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A NEW PROOF THAT THE FIRST COHOMOTOPY GROUP OF A UNICOHERENT SPACE IS TRIVIAL

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1. Introduction

The first cohomotopy group of a topological space is defined as follows. The unit circle in the complex plane is a topological group under the multiplication of complex numbers. The set $F(X)$ of all mappings of X into S^1 inherits the structure of an Abelian group from S^1. The set $H(X)$ of all homotopically constant mappings of X into S^1 is a subgroup of $F(X)$. The *first cohomotopy group of* X is the quotient group

$$\Pi^1(X) = F(X)/H(X)$$

(p.45 of [24]; pp. 61, 62 of [26](1)). An element f of $F(X)$ belongs to $H(X)$ if and only if it has a lifting - that is, a real-valued mapping ϕ defined on X such that

$$f(x) = e^{i\phi(x)}, \qquad x \in X.$$

(theorem 1 in [7] (see footnote (3) below); theorem (6.2), p.226 of [27]; corollary to theorem 7.3 in [26]). By virtue of this equivalence, the most effective way of proving the separation properties of the plane by its subsets is through the use of the first cohomotopy group. This was first done by Eilenberg in [7], the first two parts of which formed his doctoral thesis. Many of the results in [7] appear in [5], [19], [20], [26], [27].

A connected space X is said to be *unicoherent* if, whenever M, N are connected closed sets such that $X = M \cup N$, $M \cap N$ is connected. It is well-known that the unicoherence of a connected locally connected space is equivalent to any one of the following separation properties of the space by its closed subsets:

(1.1) if A, B are disjoint closed sets and $A \cup B$ separates X, then so does A or B,

(1.2) if A, B are disjoint closed sets and $A \cup B$ separates p, q in X, then so does A or B,

(1.3) if A is a closed set which separates X, then so does some component B of A,

(1.4) if A is a closed set which separates p, q in X, then so does some component B of A.

These are the so-called *Phragmen-Brouwer* properties(2), which are very useful in plane topology.

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(1) The notation $H^1(X)$ is used in [26], because, for a compact Hausdorff space X, the first Čech cohomology group of X and the first cohomotopy group of X coincide.

(2) See $\mathcal{U}_3, \mathcal{U}_4$ in [17], I,I′, p.47 of [29], properties (v), (vi) in theorem 1 of [25], and properties (ii), (iii) in theorem 2 of [14]. See [28], [29], [11], [12] for nomenclature.

A fundamental result in Eilenberg's treatment of plane topology in [7] is that, *for a connected locally connected metric space X, a necessary and sufficient condition that X be unicoherent is that* $\Pi^1(X) = 0$(3). The sufficiency of the condition that $\Pi^1(X) = 0$ is easily seen by making use of the metric(4) (theorem 2 of [7]; 25.2.1 of [5]). The purpose of this note is to give a new proof of the necessity of the condition that $\Pi^1(X) = 0$, using a well-known characterization of unicoherthe frontiers of connected open sets in Stone [25] (proposition 2.1 below). It is considerably simpler than both Eilenberg's original proof (theorem 3 of [7]) and Čech's proof (25.2.2 of [5]),(5) which as far as the author is aware are the only ones that have been published, in that the open covering of X involved contains a *unique* simple chain between any pair of points in X.

A course in plane topology using Eilenberg's methods makes a good introduction to algebraic topology, and it was used for this purpose by both Čech and Wall in teaching students in Prague and Cambridge, respectively (see prefaces to [5], [26]). However, in using the first cohomotopy group to prove the main separation properties of the plane by its compact subsets, culminating in Eilenberg's proof of the Alexander duality theorem in the plane, Wall nowhere in [26] makes mention of the above result. The proof below can be easily understood in the classroom (the open covering is easy to depict on the blackboard - its nerve is an acyclic 'graph'), and it is hoped that it will contribute to making the result and the Phragmen-Brouwer properties of a unicoherent space more widely recognized as an integral part of a course in plane topology, as they are not as commonly known to topologists and complex analysts as they should be.

2. The Proof

There are several characterizations of unicoherence in terms of open sets, some of which are difficult to prove (e.g., theorem 3 of [25]; the theorem of [13]). The one that we need, namely theorem 1(i), (iv) of [25], is easily proved (i.e., it is a student exercise to prove it directly, without passing through the cycle of statements in theorem 1 of [25]):

PROPOSITION (2.1). *A connected locally connected space X is unicoherent if and only if, for each pair of connected open sets G, H such that $(FrG) \cap (FrH) = \emptyset$, $G \cap H$ is connected.*

We shall also frequently use the following obvious fact about locally connected spaces:

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(3) A space X is said to have *property (b)* in [7] if every mapping of X into S^1 has a lifting. This is equivalent to saying that $\Pi^1(X) = 0$ (see above).

