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HYDRODYNAMIC AND FLUCTUATION LIMITS OF BRANCHING PARTICLE SYSTEMS WITH CHANGES OF MASS

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1. Introduction

Branching diffusions are models of particle systems which evolve in space by random migration and branching. The usual object of study is the counting measure-valued process determined by the locations of the particles present at each time, assuming implicitly that all the particles have mass 1. In this paper we consider a "branching mass" model, where each particle has its own mass, and when a particle branches the mass of each of its offspring is proportional to the mass of the parent and depends on the number of the offspring produced. An example of this model is a system of small spheres in R^3 (e.g. a vapor or a powder) such that when a sphere of surface S splits into n equal spheres with conservation of volume, each one of the new spheres has a surface equal to $Sn^{-2/3}$ and one is interested in the space-time distribution of surface of the system (the surface of a sphere is taken as the "mass" of the particle). We consider the measure-valued process determined by the locations and the masses of the particles present at each time. This mass process is not Markovian. We will present laws of large numbers (hydrodynamic limits) and fluctuation limits of this process under different rescalings. The fluctuation limits are Markovian generalized Gaussian Ornstein-Uhlenbeck processes. These results include the known ones in the special case when all the particles have mass 1.

2. The model, notation

The system consists of particles in Euclidean space R^d which evolve as follows. At time $t = 0$ the particles are distributed according to a Poisson random field on $B \in \mathcal{B}(R^d)$ (the Borel sets) with intensity $\gamma \geq 0$. As time elapses, each particle independently migrates according to a symmetric stable process with exponent $\alpha \in (0, 2]$, and after an exponentially distributed lifetime with parameter V it branches, producing n offspring with probability p_n, $n = 0, 1, \ldots$. The offspring are born at the same site where their parent branches, and they also migrate and branch as described. In addition, particles immigrate into R^d according to a (space-time) Poisson random field on $\mathcal{C} \in \mathcal{B}(R^d \times R_+)$ (assumed to have a smooth boundary) with intensity $\beta \geq 0$, and each immigrant particle also evolves as above. The initial and the immigration Poisson fields are independent. Let $N \equiv \{N_t,\ t \geq 0\}$ denote the counting measure-valued process defined by

$$N_t = \sum_{i=1}^{\infty} \delta_{x_i},$$

where $\{x_i\}_{i=1}^{\infty}$ are the locations of the particles present at time t. Up to this point we have the usual model. We assume in addition that each particle has a (positive) mass of its own, and when a particle of mass a produces n (> 0) offspring, the mass of each of the offspring is ac_n where $c_n \geq 0$. For simplicity we assume that all initial and all immigrant particles have mass 1. Let $M \equiv \{M_t,\ t \geq 0\}$ denote the measure-valued process defined by

$$M_t = \sum_{i=1}^{\infty} a_i \delta_{x_i} \ \text{ if } \ N_t = \sum_{i=1}^{\infty} \delta_{x_i},$$

where a_i is the mass of the particle at x_i, $i = 1, 2, \ldots$ The process M is not Markovian (looking at a point mass we do not know if it is a single particle or if a branching has occured at that site, and therefore the past and the future are not independent conditioned upon the present), but the pair (N, M) is Markovian (N tells us if a branching has occured). We will also need the measure-valued processes $M^2 \equiv \{M_t^2,\ t \geq 0\}$ and $M^4 \equiv \{M_t^4,\ t \geq 0\}$ defined by

$$M_t^2 = \sum_{i=1}^{\infty} a_i^2 \delta_{x_i} \text{ and } M_t^4 = \sum_{i=1}^{\infty} a_i^4 \delta_{x_i} \text{ if } M_t = \sum_{i=1}^{\infty} a_i \delta_{x_i},$$

and we note that (N, M, M^2, M^4) is Markovian.

We assume that the mass produced by the branching, relative to the mass of the parent, has finite third moment, and we denote the first three moments by

$$m_1 = \sum_{n=0}^{\infty} nc_n p_n, \quad m_2 = \sum_{n=0}^{\infty} (nc_n)^2 p_n, \quad m_3 = \sum_{n=0}^{\infty} (nc_n)^3 p_n,$$

We also write $q_1 = \sum_{n=0}^{\infty} nc_n^2 p_n$.

It can be shown that the processes N, M, M^2, M^4 have paths in $D(R_+, \mathcal{S}'(R^d))$, the space of right-continuous with left limits functions from R_+ into $\mathcal{S}'(R^d)$, where $\mathcal{S}'(R^d)$ is the space of tempered distributions, i.e., the dual of $\mathcal{S}(R^d)$, the space of infinitely differentiable rapidly decreasing functions from R^d into R. As is well known, these are appropriate spaces for studying weak convergence of fluctuation processes of particle systems. The topologies on the spaces $\mathcal{S}(R^d)$, $\mathcal{S}'(R^d)$ are well-known. The space $D(R_+, \mathcal{S}'(R^d))$ is endowed with a Skorokhod-type topology [19,17].

The following notation will be used.

$\langle \cdot, \cdot \rangle$: the canonical bilinear form on $\mathcal{S}'(R^d) \times \mathcal{S}(R^d)$.

