Barz) *Let $\Pi : X \to Y$ a ramified covering of Y. If Y is Stein, so is X.*

Applying this result we obtain the next generalization of Theorem (3).

THEOREM (5). *Let $\Pi : X \to Y$ be a ramified covering of a Stein space Y and suppose that X is normal. If $\Omega_1 \subset \Omega_2 \subset \cdots$ is a nested sequence of open Stein subsets of X and $\Omega := \bigcup_{j=1}^{\infty} \Omega_j$ is irreducible and relatively compact in X, then Ω is a domain of holomorphy.*

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