(4) If X is a connected normal space, the same inference can be drawn by using Tietze's extension theorem for real-valued mappings (exercise 2(d) with $n = 1$, p.82 of [26]).

(5) Although all the spaces in [7] and [5] are assumed to be metric spaces, the metric is not used in either of these proofs. (It is also not used in the proof of theorem 1 of [7]).

PROPOSITION (2.2). *If H is a union of components of an open set G in a locally connected space X, then $FrH \subset X - G$.*

THEOREM (2.3). *If X is a connected locally connected unicoherent space, then* $\Pi^1(X) = 0$.

Proof. We denote the exponential mapping by

$$\exp : \mathbf{R} \to S^1, \quad t \mapsto e^{it},$$

and we let I, J be two open arcs which cover S^1 such that $1 \in I - J$.

In order to show that $\Pi^1(X) = 0$, we show that every mapping from X into S^1 has a lifting. Thus let $f : X \to S^1$ be a mapping and let $U = f^{-1}(I)$, $V = f^{-1}(J)$. Since X is locally connected, $K = \{U_i\}_i \cup \{V_j\}_j$ is an open covering of X, where $\{U_i\}_i$, $\{V_j\}_j$ are the collections of components of U, V, respectively. Select $x_0 \in X$, and let W_n be the set of all $x \in X$ such that there is a simple chain in $\mathbf{K}$ from x_0 to x of length at most n, for $n = 1, 2, 3, \ldots$. Then X is expressible as the union of the increasing sequence

$$W_1 \subset W_2 \subset W_3 \subset \ldots \subset W_n \subset W_{n+1} \subset \ldots$$

of connected open sets (theorem (3.4) of [9]). We construct a lifting θ of f by inductively constructing a lifting θ_n of $f|W_n$ such that $\theta_{n+1}|W_n = \theta_n$, for $n = 1, 2, 3, \ldots$.

Since W_1 is the component of U that contains x_0, $FrW_1 \subset X - U$ by (2.2); i.e., $FrW_1 \subset V$. Let C be the component of $\exp^{-1}(I)$ that contains 0. Since $\exp|C : C \to I$ is a homeomorphism,

$$\theta_1 = (\exp|C)^{-1} \circ (f|W_1) : W_1 \to \mathbf{R}$$

is a lifting of $f|W_1$.

Now let $\{V_{j_l}\}_l$ be the subcollection of $\{V_j\}_j$ consisting of all components of V which meet FrW_1. Clearly

$$W_2 = W_1 \cup \bigcup_l V_{j_l},$$

and also $FrW_2 \subset U$. To see this observe that $FrW_2 \subset Fr(\bigcup_l V_{j_l})$, because $FrW_1 \subset \bigcup_l V_{j_l})$; thus $FrW_2 \subset U$, because $Fr(\bigcup_l V_{j_l}) \subset X - V$ by (2.2). In order to construct the lifting of $f|W_2$, observe that $(FrV_{j_l}) \cap (FrW_1) = \emptyset$, because $FrV_{j_l} \subset X - V$ by (2.2) and $FrW_1 \subset V$. Thus $V_{j_l} \cap W_1$ is connected by (2.1), because V_{j_l}, W_1 are connected open sets. Let D_l denote the component of $\exp^{-1}(J)$ that contains $\theta_1(V_{j_l} \cap W_1)$. Since expa homeomorphism,

$$\psi_l = (\exp|D_l)^{-1} \circ (f|V_{j_l}) : V_{j_l} \to \mathbf{R}$$

is a lifting of $f|V_{j_l}$, and it coincides with θ_1 on $V_{j_l} \cap W_1$. Since this holds for all l,

$$\theta_2 = \theta_1 \cup \bigcup_l \psi_l : W_2 \to \mathbf{R}$$

is a well-defined lifting of $f|W_2$.

We carry out the next step of the induction, which is the same as the previous one, except for the interchange of letting.