$\Delta_\alpha \equiv -(-\Delta)^{\alpha/2}$: the infinitesimal generator of the symmetric stable process with exponent $\alpha \in (0, 2]$.

$A_1 = V(m_1 - 1), \quad A_2 = V(q_1 - 1)$.

$\mathcal{T} \equiv \{\mathcal{T}_t,\ t \geq 0\}$: the semigroup generated by Δ_α.

$\mathcal{U} \equiv \{\mathcal{U}_t,\ t \geq 0\}$: the semigroup defined by $\mathcal{U}_t = e^{A_1 t}\mathcal{T}_t$, with generator $\mathcal{A}_1 = \Delta_\alpha + A_1$.

$\mathcal{V} \equiv \{\mathcal{V}_t,\ t \geq 0\}$: the semigroup defined by $\mathcal{V}_t = e^{A_2 t}\mathcal{T}_t$, with generator $\mathcal{A}_2 = \Delta_\alpha + A_2$.

$F(\phi, \psi)$ denotes the continuous bilinear form defined on $\mathcal{S}(R^d) \times \mathcal{S}(R^d)$ by

$$F(\phi, \psi) = \Delta_\alpha(\phi\psi) - \phi\Delta_\alpha\psi - \psi\Delta_\alpha\phi + V(m_2 - 2m_1 + 1)\phi\psi.$$

For $\alpha = 2$ the operators Δ_α and $\mathcal{T}_t$ map $\mathcal{S}(R^d)$ into itself, but for $\alpha < 2$ they do not, and it is necessary to introduce the following spaces (see [6] for details). Let $\phi_p(x) = (1 + \|x\|^2)^{-p},\ x \in R^d,\ (p > 0)$.

$C_p(R^d) = \{\phi \in C(R^d) : \|\phi\|_p < \infty\}$: Banach space with norm $\|\phi\|_p = \sup_{x \in R^d} |\phi(x)/\phi_p(x)|$.

$\mathcal{M}_p(R^d)$ = the non negative Radon measures μ on R^d such that $\int \phi_p d\mu < \infty$.

The spaces $\mathcal{M}_p(R^d)$ and $C_p(R^d)$ are in dualtity (the duality is also denoted by $\langle \cdot, \cdot \rangle$).

The space $\mathcal{M}_p(R^d)$ contains the Lebesgue measure for $p > d/2$, and the processes N, M, M^2, M^4 take values in $\mathcal{M}_p(R^d)$.

$\mathcal{T}_t$ maps $C_p(R^d)$ into itself, and for $d/2 < p < (d+\alpha)/2$, Δ_α and $\mathcal{T}_t$ map $\mathcal{S}(R^d)$ continuously into $C_p(R^d)$, and $t \mapsto \mathcal{T}_t\phi$ is a continuous curve in $C_p(R^d)$ for $\phi \in \mathcal{S}(R^d)$.

Hence all the expressions below are well-defined, in particular $F(\mathcal{U}_s\phi, \mathcal{U}_t\psi)$ for $\phi, \psi \in \mathcal{S}(R^d)$ (it can be shown that $(\mathcal{T}_s\phi)(\mathcal{T}_t\psi)$ belongs to the domain of Δ_α).

3. Laws of large numbers and fluctuation limits

We consider the following three rescalings of the mass process M, denoting by $K > 0$ the scaling parameter which tends to ∞.

(1) **High density.** The initial and immigration intensities are given by $K\gamma$ and $K\beta$, respectively. We denote by $M^{(1),K} \equiv \{\langle M_t^{(1),K}, \phi\rangle,\ \phi \in \mathcal{S}(R^d),\ t \geq 0\}$ the process with these intensities.

(2) **Space scaling.** The space scaling is given by $x \mapsto Kx$. We denote by $M^{(2),K} \equiv \{\langle M_t^{(2),K}, \phi\rangle,\ \phi \in \mathcal{S}(R^d),\ t \geq 0\}$ the rescaled process, where

$$\langle M_t^{(2),K}, \phi\rangle \equiv \langle M_t, \phi(\cdot/K)\rangle.$$

In this case we assume that the initial set B and the immigration set $\mathcal{C}$ satisfy $KB = B$ and $K\mathcal{C}_t = \mathcal{C}_t$ for all $K > 0$ and $t > 0$, where $\mathcal{C}_t = \{x \in R^d | (x,t) \in \mathcal{C}\}$ is the t-section of $\mathcal{C}$.

(3) **Space-time scaling.** The space-time scaling is given by $(x,t) \mapsto (Kx, K^\alpha t)$. We denote by $M^{(3),K} \equiv \{\langle M_t^{(3),K}, \phi\rangle,\ \phi \in \mathcal{S}(R^d),\ t \geq 0\}$ the rescaled process, where

$$\langle M_t^{(3),K}, \phi\rangle = \langle M_{K^\alpha t}, \phi(\cdot/K)\rangle.$$

