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A NEW INVERSION FORMULA AND ABELIAN THEOREMS FOR A GENERALIZED HANKEL TRANSFORM

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1. Introduction

In [1], Agarwal defined a generalization of the well-known Hankel transformation

$$h_\lambda(f(x))(y) = \int_0^\infty \sqrt{xy}\, J_\lambda(xy) f(x)\,dx \tag{1}$$

by means of the integral equation

$$F(y) = h_{\lambda,\mu}(f(x))(y) = 2^{-\lambda} \int_0^\infty (xy)^{\lambda+(1/2)} J_\lambda^\mu\left(\frac{x^2y^2}{4}\right) f(x)\,dx, \tag{2}$$

where J_λ^μ denotes the Bessel-Wright function (see [14]) defined by the series

$$J_\lambda^\mu(z) = \sum_{r=0}^\infty \frac{(-z)^r}{r!\Gamma(1+\lambda+\mu r)}, \qquad \mu > 0 \text{ and } \lambda > -1$$

Notice that when $\mu = 1$, (2) reduces to (1). There is an inversion formula for the $h_{\lambda,\mu}$ transformation, given also by Agarwal [1]. He proved that if $F(y)$ is defined by (2), then

$$f(x) = 2^{2-\frac{2}{\mu}-2\frac{\lambda}{\mu}+\lambda} \frac{1}{\mu} \int_0^\infty (xy)^{-\lambda-\frac{3}{2}+2\frac{1+\lambda}{\mu}} J^{1/\mu}_{-1+(1+\lambda)/\mu}\left(\left(\frac{x^2y^2}{4}\right)^{1/\mu}\right) F(y)\,dy$$

provided that $\lambda > -1$ and $\mu > 0$.

The main result of this paper is a new inversion formula for the generalized Hankel transformation $h_{\lambda,\mu}$. We also establish abelian theorems for the $h_{\lambda,\mu}$ transforms which incorporate abelian theorems for the aforesaid Hankel transformation (1) as special cases (see [17]). We note that the Bessel-Wright function is a very specialized case of Fox's function (see [6]). For the H-transformation some results similar to the ones shown here for the $h_{\lambda,\mu}$ transformation have been proved in the last years by several authors (see [3], [4] [5] and [7]).

Throughout this paper the asymptotic behaviours of the function J_λ^μ will be used. Wright [15] proved that

$$J_\lambda^\mu(x) \cong O\left[x^{-k(\frac{1}{2}+\lambda)} \exp\left(\frac{(\mu x)^k}{\mu k} \cos(\pi k)\right)\right], \quad \text{as } x \to \infty \tag{3}$$

where $k = \frac{1}{\mu+1}$ and

$$J_\lambda^\mu(x) \cong O(1), \quad \text{as } x \to 0 \tag{4}$$

provided that $\lambda > -\frac{1}{2}$ and $\mu > 0$.

Moreover, the following formula (see [8]) will be employed

$$\mu x^{(\mu-\lambda)/\mu} J_\lambda^\mu(x) = J_{\lambda-1}^\mu(x), \quad \lambda > 0 \text{ and } \mu > 0. \tag{5}$$

2. An inversion formula for the $h_{\lambda,\mu}$ transformation

In this section we give an operational inversion formula for the $h_{\lambda,\mu}$ transformation following a procedure similar to the one used by Nasim [10].

The space E_0 of functions, introduced in [13], consists of all functions $E(s)$ of the form

$$E(s) = e^{bs} \prod_{k=1}^{\infty} \left(1 - \frac{s}{a_k}\right) \exp\left(\frac{s}{a_k}\right),$$

where a_k and b are real and finite, $\sum_{k=1}^{\infty} a_k^{-2} < \infty$ and s is a complex number.

The next result due to Widder [13], will be useful in the sequel.

LEMMA (1). *If $k(x)$, $0 < x < \infty$, is a complex valued function and*

i) $E(s) = \left(\int_0^\infty k(x) x^{s-1} dx\right)^{-1}$ *is in* E_0,

ii) $\phi(t)$ *is bounded and continuous in* $(0, \infty)$, *and*

iii) $f(x) = \int_0^\infty \frac{1}{t} k(\frac{x}{t}) \phi(t) dt, \quad x > 0$

then for almost all x

$$E(\vartheta) f(x) = \phi(x)$$

where $\vartheta = -x\frac{d}{dx}$. Here $E(\vartheta)$ must be understood as in [10]. □

Interesting remarks about the algebra of operators involving ϑ can be found in papers of Nasim [10] and Widder [13], among others.