Let $\{U_{i_k}\}_k$ be the subcollection of $\{U_i\}_i$ consisting of all the components of U which meet FrW_2. Clearly

$$W_3 = W_2 \cup \bigcup_k U_{i_k},$$

and also $FrW_3 \subset V$. To see this observe that $FrW_3 \subset Fr(\bigcup_k U_{i_k})$, because $FrW_2 \subset \bigcup_k U_{i_k}$; thus $FrW_3 \subset V$, because $Fr(\bigcup_k U_{i_k}) \subset X - U$ by (2.2). In order to construct the lifting of $f|W_3$, observe that $(FrU_{i_k}) \cap (FrW_2) = \emptyset$, because $FrU_{i_k} \subset X - U$ by (2.2) and $FrW_2 \subset U$. Thus $U_{i_k} \cap W_2$ is connected by (2.1), because U_{i_k}, W_2 are connected open sets. Let C_k be the component of $exp^{-1}(I)$ that contains $\theta_2(U_{i_k} \cap W_2)$. Since $exp|C_k : C_k \to I$ is a homeomorphism,

$$\phi_k = (exp|C_k)^{-1} \circ (f|U_{i_k}) : U_{i_k} \to \mathbf{R}$$

is a lifting of $f|U_{i_k}$, and it coincides with θ_2 on $U_{i_k} \cap W_2$. Since this holds for k,

$$\theta_3 = \theta_2 \cup \bigcup_k \phi_k : W_3 \to \mathbf{R}$$

is a well-defined lifting of $f|W_3$.

Continuing in this way, we obtain a lifting $\theta_n : W_n \to \mathbf{R}$ of $f|W_n$ such that $\theta_{n+1}|W_n = \theta_n$, for $n = 1, 2, 3 \ldots$. Since, as previously mentioned,

$$X = \bigcup_{n=1}^{\infty} W_n,$$

$$\theta = \bigcup_{n=1}^{\infty} \theta_n : X \to \mathbf{R}$$

is a well-defined lifting of f. Q.E.D.

3. Comments on the proofs of Eilenberg and Čech

We outline the proofs of the above theorem given by Eilenberg in [7] and Čech in [5] in order to comment on the minimum number of open arcs required to cover S^1 in each case. We begin with the antecedents in [2] of Eilenberg's proof, changing the original notation in [2], [7], [5] only when it is necessary for the purposes of explanation, as indicated in the footnotes below.

Let X be a quasi-Peano space, as defined by Borsuk in [2] (i.e., a connected locally connected metric space which is an absolute G_δ); it is completely metrizable and locally arcwise connected, but not necessarily locally compact. For a mapping $f : X \to S^1$ and a simple arc L in X with initial point a and terminal point b, let

$$I_a^b(f, L) = \phi(b) - \phi(a),$$

where ϕ is a lifting of $f|L$[6]. Borsuk showed in theorem 34 of [2] that, *for a quasi-Peano space X, a necessary and sufficient condition that X be unicoherent is that $I_a^b(f, L) = I_a^b(f, L')$, for any mapping $f : X \to S^1$ and any pair of simple arcs L, L' with the same initial point a and the same terminal point b.* The proof of the necessity of this condition consists of showing that $I(f, \Omega) = 0$, for any mapping $f : X \to S^1$ and any oriented simple closed curve Ω in X. This is accomplished by supposing that $I(f, \Omega) \neq 0$ and then performing a sequence of modi$(f_1, \Omega_1), (f_2, \Omega_2), \ldots, (f_n, \Omega_n)$[7] on the pair (f, Ω) such that $I(f_r, \Omega_r) \neq 0$, for $r = 1, 2, \ldots, n$, until at the last stage it can be seen that $I(f_n, \Omega_n) \neq 0$ is contradictory. The sufficiency of the condition is proved using the notion of a retract, which was introduced by Borsuk in his doctoral thesis (see [1]).

Three years after the appearance of [2], the first cohomotopy group was introduced by Bruschlinsky in [4] (sometimes it is called the 'Bruschlinsky' group, as in [10]). Two years later Eilenberg's doctoral thesis appeared in [7]. Both Eilenberg and Borsuk were students of Kuratowski (see p. 134 of [21]), but Borsuk was Eilenberg's " 'guru' and inspiration", in the words of [8].

For each pair of complex numbers $z_1, z_2 \in S^1$ such that $|z_1 - z_2| < 2$, denote by $[z_1, z_2]$ the unique number in the interval $[0, \pi)$ such that

$$e^{i[z_1, z_2]} = z_2/z_1.$$

In Eilenberg's proof - the proof of theorem 3 of [7] - S^1 is covered with a simple circular chain of open arcs $L_1, L_2, \ldots, L_n$ (i.e., $L_i \cap L_j \neq \emptyset$ if and only if $|i-j| \leq 1$, the indices being reduced mod n), whose closures also form a simple circular chain, such that $\operatorname{diam}(L_i) < 1/2$, for $i = 1, 2, \ldots, n$. Let $f : X \to S^1$ be a mapping of the connected locally connected unicoherent space X into S^1, and let $\mathbf{K}$ denote the open covering of X consisting of all the components of all the sets $f^{-1}(L_i)$, for $i = 1, 2, \ldots, n$. Let $x', x'' \in X$ and let $A_1, A_2, \ldots, A_p$ be a chain in $\mathbf{K}$ from x' to x''. Select a point $x_i \in A_i$, for $i = 1, 2, \ldots, p$, and consider the number