The immigration intensity is given by β/K^α, the sets B and $\mathcal{C}$ have the same properties as in (2) and $\mathcal{C}_t \to Q \in \mathcal{B}(R^d)$ as $t \to \infty$, the parameter α satisfies the condition $\alpha < d$, and the change of mass and the branching law may depend on K as follows:

$$m_1^K = 1 + a_K/K^\alpha,\ q_1^K = 1 + b_K/K^\alpha,\ a_K \to a \in R,\ b_K \to b \in R,$$
$$m_2^K \to m_2 \geq 1 \text{ as } K \to \infty, \text{ and } \sup_{K\geq 1} m_3^K < \infty.$$

The fluctuation processes $X^{(1),K}$, $X^{(2),K}$, $X^{(3),K}$ corresponding to these three rescalings are defined by

$$\begin{aligned} X^{(1),K} &= K^{-1/2}(M^{(1),K} - EM^{(1),K}), \\ X^{(2),K} &= K^{-d/2}(M^{(2),K} - EM^{(2),K}), \\ X^{(3),K} &= K^{-(d+\alpha)/2}(M^{(3),K} - EM^{(3),K}). \end{aligned}$$

We remark that these processes are not Markovian.

THEOREM (1). (Laws of large numbers). *For each $t \geq 0$ and $\phi \in \mathcal{S}(R^d)$,*

$$K^{-1}\langle M_t^{(1),K}, \phi\rangle \to \gamma \int_B \mathcal{U}_t\phi(x)dx + \beta\int_0^t\int_{\mathcal{C}_r} \mathcal{U}_{t-r}\phi(x)dxdr,$$

$$K^{-d}\langle M_t^{(2),K}, \phi\rangle \to \gamma e^{A_1 t}\int_B \phi(x)dx + \beta e^{A_1 t}\int_0^t e^{-A_1 r}\int_{\mathcal{C}_r}\phi(x)dxdr,$$

$$K^{-d}\langle M_t^{(3),K}, \phi\rangle \to \gamma e^{Vat}\int_B \mathcal{T}_t\phi(x)dx + \beta\int_0^t e^{Var}\int_Q \mathcal{T}_r\phi(x)dxdr$$

in L^2 as $K \to \infty$.

THEOREM (2). (Fluctuation limits). $X^{(1),K} \Rightarrow X^{(1)}$, $X^{(2),K} \Rightarrow X^{(2)}$, $X^{(3),K} \Rightarrow X^{(3)}$ *(weak convergence in $D(R_+, \mathcal{S}'(R^d))$ as $K \to \infty$, where $X^{(1)}$, $X^{(2)}$, $X^{(3)}$ are continuous, centered, Gauss-Markov processes which satisfy the respective generalized Langevin equations (see Remark 5)*

$$\begin{aligned} dX_t^{(1)} &= (\Delta_\alpha + A_1)X_t^{(1)}dt + dW_t^{(1)}, \quad t > 0, \\ X_0^{(1)} &= \gamma^{1/2}W_B, \end{aligned} \tag{1}$$

where W_B is the standard Gaussian white-noise on B, and $W^{(1)}$ is the $\mathcal{S}'(R^d)$-Wiener process with

$$\begin{aligned} Q_u^{(1)}(\phi,\psi) = {} & \gamma\int_B \mathcal{V}_u F(\phi,\psi)(x)dx \\ & + \beta\{\int_{\mathcal{C}_u}\phi(x)\psi(x)dx + \int_0^u\int_{\mathcal{C}_r}\mathcal{V}_{u-r}F(\phi,\psi)(x)dxdr\}, \end{aligned}$$

(2)
$$dX_t^{(2)} = A_1 X_t^{(2)} dt + dW_t^{(2)}, \quad t > 0,$$
$$X_0^{(2)} = \gamma^{1/2} W_B,$$

where W_B is the standard Gaussian white-noise on B, and $W^{(2)}$ is the $\mathcal{S}'(R^d)$-Wiener process with

$$Q_u^{(2)}(\phi,\psi) = \gamma V(m_2 - 2m_1 + 1)e^{A_2 u}\int_B \phi(x)\psi(x)dx$$
$$+\beta\{\int_{C_u}\phi(x)\psi(x)dx + V(m_2 - 2m_1 + 1)\int_0^u e^{A_2(u-r)}\int_{C_r}\phi(x)\psi(x)dxdr\},$$

(3)
$$dX_t^{(3)} = (\Delta_\alpha + Va)X_t^{(3)} dt + dW_t^{(3)}, \quad t > 0,$$
$$X_0^{(3)} = 0,$$

where $W^{(3)}$ is the $\mathcal{S}'(R^d)$-Wiener process with

$$Q_u^{(3)}(\phi,\psi) = \gamma V(m_2 - 1)e^{Vbu}\int_B \mathcal{T}_u(\phi\psi)(x)dx$$
$$+\beta V(m_2 - 1)\int_0^u e^{Vbr}\int_Q \mathcal{T}_r(\phi\psi)(x)dxdr.$$

Remarks

(1) The present results yield the known ones by setting $c_n = 1$ for all n (see [1,4,7,8,11,12,13,14,15,16]. López-Mimbela [18] extended these known results to multitype branching particle systems; it would be interesting to study multitype systems with changes of mass.

(2) For the high density limits the branching law needs only have finite second moment.

(3) In the space-scaling limits the effect of the particle motion vanishes, and the results are the same as for the branching particle system in R^d with changes of mass and no spatial migration.

