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SOLUTION OF THE SPECTRAL ANALYSIS PROBLEM ON THE SPACE OF HOROCYCLES

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1. Notation and preliminaries.

Let G be a connected semisimple Lie group of noncompact type and of finite center. Denote by $G = K\ A\ N$ the Iwasawa decomposition of the group G. By $\mathfrak{g}, \mathfrak{k}, \mathfrak{a}, \mathfrak{n}$ we denote the Lie algebras of G, K, A and N, respectively. If $\Delta \subset \mathfrak{a}^*$ is the set of restricted roots, then for an appropiate order denoted by $<$ we have

$$\mathfrak{n} = \sum_{\alpha>0} \mathfrak{g}_\alpha, \quad \text{where}$$

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} | [H, X] = \alpha(H)X, \quad H \in \mathfrak{a}\}.$$

Let us put $m_\alpha := \dim \mathfrak{g}_\alpha$. We denote also $\rho := \frac{1}{2} \sum_{\alpha>0} m_\alpha \alpha$.

In what follows we shall assume that the split-rank of G is equal to one, that is $\dim A = 1$. This being the case $\Delta = \{\alpha, -\alpha\}$ or $\Delta = \{\alpha, 2\alpha, -\alpha, -2\alpha\}$. The centralizer of K in A will be denoted by M and the normalizer by M'. The Weyl group $W := M'/M$ acts in a natural way of A as well as on $\mathfrak{a}$ and in the rank one case it contains only the identity and the reflection $A \ni a \to a^{-1} \in A$.

The components of the Iwasawa decomposition of the element $g \in G$ are denoted as

$$g = K(g) \exp H(g)\ N(g). \tag{1.1}$$

Since A is a vector group the exponential mapping is a homeomorphism of $\mathfrak{a}$ onto A. The inverse mapping is denoted by $\log : A \to \mathfrak{a}$.

We denote by $\mathfrak{a}_c^*$ the space of all complex valued R-linear functionals on $\mathfrak{a}$. The homogeneous space $X := G/K$ is the symmetric space of the rank one associated to the group G and the manifold $\Xi := G/MN$ is called the dual manifold of X or the space of horocycles.

By a horocycle in X we mean a subset of the form $gNK \subseteq X = G/K$. If we denote $\xi_0 := NK$ then each horocycle may be represented in the form $\xi = g\xi_0$. One proves that the isotropy subgroup of the point ξ_0 is just equal to MN hence the space of all horocycles in X can be identified with the homogeneous space $\Xi = G/MN$.

On the other hand the points of the space X can be interpreted as submanifolds in Ξ. Set $x_0 := K\xi_0$. Since the isotropy subgroup of x_0 is equal to K we may identify X with the set of all orbits in Ξ of the form $gx_0 = gK$.

It follows by the Iwasawa decomposition that the mapping: $K/M \times A \ni (kM, a) \to ka\xi_0 \in \Xi$ is a smooth parametrization of the space Ξ. The manifold $B := K/M = G/MAN$ is called the boundary of the symmetric space X.

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The Haar measures on A, K, G are denoted by da, dk, dg respectively; the measure dk is supposed to be normalized.

For a given smooth manifold Y we denote by $\mathcal{E}(Y)$ the space of all infinitely differentiable functions on Y with its usual Fréchet topology of the uniform convergence of all derivates on compact sets. If (G, Y) is a Lie transformation group the natural action of G on $\mathcal{E}(Y)$ is defined as:

$$L_g f(x) := f(g^{-1}x), \quad g \in G,\ x \in Y,\ f \in \mathcal{E}(Y).$$

We shall also deal with the space $\mathcal{D}(Y)$ of the Schwartz test functions on Y which is obviously an invariant subspace of $\mathcal{E}(Y)$. By $\mathcal{D}'(Y)$ we denote the space of all distributions on Y. The action of the group on $\mathcal{D}'(Y)$ is defined by the formula:

$$L_g T(f) := T(L_{g^{-1}} f), \quad f \in \mathcal{D}(Y),\ T \in \mathcal{D}'(Y).$$

The dual space $\mathcal{E}'(Y)$ is identified with the space of all compactly supported distributions on Y.