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(6) Notice however, that the notion of a lifting is nowhere made explicit in [2]. Thus an *ad hoc* definition of the number $I_a^b(F, L)$ is given in [2].

(7) The subscripts used here do not coincide with those used in [2].

$$I(x', x'') = [f(x'), f(x_1)] + \sum_{i=1}^{p-1} [f(x_i), f(x_{i+1})] + [f(x_p), f(x'')].$$

It is easily seen that $I(x', x'')$ does not depend on the choice of the points $x_1, x_2, \ldots, x_p$. Eilenberg's proof consists of showing that $I(x', x'')$ does not depend on the choice of the chain $A_1, A_2, \ldots, A_p$. This is equivalent to showing that, if $x' = x''$, then $I(x', x'') = 0$, whatever chain (now a circular chain) is selected. The defining property of unicoherence - rather than any property equivalent to it - is used to prove this. A lifting of f may now be defined using the function $I(x', x'')$.

Notice that Eilenberg's proof depends on $L_j \cup L_{j+1}$ being contained in an open semicircle of S^1, for $j = 1, 2, \ldots, n$ (the indices being reduced mod n). This requires the covering of S^1 to have at least *five* open arcs. Notice also, however, that Eilenberg's entire proof can in fact be carried out with a covering of S^1 by five open arcs having diameters $\leq \sqrt{2}$, for example.

Čech's proof of the theorem was found among his posthumous papers (see preface to [5]). In this proof - the proof of theorem 25.2.2 of [5] - S^1 is covered with the four semicircular open arcs contained in two consecutive quadrants in the complex plane. Let $f : X \to S^1$ be a mapping from the connected locally connected(8) unicoherent space X into S^1. The inverse images of the four open arcs under f are denoted by Q_1, Q_2, Q_3, Q_4. Let $\mathcal{Q}_i$ be the collection of all components of Q_i, and let $\mathbf{K} = \bigcup_{i=1}^{4} \mathcal{Q}_i$.(9) It is first shown that every simple circular chain $V_1, V_2, \ldots, V_m$ in the open covering $\mathbf{K}$ of X is in fact contained in some $\mathbf{K} - \mathcal{Q}_i$.(10) This is done by using one of the Phragmen-Brouwer properties, namely (1.4) above.(11) Let

$$v = (exp\,|(-\pi, \pi))^{-1} : S^1 - \{-1\} \to (-\pi, \pi);$$

thus, if $V \in \mathbf{K}$ and $y', y'' \in V$, then $v(f(y'')/f(y'))$ is defined, because $f(y') + f(y'') \neq 0$. Let $V_1, V_2, \ldots, V_m$ be a circular chain in $\mathbf{K}$, and let $y_r \in V_r \cap V_{r+1}$ (the indices being reduced mod m). Using the fact any simple circular chain

(8) There is a misprint in the hypothesis of 25.2.2 of [5]: for "locally compact" read "locally connected".

(9) This notation is not used in [5]. In this outline we have accordingly suppressed mention of the set M and the sets M_λ, for $\lambda = 1, 2, 3, 4$, used in [5]; thus V_r here coincides with $V(x_r)$, for $x_r \in M$, in [5].

(10) This part of Čech's argument can be carried out to obtain the same conclusion if S^1 is covered by any three open arcs, no two of which cover S^1, instead of the four semicircular open arcs defined here.

(11) Notice that, although 25.1.2 of [5] is quoted in the proof of 25.2.2, it is only used to deduce that a connected subset S of $B[V(x_s)]$ separates x_s, x_t in X (notation of [5]), and (1.4) above may also be used to deduce this.

is contained in some $\mathbb{K} - \mathcal{Q}_i$, it is shown that

$$v(f(y_m)/f(y_1)) = \sum_{r=1}^{m-1} v(f(y_{r+1}/f(y_r)).$$

A point $a \in X$ and a real number α are now chosen such that $f(a) = e^{i\alpha}$. For each point $y \in X$, there is a simple chain $V_1, V_2, \ldots, V_m$ in $\mathbb{K}$ from a to y. Let $a = y_0, y_1, y_2, \ldots, y_{m-1}, y_m = y$ be a sequence of points such that $y_r \in V_r \cap V_{r+1}$, for $r = 1, 2, \ldots, m-1$, and put

$$\psi(y) = \alpha + \sum_{r=1}^{m} v(f(y_r)/f(y_{r-1})).$$

Notice that Čech's proof can in fact be carried out if S^1 is covered with just three open arcs each one of which is contained in an open semicircle (see footnote (7)).