(4) In the space-time scaling limit, $X_t^{(3)} = 0$ for all t in case $m_2 = 1$.

(5) The generalized Langevin equations of the form $dX_t = \mathcal{A}X_t dt + dW_t$ in Theorem 2 are interpreted as

$$\langle X_t,\phi\rangle = \langle X_0,\phi\rangle + \int_0^t \langle X_s,\mathcal{A}\phi\rangle ds + \langle W_t,\phi\rangle, \; t > 0,$$

for each $\phi \in \mathcal{S}(R^d)$, when $\mathcal{A}$ maps $\mathcal{S}(R^d)$ into itself. An (inhomogeneous) $\mathcal{S}'(R^d)$-Wiener process W is a continuous, centered, Gaussian process whose covariance functional is given by

$$\mathrm{Cov}(\langle W_s,\phi\rangle,\langle W_t,\psi\rangle)=\int_0^{s\wedge t} Q_u(\phi,\psi)du,\quad \phi,\psi\in\mathcal{S}(R^d),$$

where $Q_u(\phi,\psi)$ is a continuous, symmetric, positive bilinear form for each $u\in R_+$, and a Borel-measurable, locally finite function of u for each $\phi,\psi\in\mathcal{S}(R^d)$ (see [1,2,3]). If $\mathcal{A}$ does not map $\mathcal{S}(R^d)$ into itself (e.g. $\mathcal{A}=\Delta_\alpha,\ \alpha<2$), the equation is interpreted in a generalized sense (see [5,6]).

Proofs

The methods of proof are basically the same ones that have been used to study the asymptotic behavior of particle systems whose fluctuation processes have generalized processes as limits (e.g. [8,9,18]); when the limit process is continuous, a new approach can be used thanks to a recent result of Aldous (see [9,10]). Hence we will restrict ourselves to those aspects that are special to the present model, which are the computation of the mean and covariance functionals of the mass process M, and their limits under the rescalings.

LEMMA (1). *For all* $0\le s\le t$, *and* $\phi,\psi\in\mathcal{S}(R^d)$,

$$E\langle M_t,\phi\rangle=\gamma\int_B \mathcal{U}_t\phi(x)dx+\beta\int_0^t\int_{\mathcal{C}_r}\mathcal{U}_{t-r}\phi(x)dxdr, \tag{1}$$

$$\begin{aligned}\mathcal{K}_M(s,\phi;t,\psi)\equiv{}&\mathrm{Cov}(\langle M_s,\phi\rangle,\langle M_t,\psi\rangle)\\ ={}&\gamma\{\int_B(\mathcal{U}_s\phi(x))(\mathcal{U}_t\psi(x))dx\\ &+\int_0^s\int_B\mathcal{V}_rF(\mathcal{U}_{s-r}\phi,\mathcal{U}_{t-r}\psi)(x)dxdr\}\\ &+\beta\{\int_0^s\int_{\mathcal{C}_r}(\mathcal{U}_{s-r}\phi(x))(\mathcal{U}_{t-r}\psi(x))dxdr\\ &+\int_0^s\int_0^{s-r}\int_{\mathcal{C}_r}\mathcal{V}_uF(\mathcal{U}_{s-r-u}\phi,\mathcal{U}_{t-r-u}\psi)(x)dxdudr\}.\end{aligned} \tag{2}$$

Proof. The processes N, M, M^2, M^4 can be written as $N=N^a+N^b$, $M=M^a+M^b$, $M^2=M^{2a}+M^{2b}$, $M^4=M^{4a}+M^{4b}$, where N^a, M^a, M^{2a}, M^{4a} are the contributions of the initial particles, and N^b, M^b, M^{2b}, M^{4b} contain those of the immigrant particles. Hence, by the independence of the two contributions we have

$$E\langle M_t,\phi\rangle=E\langle M_t^a,\phi\rangle+E\langle M_t^b,\phi\rangle, \tag{3}$$

$$\begin{aligned}\mathrm{Cov}(\langle M_s,\phi\rangle,\langle M_t,\psi\rangle)={}&\mathrm{Cov}(\langle M_s^a,\phi\rangle,\langle M_t^a,\psi\rangle)\\ &+\mathrm{Cov}(\langle M_s^b,\phi\rangle,\langle M_t^b,\psi\rangle).\end{aligned} \tag{4}$$

We will compute first the mean and covariance of M^a for a general initial configuration of particles of mass 1; this will allow us to obtain the moments of both M^a with initial Poisson measure, and M^b.