In the present paper we are studying the invariant closed subspaces of the space $\mathcal{E}(\Xi)$. For the group $G = SO(2,1)$ in which case the manifold Ξ is simply the upper light cone in $\mathbf{R}^3$ the problem of the spectral analysis on Ξ was solved in the author's paper [7]. A family of elementary functions was defined on Ξ and it was proved that each closed and invariant subspace of $E(\Xi)$ contains some element of the family. The analogous family of elementary functions can be defined in the general case and the same theorem is valid (see Thm 4.2).

The space Ξ, besides the structure of the homogeneous space of G is a subject of an additional action of the subgroup A. If $y = kaMN$ and $a_1 \in A$ then the element

$$\tau(a_1)y := kaa_1MN \in \Xi \tag{1.2}$$

is well defined. The operation $\tau : A \to \operatorname{Aut} \Xi$ satifies

$$\tau(ab) = \tau(a)\tau(b), \quad a, b \in A \tag{1.3}$$

and commutes with the natural action of G on Ξ.

Unfortunately the G-invariant spaces of functions on Ξ are not τ-invariant in general. As a countrexample may serve the kernel of the dual Radon transform $B : \mathcal{E}(\Xi) \to \mathcal{E}(X)$ defined by the formula: $Bf(gK) := \int_K f(gkMN)dk$, $f \in \mathcal{E}(\Xi)$. This example is explained in [7] for $G = SO(2,1)$, nevertheless the fact is general.

The G-invariance of a subspace $V \subset \mathcal{E}(\Xi)$ implies however some additional property of V with respect to the action τ.

We shall prove that a non-trivial G-invariant subspace $V \subset \mathcal{E}(\Xi)$ contains a non-trivial subspace V_1 such that for every $f \in V_1$ and every $\varphi \in \mathcal{D}(A)$ which satisfies

$$\varphi(a^{-1}) = e^{-\rho(\log a)}\varphi(a) \tag{1.4}$$

the function $f \times \varphi := \int_A \varphi(a)\tau(a^{-1})f\, da$ belongs to V_1.

This property permits us to identify a subspace of V with a subspace of $\mathcal{E}(\mathbf{R})$ which is invariant under convolutions with even test functions. The problem of spectral analysis for such spaces was solved in [7] and in this way we obtain the principal result of the present paper.

Conjecture. Every closed G-invariant subspace $V \subset \mathcal{E}(\Xi)$ is invariant under the application $f \to f \times \varphi$ for $\varphi \in \mathcal{D}(A)$ satisfying (1.4).

We refer the reader to [2] and [6] for an exhaustive account of the material presented above.

2. On the G-invariance and τ-invariance.

The differential operators which commute with the regular representation L were described by S. Helgason [1]. Let $\mathfrak{A}(\mathfrak{g})$ denote the universal enveloping algebra of the Lie algebra g. An element $X \in \mathfrak{g}$ represents a right-invariant differential operator on G by the formula:

$$D(X)f(g) := \lim_{t \to 0} f(\exp tX\, g).$$

Let $\mathfrak{g}^c$ denote the complexification of $\mathfrak{g}$. The representation $\mathfrak{g} \ni X \to D(X)$ extends to a representation of the algebra $\mathfrak{A}(\mathfrak{g})$ on $\mathcal{E}(G)$. The elements of the center $\mathfrak{Z}(\mathfrak{g}^c)$ of $\mathfrak{A}(\mathfrak{g}^c)$ commute with the left and with the right translations, hence for any $Y \in \mathfrak{Z}(\mathfrak{g}^c)$ we have

$$D(Y)f(g) = D(Y)(L_{g^{-1}}f)(e).$$

If $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ then in a natural way $\mathfrak{A}(\mathfrak{h}^c)$ may be treated as a subalgebra of $\mathfrak{A}(\mathfrak{g}^c)$ and the elements of $\mathfrak{A}(\mathfrak{g}^c)$ acts as differential operators on $\mathcal{E}(G)$. The theorem of Helgason identifies the space $\mathbf{D}(\Xi)$ of differential operator on Ξ commuting with the action of G, with $\mathfrak{A}(\mathfrak{a}^c)$ in the following sense. For every $D \in \mathbf{D}(\Xi)$ there exists a unique element $X \in \mathfrak{A}(\mathfrak{a}^c)$ such that for every $f \in \mathcal{E}(G/MN)$

$$Df(gMN) = (D(X)L_{g^{-1}}f)(eMN) := \mathcal{R}(X)f(gMN). \tag{2.1}$$

The elements of the center $\mathfrak{Z}(\mathfrak{g}^c)$ leave invariant the space $\mathcal{E}(G/MN) \subset \mathcal{E}(G)$ hence they define differential invariant operators on $\mathcal{E}(\Xi)$. Let us denote by $\mu(Y)$ the element of $\mathfrak{A}(\mathfrak{a}^c)$ such that

$$D(Y)f = R(\mu(Y))f, \quad Y \in \mathfrak{Z}(\mathfrak{g}^c),\ f \in \mathcal{E}(\Xi). \tag{2.2}$$