In neither of the proofs of Eilenberg or Čech is there any possibility that the open covering $\mathbb{K}$ of X will in general contain a *unique* simple chain between any two points in X, as is the case with our open covering $\mathbb{K}$ of X. This is the feature of our proof that makes it readily underarises from basing it on proposition (2.1) above, by virtue of which we are able to use just *two* open arcs to cover S^1. No greater number will do.

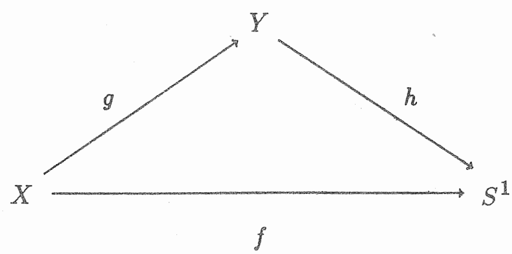
4. Eilenberg's "very simple" proof for a Peano continuum

Borsuk strengthened theorem 34 of [2] for a Peano continuum (i.e., a connected locally connected compact metric space) in [2]. Namely, he showed in theorem 38 of [2] that a necessary and sufficient condition for a Peano continuum X to be unicoherent is that the function space $(S^1)^X$, with the supremum metric, be connected. Eilenberg obtained in [6] what he called a "very simple" proof of the necessity of this condition (see pp. 292, 295 of [6]). This proof uses results in continua theory in the literature of the 1920's and 1930's, namely [2], [14], [17], [20], [21], which are no longer contemporary sources. Whyburn gives the same proof in theorem (7.4), p.228 of [25](12), but omits the reasons and references for some of the main statements in the argument. We conclude by giving this earlier proof of Eilenberg in order to point out its analogy with our proof above.

Let X be a unicoherent Peano continuum and let $f : X \to S^1$ be a non-constant mapping. Let Y be the quotient space formed from the decomposition

(12) In theorem(7.4), p.228 of [27], Whyburn uses Eilenberg's argument to show that, for a Peano continuum X, a necessary and sufficient condition that X be unicoherent is that X have property (b) (see footnote (3) above). This condition is equivalent to saying that $(S^1)^X$ is connected, because X is compact (theorem 9 of [7]; corollary (6.11), p.226 of [27]). Notice that a different proof of this theorem (for a Peano continuum) is given by Kuratowski in theorem 3, p.438 of [20].

of X into the components of the inverse point-images of f, and consider Y as a Peano continuum (i.e., endow Y with some metric). Let $g : X \to Y$ be the quotient mapping and let $h : Y \to S^1$ be the unique mapping such that the diagram



commutes. Then g is monotone (definition ((4.1), p.127 of [27]), h is light (definition (4.4), p. 130 of [27]), and $f = h \circ g$ is the monotone-light factorization of f ((4.1),pp. 141-143 of [27]; theorem 1 of [6], where the monotone-light factorization was established for the first time). Since the inverse point-images of h are 0-dimensional and S^1 is 1-dimen1-dimensional (theorem V1.7 of [16]). Since g is monotone and X is unicoherent, so is Y (consequence of (2.2), p. 138 of [27] or theorem 9, p. 131 of [20]). As a 1-dimensional unicoherent Peano continuum, Y is a dendrite (corollary 8, p. 442 of [20][13]; definition on p.300 of [20] or p.88 of [27]). Since there is a unique arc between any two points in Y ((1.2), p.89 of [27]), h has a lifting. Thus f also has a lifting, and consequently f is homotopically constant. This shows that $(S^1)^X$ is connected.

The analogy that we wish to point out between this proof and ours is that in each case the lifting is performed through the use of an acyclic construct: in Eilenberg's case a dendrite (i.e., an acyclic Peano continuum) and in ours an open covering whose nerve is an acyclic 'graph'. Our proof may thus be viewed as a return to using this "very simple" idea of Eilenberg in [6], which predates [7], in the absence of metrizability and compactness.

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(13) This can quite easily be proved as a student exercise, without recourse to Whyburn's theory of cyclic elements, which is used in the proof in [20].

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