For the $h_{\lambda,\mu}$ transformation we have the following inversion theorem.

THEOREM (1). *Let $\lambda > -\frac{1}{2}$ and $0 < \mu \leq 1$. If $f(x)$ is a bounded continuous and absolutely integrable function on $(0, \infty)$ and $F(y)$ is the $h_{\lambda,\mu}$-transform of $f(x)$, then for almost $x \in (0, \infty)$*

$$E(\vartheta) R(x) = f(x)$$

where $E(\vartheta) = \Gamma\left(\frac{\lambda}{2} + \frac{3}{4}\right) \frac{2^\vartheta}{\Gamma(\frac{\lambda}{2} - \frac{\vartheta}{2} + \frac{3}{4})}$ *is understood as in* [10] *and*

$$R(x) = \frac{2^{(3/2)+\lambda}}{\Gamma\left(\frac{\lambda}{2}+\frac{3}{4}\right)\mu}\int_0^\infty (xy)^{-\lambda-(3/2)+2(1+\lambda)/\mu}\exp\left(-(xy)^{2/\mu}\right)F(y)dy.$$

Proof: According to the asymptotic expressions (3) and (4) for the function J_λ^μ we can deduce that $z^{\lambda+(1/2)}J_\lambda^\mu\left(\frac{z^2}{4}\right)$ is bounded. Hence $F(y)$ is a bounded and continuous function on $(0,\infty)$ and $R(x)$ is well defined for every $x \in (0,\infty)$.

By applying Fubini's theorem, we get

$$(6)\quad R(x) = \frac{2^{(3/2)}+\lambda}{\Gamma\left(\frac{\lambda}{2}+\frac{3}{4}\right)\mu}\int_0^\infty (xy)^{-\lambda-(3/2)+2(1+\lambda)/\mu}\exp\left(-(xy)^{2/\mu}\right)F(y)dy$$

$$= \frac{2^{(3/2)+\lambda}}{\Gamma\left(\frac{\lambda}{2}+\frac{3}{4}\right)\mu}\int_0^\infty f(t)dt$$

$$\int_0^\infty (xy)^{-\lambda-(3/2)+2(1+\lambda)/\mu}\exp\left(-(xy)^{2/\mu}\right)(ty)^{\lambda+(1/2)}J_\lambda^\mu\left(\frac{t^2y^2}{4}\right)dy$$

By virtue of the following transform formula

$$h_{\lambda,\mu}\left(x^{-\lambda-(3/2)+2(1+\lambda)/\mu}\exp\left(-x^{2/\mu}\right)\right)(y) = \mu 2^{-\lambda-1}y^{\lambda+(1/2)}\exp\left(-\frac{y^2}{4}\right),$$

from (6) we can infer

$$R(x) = \frac{2^{(3/2)-1}}{\Gamma\left(\frac{\lambda}{2}+\frac{3}{4}\right)}x^{-\lambda-(3/2)}\int_0^\infty f(t)t^{\lambda+(1/2)}\exp\left(-t^2/(4x^2)\right)dt$$

or, in other words,

$$R(x) = \int_0^\infty \frac{1}{t}K\left(\frac{x}{t}\right)f(t)dt,$$

where $K(x) = \frac{2^{(3/2)-1}}{\Gamma(\frac{\lambda}{2}+\frac{3}{4})}x^{-\lambda-(3/2)}\exp\left(-1/(4x^2)\right)$. If we define

$$E(s) = \left(\int_0^\infty K(x)x^{s-1}dx\right)^{-1} = \Gamma\left(\frac{\lambda}{2}+\frac{3}{4}\right)\frac{2^s}{\Gamma\left(\frac{\lambda}{2}-\frac{s}{2}+\frac{3}{4}\right)}$$

then $E(s) \in E_0$. Therefore, according to Lemma 1, we can conclude that:

$$E(\vartheta)R(x) = f(x) \text{ almost all } x \in (0,\infty)$$

Here, $E(\vartheta)$ must be understood as in [10]. □

3. Abelian theorems for the $h_{\lambda,\mu}$-transformation

We now establish initial and final value theorems for the $h_{\lambda,\mu}$ transformations. Throughout this section $F(y)$ denotes the $h_{\lambda,\mu}$ transform of $f(x)$ and

$$H(\lambda,\mu,\eta)=2^{-\eta+(1/2)}\frac{\Gamma\left(\frac{\lambda}{2}-\frac{\eta}{2}+\frac{3}{4}\right)}{\Gamma\left(1+\lambda-\mu\left(\frac{\lambda}{2}+\frac{3}{4}-\frac{\eta}{2}\right)\right)},\quad \text{for } \eta<\lambda+\frac{3}{2}.$$