The image of the operator $\mu : \mathfrak{Z}(\mathfrak{g}^c) \to \mathfrak{A}(\mathfrak{a}^c)$ can be recognized with the aid of the Harish-Chandra isomorphism. In order to describe it we need more information about the structure of $\mathfrak{g}^c$ and $\mathfrak{Z}(\mathfrak{g}^c)$.

Let $\mathfrak{j} \subset \mathfrak{g}^c$ be an extension of $\mathfrak{a}^c$ to a Cartan subalgebra of $\mathfrak{g}^c$. Denote by Φ the root system of the pair $(\mathfrak{g}^c, \mathfrak{j})$ and let Φ_+ be the set of positive roots for some fixed order in $\mathfrak{j}^*$. Then $\Phi_+ = P_+ \cup P_-$ where P_+ consists of roots which are not identically zero on $\mathfrak{a}$ and P_- consists of the root which vanish on $\mathfrak{a}$. Denoting by $\mathfrak{g}^c_\alpha$ the root space of α in $\mathfrak{g}^c$ we introduce $\mathfrak{n}^c_+ := \sum_{\alpha\in P_+} \mathfrak{g}^c_\alpha$ and $\mathfrak{n}_+ := \mathfrak{g} \cap \mathfrak{n}^c_+$.

The order in $\mathfrak{j}^*$ can be introduced in such a way that $\mathfrak{n}_+ = \mathfrak{n}$. The complexification of the Lie algebra $\mathfrak{m}$ of the group M has then the form:

$$\mathfrak{m}^c = \mathfrak{j} \cap \mathfrak{k}^c + \sum_{\alpha\in P_-} (\mathfrak{g}^c_\alpha + \mathfrak{g}^c_{-\alpha}).$$

We denote also:

$$\rho := \frac{1}{2} \sum_{\alpha\in\Phi_+} \alpha.$$

The restriction of the function ρ to the algebra $\mathfrak{a}$ is equal to the function ρ introduced in the first section.

Let $\hat{W}$ be the Weyl group of the pair $(\mathfrak{g}^c, \mathfrak{j})$. The group $\hat{W}$ acts in a natural way on $\mathfrak{j}$ and $\mathfrak{j}^*$. The natural injection $\mathfrak{a}^c \to \mathfrak{j}$ intertwines the action of W with the action of $\hat{W}$. The algebra $\mathfrak{A}(\mathfrak{j})$ can be identified with the symmetric tensor product $S(\mathfrak{j})$ because $\mathfrak{j}$ is abelian. With the aid of the Killing form we can also identify j with $\mathfrak{j}^*$ and in this way the space $S(\mathfrak{j})$ can be interpreted as the space of polynomials on $\mathfrak{j}$. We denote by $S(\mathfrak{j})^{\hat{W}}$ the space of $\hat{W}$-invariant elements in $S(\mathfrak{j})$, which is isomorphism to the space of $\hat{W}$-invariant polynomials on $\mathfrak{j}$.

Now, consider $Z \in \mathfrak{Z}(\mathfrak{g}^c)$. According to Lemma 2.3.3.4 in [6] there exists a unique $Y \in S(\mathfrak{j})$ and Q contained in the ideal of $\mathfrak{A}(\mathfrak{g}^c)$ generated by the space $\sum_{\alpha\in\Phi_+} \mathfrak{g}^c_\alpha$, such that $Z = Y + Q$. The differential operators $\mathscr{R}(X_\alpha)$, $X_\alpha \in \mathfrak{g}^c_\alpha$, $\alpha \in \Phi_+$, vanish on the functions which are MN-invariant, hence $\mathscr{R}(Z) = \mathscr{R}(Y)$ on $\mathscr{E}(\Xi)$.