The process $(N^a, M^a, M^{2a}, M^{4a})$ with values in $(\mathcal{M}_p(R^d))^4$ is Markovian, and its infinitesimal generator $\mathcal{L}$ has the following forms on the functions of the types

$$g(\mu_1,\mu_2,\mu_3,\mu_4) = G(\langle \mu_2, \phi\rangle) \text{ and } h(\mu_1,\mu_2,\mu_3,\mu_4) = H(\langle \mu_3, \phi\rangle),$$

for $\mu_1,\mu_2,\mu_3,\mu_4 \in \mathcal{M}_p(R^d)$, where $\phi \in \mathcal{S}(R^d)$, and $G, H \in C^3(R)$ with $G''' = H''' = 0$:

$$\begin{aligned}\mathcal{L}g(\mu_1,\mu_2,\mu_3,\mu_4) =\ & G'(\langle \mu_2,\phi\rangle)\langle \mu_2, \Delta_\alpha \phi\rangle + \frac{1}{2}G''(\langle \mu_2,\phi\rangle)\langle \mu_3, \Delta_\alpha \phi^2 - 2\phi\Delta_\alpha\phi\rangle \\ & + V\sum_{n=0}^{\infty} p_n[\sum_{i=1}^{\infty} G(\langle \mu_2,\phi\rangle + (nc_n - 1)a_i\phi(x_i)) - G(\langle \mu_2,\phi\rangle)],\end{aligned} \tag{5}$$

$$\begin{aligned}\mathcal{L}h(\mu_1,\mu_2,\mu_3,\mu_4) =& H'(\langle \mu_3,\phi\rangle)\langle \mu_3, \Delta_\alpha \phi\rangle + \frac{1}{2}H''(\langle \mu_3,\phi\rangle)\langle \mu_4, \Delta_\alpha \phi^2 - 2\phi\Delta_\alpha\phi\rangle \\ & + V\sum_{n=0}^{\infty} p_n[\sum_{i=1}^{\infty} H(\langle \mu_3,\phi\rangle + (nc_n^2 - 1)a_i^2\phi(x_i)) - H(\langle \mu_3,\phi\rangle)],\end{aligned} \tag{6}$$

where $\mu_1 = \sum_{i=1}^\infty \delta_{x_i}$, $\mu_2 = \sum_{i=1}^\infty a_i\delta_{x_i}$, $\mu_3 = \sum_{i=1}^\infty a_i^2\delta_{x_i}$, and $\mu_4 = \sum_{i=1}^\infty a_i^4\delta_{x_i}$.

For $G(u) = H(u) = u$ in (5) and (6), it follows from the Markov property that the processes

$$Y_t(\phi) \equiv \langle M_t^a, \phi\rangle - \int_0^t \langle M_s^a, \mathcal{A}_1\phi\rangle ds, \quad t \ge 0, \tag{7}$$

$$Z_t(\phi) \equiv \langle M_t^{2a}, \phi\rangle - \int_0^t \langle M_s^{2a}, \mathcal{A}_2\phi\rangle ds, \quad t \ge 0, \tag{8}$$

are martingales. From (7), $EY_t(\phi) = EY_0(\phi) = E\langle M_0^a, \phi\rangle$, and therefore we have the equation

$$\frac{d}{dt}E\langle M_t^a, \phi\rangle = E\langle M_t^a, \mathcal{A}_1\phi\rangle, \quad t \ge 0,$$

whose solution is

$$E\langle M_t^a, \phi\rangle = E\langle M_0^a, \mathcal{U}_t\phi\rangle, \quad t \ge 0.$$

Similarly, (8) yields

$$E\langle M_t^{2a}, \phi\rangle = E\langle M_0^{2a}, \mathcal{V}_t\phi\rangle, \quad t \ge 0.$$

Since $N_0 = M_0^a = M_0^{2a}$, because initial particles have mass 1, we then have

$$E\langle M_t^a,\phi\rangle = E\langle N_0^a,\mathcal{U}_t\phi\rangle,\quad t\geq 0. \tag{9}$$

$$E\langle M_t^{2a},\phi\rangle = E\langle N_0^a,\mathcal{V}_t\phi\rangle,\quad t\geq 0. \tag{10}$$

The increasing process of the martingale (7) is given by

$$\int_0^t [\mathcal{L}((\langle M_s^a,\phi\rangle^2) - 2\langle M_s^a,\phi\rangle\mathcal{L}((\langle M_s^a,\phi\rangle)]ds,\quad t\geq 0;$$

hence, taking $G(u)=u^2$ in (5) we have that

$$Y_t'(\phi)\equiv Y_t(\phi)^2 - \int_0^t \langle M_s^{2a},F(\phi,\phi)\rangle ds,\quad t\geq 0, \tag{11}$$

is a martingale, where $F(\phi,\phi)$ is defined in section 2. From (11) and (7),

$$EY_t'(\phi) = EY_0'(\phi) = EY_0(\phi)^2 = E\langle M_0^a,\phi\rangle^2 = E\langle N_0^a,\phi\rangle^2.$$

Then from (11) and (10) we have

$$\begin{aligned} EY_t(\phi)^2 &= E\langle N_0^a,\phi\rangle^2 + \int_0^t E\langle M_s^{2a},F(\phi,\phi)\rangle ds \\ &= E\langle N_0^a,\phi\rangle^2 + \int_0^t E\langle N_0^a,\mathcal{V}_sF(\phi,\phi)\rangle ds,\quad t\geq 0. \end{aligned} \tag{12}$$

On the other hand, from (7),

$$\begin{aligned} EY_t(\phi)^2 &= E\langle M_t^a,\phi\rangle^2 + 2\int_0^t\int_s^t E\langle M_s^a,\mathcal{A}_1\phi\rangle\langle M_r^a,\mathcal{A}_1\phi\rangle drds \\ &\quad - 2\int_0^t E\langle M_s^a,\mathcal{A}_1\phi\rangle\langle M_t^a,\phi\rangle ds. \end{aligned} \tag{13}$$