THEOREM (2). (initial value theorem): *Let* $0<\mu\leq 1$ *and* $1<\eta<\lambda+\frac{3}{2}$. *If* $f(x)$ *is an absolutely integrable function on every interval* (X,∞) $(X>0)$ *and* $\lim_{x\to 0} x^\eta f(x)=\alpha$, *then*

$$\lim_{y\to\infty} y^{1-\eta}F(y)=\alpha H(\lambda,\mu,\eta).$$

Proof: By virtue of the asymptotic behaviours (3) and (4) it follows that

$$\left|\int_0^\infty (xy)^{\lambda+(1/2)}J_\lambda^\mu\left(\frac{x^2y^2}{4}\right)f(x)dx\right|\leq C\left(\int_0^1 (xy)^{\lambda+(1/2)}|f(x)|dx+\int_1^\infty |f(x)|dx\right)$$

for a certain C positive constant. Hence, since $x^\eta f(x)$ is bounded near the origin $\eta<\lambda+\frac{3}{2}$ and $f(x)$ is absolutely integrable on (X,∞) with $X>0$, the function $F(y)$ is well defined for every $y>0$.

From [9, p.29], we have

$$\int_0^\infty z^{\lambda-\eta+(1/2)}J_\lambda^\mu\left(\frac{z^2}{4}\right)dz=2^{-\eta+(1/2)}\frac{\Gamma\left(\frac{\lambda}{2}-\frac{\eta}{2}+\frac{3}{4}\right)}{\Gamma\left(1+\lambda-\mu\left(\frac{\lambda}{2}+\frac{3}{4}-\frac{\eta}{2}\right)\right)}. \tag{7}$$

From (7) we can deduce

$$\begin{aligned}&|y^{1-\eta}F(y)-\alpha H(\lambda,\mu,\eta)|\\ &\leq 2^{-\lambda}\left(\sup_{0<x<X}|f(x)x^\eta-\alpha|\int_0^\infty\left|z^{\lambda-\eta+(1/2)}J_\lambda^\mu\left(\frac{z^2}{4}\right)\right|dz\right.\\ &\left.+y^{1-\eta}\int_X^\infty\left|(xy)^{\lambda-\eta+(1/2)}J_\lambda^\mu\left(\frac{x^2y^2}{4}\right)\right|\left|f(x)-\alpha x^{-\eta}\right|dx\right)\end{aligned} \tag{8}$$

for every $X>0$.

Since $1<\eta<\lambda+\frac{3}{2}$ and $0<\mu\leq 1$, the first integral on the right hand side of (8) is convergent. Therefore, given an $\varepsilon>0$, the first term on the right hand side of (8), which is independent of y, can be made less than $\frac{\varepsilon}{2}$ by choosing X small enough. Moreover $z^{\lambda+(1/2)}J_\lambda^\mu\left(\frac{z^2}{4}\right)$ is bounded on $z\in(0,\infty)$ for $\lambda>-\frac{1}{2}$ and $0<\mu\leq 1$, and $f(x)-\alpha x^{-\eta}$ is absolutely integrable on (X,∞) provided that $\eta>1$. Hence keeping X fixed the second term on the right hand side of (8) can be made less than $\frac{\varepsilon}{2}$ for all sufficiently large y.

The last theorem can be generalized as follows:

THEOREM (3). *Let* $0 < \mu \leq 1$ *and* $1 < \eta < \lambda + \frac{3}{2}$. *Let* $f(t)$, $0 < t < \infty$, *be a complex valued p-times continuously differentiable function with* $p \geq 1$, *such that*

$$x^{2(\lambda+p+1)/\mu-(3/2)-\lambda-p}\left(x^{1-(2/\mu)}\frac{d}{dx}\right)^{p}\left(x^{\lambda+(3/2)-2(\lambda+1)/\mu}f(x)\right)$$

is absolutely integrable over (X,∞) *for all* $X > 0$. *If*

i) $\lim_{x\to 0} x^{2(\lambda+r)/\mu}\left(x^{1-(2/\mu)}\frac{d}{dx}\right)^{r-1}\left(x^{\lambda+(3/2)-2(\lambda+1)/\mu}f(x)\right) = 0,$

for $r = 1, \ldots, p$,

ii) $x^{2(\lambda+r)/\mu-(2(\lambda+r)+1)/(\mu+1)}\exp\left(dx^{2/(\mu+1)}\right)$.