Let γ be the automorphism of $S(\mathfrak{j})$ which on $H \in \mathfrak{j}$ takes the form $\gamma(H) = H + \rho(H)$. As proved in Prop. 2.3.3.6 in [6] there exists $T \in S(\mathfrak{j})^{\hat{W}}$ such that $Y = \gamma(T)$ and the mapping $Y \to \gamma^{-1}(Y)$ is onto $S(\mathfrak{j})^{\hat{W}}$. In this way we have obtained:

LEMMA (2.1). *For every $\hat{W}$-invariant polynomial T on $\mathfrak{j}$ there exists $Z \in \mathfrak{Z}(\mathfrak{g}^c)$ such that $Z = Y + Q$, $Y \in S(\mathfrak{j})$ and $\mathscr{R}(Z) = \mathscr{R}(Y) = \mathscr{R}(\gamma(T))$ on $\mathscr{E}(G/MN)$.*

The injection $\mathfrak{a} \to \mathfrak{j}$ induces a homomorphism of $S(\mathfrak{j})^{\hat{W}}$ into $S(\mathfrak{j})^{W}$. This mapping is onto if and only if the center $\mathfrak{Z}(\mathfrak{g})$ acting as differential operators on $\mathscr{E}(G/K)$ exhausts the whole algebra $\mathbf{D}(G/K)$ of invariant differential operators (cf. [3], Prop. 5.32). This is the case when rank $G/K = 1$, because the Laplace-Beltrami operator generates then the algebra $\mathbf{D}(G/K)$. We get

LEMMA (2.2). *If rank $G/K = 1$ then for every W-invariant polynomial T on $\mathfrak{a}$ there exists an element $Z \in \mathfrak{Z}(\mathfrak{g}^c)$ such that $\mathscr{R}(Z) = \mathscr{R}(\gamma(T))$ on $\mathscr{E}(\Xi)$.*

COROLLARY (2.3). *Let rank $G/K = 1$ and let $T \in S(\mathfrak{a}^c)^W$. Suppose that $V \subset \mathscr{E}(\Xi)$ is a G-invariant and closed subspace. Then the operator $\mathscr{R}(\gamma(T))$ conserves V.*

Proof. According to Lemma 2.2 $\mathscr{R}(\gamma(T)) = \mathscr{R}(Z)$ for some central element Z. On the other hand $\mathscr{R}(Z)f = D(Z)f$ for $f \in \mathscr{E}(G/MN)$ and $Z \in \mathfrak{Z}(\mathfrak{g}^c)$. By the very definition the operator $D(Z)$ conserves invariant closed subspaces and the proof follows.

Now, we shall look for an integral form of the invariance property expressed in Corollary 2.3. In the rank one case for every $H \in \mathfrak{a}$ the element $H^2 \in S(\mathfrak{a})$ is W-invariant. We have $\gamma(H^2) = (H + \rho(H))^2$ and the action of $H + \rho(H)$ as of a differential operator on $\mathscr{E}(\Xi)$ can be represented in the form:

$$\mathscr{R}(H + \rho(H))f = e^{-\rho}R(H)e^{\rho}f,$$

where $(e^{\rho}f)(kaMN) := e^{\rho(\log a)}f(kaMN)$.

Corollary 2.3 states that the operator $\mathscr{R}((H+\rho(H)^2) = e^{-\rho}\mathscr{R}(H)^2e^{\rho}$ leaves invariant each closed and G-invariant subspace.

Let us assume now, that $f \in \mathscr{E}(\Xi)$ admits the Taylor series expansion:

$$\tau(\exp H)f(gMN) = f(g(\exp H)MN) = \sum_{n=0}^{\infty}\left(\frac{\mathscr{R}(H)^n}{n!}f\right)(gMN),$$

where the series converges in $\mathscr{E}(\Xi)$. The functions satisfying this condition will be referred to as A-analytic on Ξ.

PROPOSITION (2.4). *Let f be an A-analytic function contained in a G-invariant closed subspace $V \subset \mathscr{E}(\Xi)$. Then the function*

$$e^{-\rho}(\tau(a) + \tau(a^{-1}))e^{\rho}f, \quad a \in A$$

also belongs to V.

Proof. Let us write

$$\begin{aligned} e^{-\rho}(\tau(\exp H) + \tau(\exp\ -H))(e^{\rho}f) &= \sum_{n=0}^{\infty} e^{-\rho}\left(\frac{\mathscr{R}(H)^n}{n!} + \frac{\mathscr{R}(-H)^n}{n!}\right)(e^{\rho}f) \\ &= \sum_{k=0}^{\infty} e^{-\rho}\left(\frac{\mathscr{R}(H)^{2k}}{(2k)!}\right)(e^{\rho}f). \end{aligned}$$

In virtue of the preceding observations every term of the series belongs to V and the proof follows.