To continue the previous calculation we will show that

$$E\langle M_s^a,\phi\rangle\langle M_t^a,\psi\rangle = E\langle M_s^a,\phi\rangle\langle M_s^a,\mathcal{U}_{t-s}\psi\rangle,\quad s\leq t. \tag{14}$$

Indeed, using the martingale (7),

$$\begin{aligned} E\langle M_s^a,\phi\rangle\langle M_t^a,\psi\rangle &= E\langle M_s^a,\phi\rangle[Y_t(\psi) + \int_0^t \langle M_r^a,\mathcal{A}_1\psi\rangle dr] \\ &= E\langle M_s^a,\phi\rangle Y_s(\psi) + \int_0^t E\langle M_s^a,\phi\rangle\langle M_r^a,\mathcal{A}_1\psi\rangle dr, \end{aligned}$$

hence we have the equation

$$\frac{d}{dt}E\langle M_s^a,\phi\rangle\langle M_t^a,\psi\rangle = E\langle M_s^a,\phi\rangle\langle M_t^a,\mathcal{A}_1\psi\rangle,\quad t\geq s,$$

and it can be verified that the solution is given by (14).

Substituting (14) into (13),

$$EY_t(\phi)^2 = E\langle M_t^a, \phi\rangle^2 + 2\int_0^t\int_s^t E\langle M_s^a, \mathcal{A}_1\phi\rangle\langle M_s^a, \mathcal{U}_{r-s}\mathcal{A}_1\phi\rangle dr ds$$
$$-2\int_0^t E\langle M_s^a, \mathcal{A}_1\phi\rangle\langle M_s^a, \mathcal{U}_{t-s}\phi\rangle ds,$$

but

$$\int_s^t \langle M_s^a, \mathcal{U}_{r-s}\mathcal{A}_1\phi\rangle dr = \langle M_s^a, \mathcal{U}_{t-s}\phi\rangle - \langle M_s^a, \phi\rangle,$$

hence

$$EY_t(\phi)^2 = E\langle M_t^a, \phi\rangle^2 - 2\int_0^t E\langle M_s^a, \mathcal{A}_1\phi\rangle\langle M_s^a, \phi\rangle ds. \tag{15}$$

From (12) and (15) we find

$$E\langle M_t^a, \phi\rangle^2 = E\langle N_0^a, \phi\rangle^2 + \int_0^t E\langle N_0^a, \mathcal{V}_s F(\phi, \phi)\rangle ds$$
$$+2\int_0^t E\langle M_s^a, \mathcal{A}_1\phi\rangle\langle M_s^a, \phi\rangle ds,$$

and by polarization we obtain the equation

$$E\langle M_t^a, \phi\rangle\langle M_t^a, \psi\rangle = E\langle N_0^a, \phi\rangle\langle N_0^a, \psi\rangle + \int_0^t E\langle N_0^a, \mathcal{V}_s F(\phi, \psi)\rangle ds$$
$$+\int_0^t [E\langle M_s^a, \mathcal{A}_1\phi\rangle\langle M_s^a, \psi\rangle + E\langle M_s^a, \phi\rangle\langle M_s^a, \mathcal{A}_1\psi\rangle] ds. \tag{16}$$

It can be verified that the solution of (16) is given by

$$E\langle M_t^a, \phi\rangle\langle M_t^a, \psi\rangle = E\langle N_0^a, \mathcal{U}_t\phi\rangle\langle N_0^a, \mathcal{U}_t\psi\rangle$$
$$+\int_0^t E\langle N_0^a, \mathcal{V}_s F(\mathcal{U}_{t-s}\phi, \mathcal{U}_{t-s}\psi)\rangle ds. \tag{17}$$

(Note that

$$\frac{d}{dt}E\langle M_0^a, \mathcal{U}_t\phi\rangle\langle M_0^a, \mathcal{U}_t\psi\rangle = E\langle M_0^a, \mathcal{U}_t\mathcal{A}_1\phi\rangle\langle M_0^a, \mathcal{U}_t\psi\rangle$$
$$+ E\langle M_0^a, \mathcal{U}_t\phi\rangle\langle M_0^a, \mathcal{U}_t\mathcal{A}_1\psi\rangle,$$

and

$$\frac{d}{dt}E\langle N_0^a, \mathcal{V}_s F(\mathcal{U}_{t-s}\phi, \mathcal{U}_{t-s}\psi)\rangle = E\langle N_0^a, \mathcal{V}_s F(\mathcal{U}_{t-s}\mathcal{A}_1\phi, \mathcal{U}_{t-s}\psi)\rangle$$
$$+ E\langle N_0^a, \mathcal{V}_s F(\mathcal{U}_{t-s}\phi, \mathcal{U}_{t-s}\mathcal{A}_1\psi)\rangle.)$$