$$\left(x^{1-(2/\mu)}\frac{d}{dx}\right)^{r-1}\left(x^{\lambda+(3/2)-2(\lambda+1)/\mu}f(x)\right)$$

is bounded over the interval (X,∞), *for certain* $d \in \mathbb{R}$, $X > 0$ *and* $r = 1, \ldots, p$, *when* $0 < \mu < 1$; *and*

$$\lim_{x\to\infty} x^{\lambda+r-(1/2)}\left(\frac{1}{x}\frac{d}{dx}\right)^{r-1}\left(x^{-\lambda-(1/2)}f(x)\right) = 0, \quad \textit{for } r = 1, \ldots, p,$$

iii) $\lim_{x\to 0} x^{\eta+2(\lambda+p+1)/\mu-(3/2)-\lambda}\left(x^{1-(2/\mu)}\frac{d}{dx}\right)^{p}\left(x^{\lambda+(3/2)-2(\lambda+1)/\mu}f(x)\right) = \alpha,$

then

$$\lim_{y\to\infty} y^{1-\eta}F(y) = \alpha(-\mu)^{p}H(\lambda+p,\mu,\eta+p).$$

Proof: According to (5) we can write

$$h_{\lambda,\mu}\{f(x)\}(y) = 2^{-\lambda-1}\mu y^{\lambda+(1/2)}$$
$$\int_0^\infty x^{\lambda+(3/2)-2(\lambda+1)/\mu}\frac{d}{dx}\left(x^{2(\lambda+1)/\mu}J^{\mu}_{\lambda+1}\left(\frac{x^2y^2}{4}\right)\right)f(x)dx$$

Integrating by parts,

$$h_{\lambda,\mu}\{f(x)\}(y) = \lim_{\substack{a\to 0\\ b\to\infty}} 2^{-\lambda-1}\mu y^{\lambda+(1/2)}\left(x^{\lambda+(3/2)}J^{\mu}_{\lambda+1}\left(\frac{x^2y^2}{4}\right)f(x)\right]_a^b$$
$$\left. -\int_a^b x^{2(\lambda+1)/\mu}J^{\mu}_{\lambda+1}\left(\frac{x^2y^2}{4}\right)\frac{d}{dx}\left(x^{\lambda+(3/2)-2(\lambda+1)/\mu}f(x)\right)dx\right)$$

We can repeat the process to obtain

$$h_{\lambda,\mu}\{f(x)\}(y) = \lim_{\substack{a\to 0\\ b\to\infty}} 2^{-\lambda} y^{\lambda+(1/2)} \left(\sum_{r=1}^{p} \left(\frac{\mu}{2}\right) r(-1)^{r-1} x^{2(\lambda+r)/\mu} J^{\mu}_{\lambda+r}\left(\frac{x^2y^2}{4}\right) \right.$$
$$\left(x^{1-(2/\mu)}\frac{d}{dx}\right)^{r-1} \left(x^{\lambda+(3/2)-2(\lambda+1)/\mu} f(x)\right)\Big]_a^b + (-1)^p \left(\frac{\mu}{2}\right)^p$$
$$\left.\int_a^b x^{2(\lambda+p+1)/\mu-1} \left(x^{1-(2/\mu)}\frac{d}{dx}\right)^p \left(x^{\lambda+(3/2)-2(\lambda+1)/\mu} f(x)\right) J^{\mu}_{\lambda+p}\left(\frac{x^2y^2}{4}\right) dx \right)$$

Hence in virtue of the hypotheses i) and ii) and the asymptotic formulae (3) and (4), we get

$$h_{\lambda,\mu}\{f(x)\}(y) = \left(-\frac{\mu}{y}\right)^p h_{\lambda+p,\mu}\left\{x^{2(\lambda+p+1)/\mu-(3/2)-\lambda-p}\right.$$
$$\left.\left(x^{1-(2/\mu)}\frac{d}{dx}\right)^p \left(x^{\lambda+(3/2)-2(\lambda+1)/\mu} f(x)\right)\right\}(y)$$

for large enough values of y.