Recall that $f \times \varphi(kaMN) := \int_A f(ka_1MN)\varphi(aa_1^{-1})da_1$.

THEOREM (2.5). *Let f be an A-analytic element of a G-invariant closed subspace $V \subset \mathscr{E}(\Xi)$. Let $\varphi \in \mathscr{D}(A)$ be an even function. Then the function $f \times (e^{-\rho}\varphi)$ belongs to V.*

Proof. According to Prop. 2.4 the function $\Xi \ni gMN \to e^{\rho(\log a_1)}f(ga_1MN)$ $+e^{-\rho(\log a_1)}f(ga_1MN)$ belongs to V for any $a_1 \in A$. On multiplying it by $\varphi(a_1)$ and integrating with respect to a_1 we also obtain an element of V.

Let us calculate:

$$\begin{aligned}\int_A f(kaa_1^{-1}MN)e^{-\rho(\log a_1)}\varphi(a_1)da_1 &= \int_A f(ka_1^{-1}MN)e^{-\rho(\log a_1 a)}\varphi(a_1 a)da_1\\ &= \int_A f(ka_1 MN)e^{-\rho(\log aa_1^{-1})}\varphi(a_1)da_1\\ &= f\times(e^{-\rho}\varphi)(kaMN).\end{aligned}$$

The second integral is equal to the latter thanks to the symmetry of the function φ and the proof follows.

We close the section with a proposition concerning the case where the subspace V contains a nontrivial K-fixed element.

Let us define for $f \in \mathcal{E}(\Xi)$:

$$Pf := \int_K L_k f\, dk. \tag{2.3}$$

The operator P is a projector onto the space of K-fixed elements in $\mathcal{E}(\Xi)$.

PROPOSITION (2.6). *Assume that $V \subset \mathcal{E}(\Xi)$ is a closed subspace which is G-invariant. If $V_0 := PV$ then for every even $\varphi \in \mathcal{D}(A)$ one has:*

$$V_0 \times (e^{-\rho}\varphi) \subset V_0.$$

Proof. Let us consider a function $u \in \mathcal{D}(K\backslash G/K)$ and $\Psi \in V_0$. The element $\phi : \Xi \ni xMN \to \int_G u(g)\psi(gxMN)dg$ belongs to V and is K-fixed, hence $\phi \in V_0$. The Iwasawa decomposition of the group G leads to the following factorization of the Haar measure dg:

$$\int_G f(g)dg = \int_K\int_A\int_N f(kan)e^{2\rho(\log a)}dk\, da\, dn, \quad \text{(see [3])}.$$

We obtain:

$$\begin{aligned}\phi(bMN) &= \int_K\int_A\int_N u(kan)\Psi(kanbMN)e^{2\rho(\log a)}dk\, da\, dn\\ &= \int_A\int_N u(an)dn\Psi(abMN)e^{2\rho(\log a)}da.\end{aligned}$$

The function on Ξ given by the formula $kaMN \to \int_N u(an)dn =: Ru(kaMN)$ is the Radon transform of the function u considered as an element of the space $\mathcal{D}(K\setminus X)$.

It is known that $R\mathcal{D}(K\setminus X)$ consists of those elements $f \in \mathcal{D}(\Xi)$ for which the function $A \ni a \to e^{\rho(\log a)}f(aMN)$ is even (see [3]).

Then we have: $\int_N u(an)dn\ e^{2\rho(\log a)} = \int_N u(a^{-1}n)dn$ and consequently $\phi(bMN) = \int_A Ru(aMN)\Psi(a^{-1}bMN)da = \Phi \times (e^{-\rho}\varphi)(bMN)$ for $\varphi(a) := Ru(aMN)e^{\rho(\log a)}$. When u ranges over $\mathfrak{D}(K \setminus X)$ then φ goes over the whole space of even test functions. The proof follows.

In what follows we denote by $\mathfrak{D}_s$, $\mathcal{E}_s$, $\mathcal{E}'_s$, etc. the space of even elements in the corresponding space.

3. The space $\mathcal{N}$.

Let us define $\mathcal{N} := \{f \in \mathcal{E}(\Xi) | \int_K f(gkx)dk = 0, \quad g \in G,\ x \in \Xi\}$.