Finally, from (14) and (17) we obtain

$$
\begin{aligned}
E\langle M_s^a,\phi\rangle\langle M_t^a,\psi\rangle &= E\langle N_0^a,\mathcal{U}_s\phi\rangle\langle N_0^a,\mathcal{U}_t\psi\rangle \\
&\quad + \int_0^s E\langle N_0^a,\mathcal{V}_r F(\mathcal{U}_{s-r}\phi,\mathcal{U}_{t-r}\psi)\rangle dr, \quad s\le t, \qquad (18)
\end{aligned}
$$

and by (9) we have

$$
\begin{aligned}
\mathrm{Cov}(\langle M_s^a,\phi\rangle,\langle M_t^a,\psi\rangle) &= \mathrm{Cov}(\langle N_0^a,\mathcal{U}_s\phi\rangle,\langle N_0^a,\mathcal{U}_t\psi\rangle) \\
&\quad + \int_0^t E\langle N_0^a,\mathcal{V}_r F(\mathcal{U}_{s-r}\phi,\mathcal{U}_{t-r}\psi)\rangle dr, \quad s\le t. \qquad (19)
\end{aligned}
$$

For N_0^a = Poisson random field with intensity γ on B, from (9) and (18) we have

$$
E\langle M_t^a,\phi\rangle = \gamma\int_B \mathcal{U}_t\phi(x)dx, \qquad (20)
$$

$$
\begin{aligned}
\mathrm{Cov}(\langle M_s^a,\phi\rangle,\langle M_t^a,\psi\rangle) &= \gamma\int_B (\mathcal{U}_s\phi(x))(\mathcal{U}_t\psi(x))dx \\
&\quad +\gamma\int_0^s\int_B \mathcal{V}_r F(\mathcal{U}_{s-r}\phi,\mathcal{U}_{t-r}\psi)(x)dxdr, \quad s\le t. \qquad (21)
\end{aligned}
$$

Now we compute the mean and the covariance of M^b. If $N_0^a=\delta_x$ (a single initial particle at x) and M_s^a is denoted by M_s^x in this case, then (18) becomes

$$
\begin{aligned}
E\langle M_s^x,\phi\rangle\langle M_t^x,\psi\rangle &= (\mathcal{U}_s\phi(x))(\mathcal{U}_t\psi(x)) \\
&\quad + \int_0^s \mathcal{V}_r F(\mathcal{U}_{s-r}\phi,\mathcal{U}_{t-r}\psi)(x)dr, \quad s\le t. \qquad (22)
\end{aligned}
$$

The random measure M_t^b can be written as $M_t^b=\sum_{i=1}^\infty M_{t-s_i}^{x_i}$, where $\{(x_i,s_i)\}_i$ are the points of the immigration Poisson random field on $\{(x,s)\in \mathcal{C}|s\le t\}$ with intensity β. Hence

$$
\mathrm{Cov}(\langle M_s^b,\phi\rangle,\langle M_t^b,\psi\rangle) = \beta\int_0^s\int_{\mathcal{C}_r} E\langle M_{s-r}^x,\phi\rangle\langle M_{t-r}^x,\psi\rangle dxdr, \quad s\le t,
$$

and substituting (22) we have

$$
\begin{aligned}
\mathrm{Cov}(\langle M_s^b,\phi\rangle,\langle M_t^b,\psi\rangle) &= \beta\int_0^s\int_{\mathcal{C}_r}(\mathcal{U}_{s-r}\phi(x))(\mathcal{U}_{t-r}\psi(x))dxdr \\
&\quad +\beta\int_0^s\int_0^{s-r}\int_{\mathcal{C}_r}\mathcal{V}_u F(\mathcal{U}_{s-r-u}\phi,\mathcal{U}_{t-r-u}\psi)(x)dxdudr, \\
&\quad s\le t. \qquad (23)
\end{aligned}
$$

Similarly, from (9) we get

(24) $$E\langle M_t^b,\phi\rangle = \beta\int_0^t\int_{\mathcal{C}_r}\mathcal{U}_{t-r}\phi(x)dxdr.$$

Finally, (20) and (24) give (1), (21) and (23) give (2), and the proof is finished.

The laws of large numbers and the fluctuation limits depend on the limits of the means and covariances under each rescaling, which are given in the next lemma.

LEMMA (2).

(25) $$\lim_{K\to\infty} K^{-1}E\langle M_t^{(1),K},\phi\rangle = E\langle M_t,\phi\rangle \text{ (given by (1))},$$

(26) $$\mathcal{K}_{X^{(1)}}(s,\phi;t,\psi) \equiv \lim_{K\to\infty}\operatorname{Cov}(\langle X_s^{(1),K},\phi\rangle,\langle X_t^{(1),K},\psi\rangle) = \mathcal{K}_M(s,\phi;t,\psi) \text{ (given by (2))},$$

(27) $$\lim_{K\to\infty} K^{-d}E\langle M_t^{(2),K},\phi\rangle = \gamma e^{A_1 t}\int_B\phi(x)dx + \beta e^{A_1 t}\int_0^t e^{-A_1 r}\int_{\mathcal{C}_r}\phi(x)dxdr,$$