To finish the proof of this theorem it is sufficient to use the Theorem 2. □

THEOREM (4). (final value theorem) *Let* $0 < \mu \leq 1$ *and* $1 < \eta < \lambda + \frac{3}{2}$. *If* $f(x)$ *is a measurable function on* $0 < x < \infty$ *such that*

i) $f(x)x^{\lambda+(1/2)}$ *is absolutely integrable on every interval* $(0, X)$ *with* $X > 0$,

and

ii) $\lim_{x\to\infty} x^{\eta} f(x) = \alpha$,

then $\lim_{y\to 0} y^{1-\eta} F(y) = \alpha H(\lambda,\mu,\eta)$.

Proof: The integral which defines $F(y)$ is absolutely convergent for every $y > 0$, as in Theorem 2.

To continue, by taking into account that the function $J^{\mu}_{\lambda}(z)$ is bounded on $z \in (0,\infty)$, provided that $\lambda > -\frac{1}{2}$ and $0 < \mu \leq 1$ and according to (7), we can write

$$\left| y^{1-\eta} F(y) - \alpha H(\lambda,\mu,\eta) \right|$$
$$\leq 2^{-\lambda} \left(\sup_{x>X} |f(x)x^{\eta} - \alpha|\, y \int_X^{\infty} \left| (xy)^{\lambda-\eta+(1/2)} J^{\mu}_{\lambda}\left(\frac{x^2y^2}{4}\right) \right| dx \right.$$
$$\left. + y^{1-\eta} \int_0^X \left| (xy)^{\lambda-\eta+(1/2)} J^{\mu}_{\lambda}\left(\frac{x^2y^2}{4}\right) \right| \left| f(x) - \alpha x^{-\eta} \right| dx \right)$$

(9)
$$\leq 2^{-\lambda} \int_0^{\infty} \left| z^{\lambda+(1/2)} J^{\mu}_{\lambda}\left(\frac{z^2}{4}\right) \right| dz \sup_{x>X} |f(x)x^{\eta} - \alpha|$$

$$+Cy^{\lambda-\eta+(3/2)}\int_0^X \left|f(x)-\alpha x^{-\eta}\right| x^{\lambda+(1/2)}dx\Big)$$

for every $X > 0$ and a suitable $C > 0$. Both integrals on (9) are convergent and the first term is independent of y. Therefore, given an $\varepsilon > 0$, the first term can be made less than $\frac{\varepsilon}{2}$ by choosing X sufficiently large. Then there will exist an $Y > 0$ such that the second term is less than $\frac{\varepsilon}{2}$ for $0 < y < Y$. □

Theorem 4 can be extended by proceeding as in Theorem 3.

THEOREM (5). *Let $0 < \mu \leq 1$ and $1 < \eta < \lambda + \frac{3}{2}$. If $f(x)$ is a p-times continuously differentiable function on $(0,\infty)$, $p \geq 1$, satisfying*

i) $x^{2(\lambda+p+1)/\mu-1}\left(x^{1-(2/\mu)}\frac{d}{dx}\right)^p\left(x^{\lambda+(3/2)-2(\lambda+1)/\mu}f(x)\right)$ *is absolutely integrable on $(0,X)$, with $X > 0$,*

ii) $\lim_{x\to 0} x^{2(\lambda+r)/\mu}\left(x^{1-(2/\mu)}\frac{d}{dx}\right)^{r-1}\left(x^{\lambda+(3/2)-2(\lambda+1)/\mu}f(x)\right) = 0,$
for $r = 1,\ldots,p$,

iii) $x^{2(\lambda+r)/\mu-(2(\lambda+r)+1)/(\mu+1)}\exp\left(dx^{2/(\mu+1)}\right)\cdot$
$\left(x^{1-(2/\mu)}\frac{d}{dx}\right)^{r-1}\left(x^{\lambda+(3/2)-2(\lambda+1)/\mu}f(x)\right)$ *is bounded over the interval (X,∞), for certain $d \geq 0$, $X > 0$ and $r = 1,\ldots,p$, when $0 < \mu < 1$; and*

$$\lim_{x\to\infty} x^{\lambda+r-(1/2)}\left(\frac{1}{x}\frac{d}{dx}\right)^{r-1}\left(x^{-\lambda-(1/2)}f(x)\right) = 0, \quad \textit{for } r = 1,\ldots,p,$$

iv) $\lim_{x\to\infty} x^{\eta+2(\lambda+p+1)/\mu-(3/2)-\lambda}\left(x^{1-(2/\mu)}\frac{d}{dx}\right)^p\left(x^{\lambda+(3/2)-2(\lambda+1)/\mu}f(x)\right) = \alpha,$

then

$$\lim_{y\to)} y^{1-\eta}F(y) = \alpha(-\mu)^p H(\lambda+p,\mu,\eta+p). \quad \square$$

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