By the very definition it follows that the space $\mathcal{N}$ is G-invariant and τ-invariant, what permits us to describe it in the spirit of the spectral synthesis. In order to reduce the question to the known theorems about the spectral synthesis in $\mathcal{E}(R)$ we decompose the space $\mathcal{E}(\Xi)$ in K-components.

As proved by B. Kostant [4], in the case of rank $G/K = 1$ every irreducible representation of K contains the trivial representation of M with multiplicity less or equal to 1, what implies in virtue of the Frobenius reciprocity the following decomposition:

$$L^2(B) = \bigoplus_{[\pi]\in\hat{K}_M} H_\pi,$$

where $\hat{K}_M$ denotes the family of equivalence classes of those irreducible representations of K which contain the one-dimensional subspace of M-fixed elements; H_π stands for the carrier space of the irreducible representation π and $[\pi]$ denotes the class of π. The trivial representation of K will be denoted by $[1]$. For a given $[\pi] \in \hat{K}_M$ let ω_π denote the zonal spherical functions of π which is an M-fixed element in H_π normalized by the condition $\omega_\pi(eM) = 1$.

The space $\mathcal{E}(B)$ admits an analogous decomposition:

$$\mathcal{E}(B) = \bigoplus_{[\pi]\in\hat{K}_M} H_\pi, \quad \text{(see [6], p. 261).} \tag{3.2}$$

The operator defined by the formula:

$$P_\pi^M := \dim H_\pi \int_K \bar{\omega}_\pi(k) L_k\, dk \tag{3.3}$$

acting on $L^2(B)$ project ortogonally the whole space onto the 1-dimensional space spanned by ω_π. The same formula makes sense as an operator in $\mathcal{E}(B)$ as well as in $\mathcal{E}(\Xi)$.

The space Ξ is isomorphic as an analytic manifold to the product $B \times A$ what implies the isomorphism of the Fréchet spaces:

$$\mathcal{E}(\Xi) = \mathcal{E}(B) \otimes \mathcal{E}(A) = \bigoplus_{[\pi]\in\hat{K}_M} H_\pi \otimes \mathcal{E}(A).$$

In particular, if $f \in \mathcal{E}(\Xi)$ is an M-invariant function we have:

$$f = \sum_{[\pi]\in\hat{K}_M} \omega_\pi \otimes f_A$$

where $\omega_\pi \otimes f_A(kaMN) := \omega_\pi(kM)f_A(a)$ and $f_A \in \mathcal{E}(A)$.

Given a subspace $V \subset \mathcal{E}(\Xi)$ we denote by V^M the subspace of M-fixed elements in V and by $V_\pi^M \subset E(A)$ the space for which

$$P_\pi^M V^M = \omega_\pi \otimes V_\pi^M. \tag{3.4}$$

The operator P_π^M commute with τ hence the τ-invariance of V implies the invariance of V_π^M with respect to translations. In particular for the space $\mathcal{N}$ we have:

$$P_\pi^M \mathcal{N} = \omega_\pi \times \mathcal{N}_\pi, \quad [\pi] \in \hat{K}_M$$

where the spaces $\mathcal{N}_\pi$ are closed in $\mathcal{E}(A)$ and translation invariant.

Since the group A is isomorphic to $\mathbf{R}$ the Spectral Synthesis Theorem of L. Schwartz is valid:

THEOREM (3.1). [5] *For every nontrivial translation invariant and closed subspace $V \subset \mathcal{E}(\mathbf{R})$ there exists a countable subset $Sp\,V \subset \mathbf{C}$ without accumulation points and a sequence of natural numbers $m(\lambda)$, $\lambda \in Sp\,V$ such that the functions of the form $x^m e^{i\lambda x}$, $\lambda \in Sp\,V$, $m \leq m(\lambda)$ belong to V and generate linearly a dense subset of V.*

The set $Sp\,V$ is called the spectrum of V.