(28) $$\mathcal{K}_{X^{(2)}}(s,\phi;t,\psi) \equiv \lim_{K\to\infty}\operatorname{Cov}(\langle X_s^{(2),K},\phi\rangle,\langle X_t^{(2),K},\psi\rangle)$$
$$= \gamma e^{A_1(s+t)}[1+V(m_2-2m_1+1)\frac{e^{(A_2-2A_1)s}-1}{A_2-2A_1}]\int_B\phi(x)\psi(x)dx$$
$$+\beta e^{A_1(s+t)}\int_0^s e^{-2A_1 r}[1+V(m_2-2m_1+1)\frac{e^{(A_2-2A_1)(s-r)}-1}{A_2-2A_1}]\cdot\int_{\mathcal{C}_r}\phi(x)\psi(x)dxdr,\quad s\le t,$$

(*the case $A_2 = 2A_1$ is included*),

(29) $$\lim_{K\to\infty} K^{-d}E\langle M_t^{(3),K},\phi\rangle = \gamma e^{Vat}\int_B \mathcal{T}_t\phi(x)dx + \beta\int_0^t e^{Vas}\int_Q \mathcal{T}_s\phi(x)dxds,$$

(30) $$\mathcal{K}_{X^{(3)}}(s,\phi;t,\psi) \equiv \lim_{K\to\infty}\operatorname{Cov}(\langle X_s^{(3),K},\phi\rangle,\langle X_t^{(3),K},\psi\rangle)$$
$$= \gamma V(m_2-1)\int_0^s\int_B e^{Vbr}\mathcal{T}_r((e^{Va(s-r)}\mathcal{T}_{s-r}\phi)(e^{Va(t-r)}\mathcal{T}_{t-r}\psi))(x)dxdr$$
$$+\beta V(m_2-1)\int_0^s\int_0^{s-r}\int_Q e^{Vbu}\mathcal{T}_u((e^{Va(s-r-u)}\mathcal{T}_{s-r-u}\phi)$$
$$(e^{Va(t-r-u)}\mathcal{T}_{t-r-u}\psi))(x)dxdudr,\quad s\le t.$$

Proof. (25) and (26) are obvious. The proofs of (27)-(30) depend on the self-similarity of the symmetric stable process with exponent $\alpha \in (0,2]$, which implies the following scaling properties of $\mathcal{T}_t$ and Δ_α: denoting $\phi^K(x) = \phi(x/K)$, $x \in R^d$, we have

$$\mathcal{T}_t\phi^K = (\mathcal{T}_{t/K^\alpha}\phi)^K, \quad \Delta_\alpha\phi^K = \frac{1}{K^\alpha}(\Delta_\alpha\phi)^K. \tag{31}$$

Introducing the space and the space-time rescalings into (1) and (2), and using (31), the limits (27)-(29) are obtained in a straightforward way.

Proofs of Theorems (1) *and* (2). Having the results of Lemma 2, the proofs of Theorems 1 and 2 can be done by the same techniques used in [8,9,18]. We will only make some comments.

The proof of Theorem 1 follows directly from (25), (27), (29), and the fact that (26), (28), (30) imply that $\mathrm{Var}\langle M_t^{(1),K},\phi\rangle = O(K)$, $\mathrm{Var}\langle M_t^{(2),K},\phi\rangle = O(K^d)$, $\mathrm{Var}\langle M_t^{(3),K},\phi\rangle = O(K^{d+\alpha})$, respectively, and therefore $K^{-2}\mathrm{Var}\langle M_t^{(1),K},\phi\rangle \to 0$, $K^{-2d}\mathrm{Var}\langle M_t^{(2),K},\phi\rangle \to 0$, and $K^{-2d}\mathrm{Var}\langle M_t^{(3),K},\phi\rangle \to 0$ (since $\alpha < d$).

The convergence proofs in Theorem 2 consist in showing weak convergence of the finite-dimensional distributions and tightness for each rescaling. The condition $m_3 < \infty$ (or $\sup_{K\geq 1} m_3^K < \infty$) is used for the convergence of $X^{(2),K}$ and $X^{(3),K}$. The limits $X^{(1)}, X^{(2)}, X^{(3)}$ are $\mathcal{S}'(R^d)$-valued continuous, centered, Gaussian processes whose covariance functionals $\mathcal{K}_{X^{(1)}}, \mathcal{K}_{X^{(2)}}, \mathcal{K}_{X^{(3)}}$ are given by (26), (28), (30), respectively. We observe that these covariances satisfy

$$\mathcal{K}_{X^{(1)}}(s,\phi;t,\psi) = \mathcal{K}_{X^{(1)}}(s,\phi;s,\mathcal{U}_{t-s}\psi), \quad s \leq t,$$
$$\mathcal{K}_{X^{(2)}}(s,\phi;t,\psi) = \mathcal{K}_{X^{(2)}}(s,\phi;s,\mathrm{e}^{A_1(t-s)}\psi), \quad s \leq t,$$
$$\mathcal{K}_{X^{(3)}}(s,\phi;t,\psi) = \mathcal{K}_{X^{(3)}}(s,\phi;s,\mathrm{e}^{V\dot{a}(t-s)}\mathcal{T}_{t-s}\psi), s \leq t.$$

This implies that $X^{(1)}, X^{(2)}, X^{(3)}$ are Markovian, and the generalized Langevin equations which govern them are derived by direct application of Theorem 4.1 in [5] (or Theorem 3.6 in [1] in the case $\alpha = 2$).

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