Our caim is now to determine the spectrum of the space $\mathcal{N}_\pi$. To this end let us consider the following family of functions on Ξ.

$$h_{\pi,\lambda}(kaMN) := \omega_\pi \otimes e^{(i\lambda-\rho)}(kaMN) = \omega_\pi(kM)e^{(i\lambda-\rho)(\log a)}. \tag{3.5}$$

The function $h_{\pi,\lambda}$ may be considered as an element of the carrier space $\mathcal{E}_\lambda$ of the representation of G induced by the character of the subgroup MAN given by the formula:

$$\chi_\lambda : man \to e^{i\lambda(\log a)}.$$

The condition $h_{\pi,\lambda} \in \mathcal{N}$ means that for every $g \in G$, $a \in A$

$$\int_K \omega_\pi(K(gk)M)e^{(i\lambda-\rho)(\log H(gka))}dk = 0$$

By applying the formula

$$\int_K F(K(gk))dk = \int_K F(k)e^{-2\rho(\log H(g^{-1}k))}dk, \quad \text{(see [6], p. 197)}$$

we obtain an equivalent condition:

$$\int_K \omega_\pi(k) e^{(-i\lambda-\rho)(\log H(gk))}\, dk = 0 \quad \text{for all } g \in G.$$

The vanishing conditions for the integrals of this type were studied by B. Kostant [4] and S. Helgason [2], [3]. Let us recall, following [2] the principal result concerning the rank one case.

THEOREM (3.2). [2] *The equation*

$$\int_K F(kM) e^{(-i\lambda-\rho)(\log H(gk))}\, dk = 0 \quad \textit{for all } g \in G \tag{3.6}$$

has a nontrivial solution $F \in \mathscr{E}(B)$ if and only if the function

$$e(\lambda) := \Gamma^{-1}\left(\frac{1}{2}\left(\frac{1}{2}\, m_\alpha + 1 + \frac{\langle i\lambda, \alpha\rangle}{\langle \alpha, \alpha\rangle}\right)\right)\Gamma^{-1}\left(\frac{1}{2}\left(\frac{1}{2}\, m_\alpha + m_{2\alpha} + \frac{\langle i\lambda, \alpha\rangle}{\langle \alpha, \alpha\rangle}\right)\right)$$

vanishes in λ.

By $\langle \cdot, \cdot \rangle$ is denoted the bilinear form on $\mathfrak{a}_c^*$ induced by the Klling form of $\mathfrak{g}$. In the sequel we denote by $\mathscr{Z}$ the set of zeroes of the function e. By the very definition of $e(\cdot)$ it follows that $\mathscr{Z} \cap (-\mathscr{Z}) = \emptyset$.

For a given $[\pi] \in \hat{K}_M$ let $v_1, \dots, v_{\dim H_\pi}$ be an orthonormal basis in H_π.

THEOREM (3.3). (see [2]) *For every $[\pi] \in \hat{K}_M$ and $\lambda \in \mathfrak{a}_c^*$ there exists a matrix $Q^\pi_{ij}(\lambda)$, $1 \le i, j \le \dim H_\pi$, such that the equation (3.6) is satisfied for some nontrivial $F \in \mathscr{E}(B)$ and for all $g \in G$ if and only if*

$$\sum_{p=1}^{\dim H_\pi} Q^\pi_{pj}(-\lambda) \int_K (v|\pi(k)v_p) F(kM) = 0 \tag{3.7}$$

for all $1 \le j \le \dim H_\pi$, $[\pi] \in \hat{K}_M$ and $v \in H_\pi$.

If we assume that $F \in H_\pi$ the integrals $\int_K (v|\sigma(k)v_p)dk$ will vanish for all $[\sigma] \neq [\pi]$ and the condition (3.7) converts in $\sum_{p=1}^{\dim H_\pi} Q^\pi_{pj}(-\lambda) \int_K (v|\pi(k)v_p)$ $F(kM)\, dk = 0$ for all $v \in H_\pi$ and $j \le \dim H_\pi$. The last condition can be satisfied for a nonzero F if the family of functions $u_j := \sum_p Q^\pi_{pj}(-\lambda) v_p$ is linearly dependent, that is if $\det Q^\pi(-\lambda) = 0$.

We have obtained:

PROPOSITION (3.4). $h_{\pi,\lambda} \in \mathscr{N}$ *iff* $\det Q^\pi(-\lambda) = 0$.

The explicit form of the functions $\det Q^\pi(\cdot)$ can be calculated (see Thm. 6.4, [2]). For our purpose the following information is of great importance:

THEOREM (3.6). *For every* $[\pi] \in \hat{K}_M$ *the space* $\mathcal{N}_\pi$ *is finite-dimensional and its spectrum is determined by the condition* $\det Q^\pi(-\lambda) = 0$. *There exists a finite sequence of natural numbers* $m_\pi(\lambda)$ *such that the functions*

$$\frac{d^p}{d\lambda^p}\, e^{(i\lambda-\rho)(\log a)}, p \leq m_\pi(\lambda), \quad \lambda \in Sp\,\mathcal{N}_\pi$$

form a base of $\mathcal{N}_\pi$.

4. Spectral analysis theorem.

The final result about the spectral analysis in the space $\mathcal{E}(\Xi)$ will be deduced with the aid of the following theorem.

THEOREM (4.1). [7] *If* $V \subset \mathcal{E}(R)$ *is a closed* $\mathcal{E}'_s(\mathbf{R})$*-convolution module then at least one of the following conditions are satisfied:*
1° *the identity function belongs to* V,
2° *some nontrivial combination of the functions* $e^{i\lambda x}, e^{-i\lambda x}$ *belongs to* V.

THEOREM (4.2). *Let* $V \subset \mathcal{E}(\Xi)$ *be a nontrivial closed and G-invariant subspace. Then V contains at least one element from the following family of functions:*

$$h^\alpha_\lambda(kaMN) := e^{(i\lambda-\rho)\log a} + \alpha e^{(-i\lambda-\rho)\log a}, \quad \lambda \in \mathfrak{a}^*_c,\ \alpha \in \mathbf{C},$$

$$h^1(kaMN) := \frac{d}{d\lambda}\, e^{(i\lambda-\rho)\log a}|_{\lambda=0},$$

$$h_{\pi,\lambda}(kaMN) := \omega_\pi(kM)e^{(i\lambda-\rho)\log a}, \quad \lambda \in \mathcal{Z} \text{ and } \det Q^\pi(-\lambda) = 0.$$

Proof. If V is nonzero then for some $[\pi] \in \hat{K}_M$ the space V_π is nontrivial. Assume first that $V \subset \mathcal{N}$. In this case V_π is finite-dimensional and spanned by analytic functions described in Thm 3.6. In particular the elements of the space $P^M_\pi V = \omega_\pi \times V_\pi$ are A-analytic and Thm 2.5 is valid. This implies that the space V_π is closed with respect to convolutions with functions of the form $e^{-\rho}\varphi$, $\varphi \in \mathfrak{D}_s(A)$.

If V is not contained in $\mathcal{N}$ then the space $V_1 := PV$ is nontrivial and in virtue of Prop. 2.6 it is also invariant under the same operation. Let us calculate for $\varphi \in \mathfrak{D}_s(A)$, $f \in V_\pi$ or V_1 (which is isormphic to $V_{[1]}$):

$$e^{\rho(\log a)}(f * (e^{-\rho}\varphi))(a) = e^{\rho(\log a)} \int_A f(b)e^{-\rho(\log ab^{-1})}\varphi(ab^{-1})\,db$$
$$= \int_A f(b)e^{\rho(\log b)}\varphi(ab^{-1})\,db = ((e^\rho f) * \varphi)(a).$$

This formula implies that the space $\tilde{V}_\pi$ of all functions of the form $e^\rho f$, $f \in V_\pi$ is a $\mathfrak{D}_s(A)$ convolution module. The application of convolution $*$: $\tilde{V}_\pi \times \mathfrak{D}_s(A) \to \tilde{V}_\pi$ is continuous with respect to the topology in $\mathfrak{D}_s$ induced by the injection $\mathcal{D}_s(A) \hookrightarrow \mathcal{E}'_s(A)$. The mapping extends continuously to the product $\tilde{V}_\pi \times \mathcal{E}'_s(A)$ and $\tilde{V}_\pi$ converts in an $\mathcal{E}'_s(A)$-convolution module. Since A is a group isomorphic to $\mathbb{R}$, Thm 4.1 can be applied. The space $\tilde{V}_\pi$ contains then a nonzero function of the form $e^{i\lambda(\log a)} + \alpha e^{-i\lambda(\log a)}$ or the function $\frac{d}{d\lambda}\, e^{i\lambda(\log a)}|_{\lambda=0}$ belongs to V. If it is the case of the space $V_{[1]} \neq 0$ then $V_{[1]}$ contains or the functions of the form h^α_λ or h^1. In the case of $V \subset \mathcal{N}$ only the frequencies $\lambda \in \mathcal{Z}$ may appear and the space V_π contains a function of the form $h_{\pi,\lambda}$ with $[\pi] \neq |1|$ such that $\det Q^\pi(-\lambda) = 0$. The proof follows.

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