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# TOPOLOGICAL GROUPS AND ACTIONS

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## Introduction

Let $(G,\cdot)$ be a topological group and $(A,+)$ an Abelian topological group. We say $G$ weakly acts on $A$ if $G$ acts on $A$ as groups and for each $a \in A$ the map $g \to ga$ is continuous from $G$ into $A$ and for each $g \in G$ the map $a \to ga$ is a continuous map of $A$ into $A$. Then continuous Cohomology groups $H^n(G,A)$ can be defined for $G$ with coefficients in $A$ for all $n \geq 0$.

In section 1 we consider compact totally disconnected groups $G$ and dense subgroups $S$ of $G$. We show that $S$ is pseudocompact if and only if: whenever $G$ weakly acts on a discret space $X$, $B$ is a countable subset of $X, \sigma$ a map of $B$ into $X$ such that for any $b_1, \ldots, b_n$ in $B$ there is a $g \in G$ such that $\sigma b_i = gb_i$, $i = 1, \ldots, n$, then there exists an $s \in S$ such that $\sigma b = sb$ for all $b \in B$.

We also show that $H^n(G,A) \simeq H^n(S,A)$ for all $n \geq 0$ in case $S$ is pseudocompact and $A$ is a discrete Abelian group on which $G$ weakly acts.

In section 2 we show first that if a topological group $G$ weakly acts on a topological Abelian group $A$ then $G$ weakly acts also on $C^1(G,A) = \{f : G \to A : f \text{ continuous and } f(1) = 0\}$.

We conclude section 2 by proving that if $G$ is such that $G^m$ is a $k$-space for all $m \in \mathbb{N}$ then $H^{n+1}(G,A) \simeq H^n(G, C^1(G,A))$ for all $n \geq 0$. These may be useful for Galois Cohomology of transcendental extension fields.

All spaces considered in this paper are assumed to be Hausdorff.

## 1.

We start with a well known

LEMMA (1.1). (1.8.2 and 1.9.3 in [12]) *Let X be a Hausdorff zero-dimensional topological space. Then the following conditions are equivalent.*

(a) *$X$ is pseudocompact*
(b) *Every countable open cover of $X$ by clopen (closed and open) sets has a finite subcover for $X$.*
(c) *Every discrete open cover of $X$ is finite.*
(d) *Every continuous map $X$ into a discrete space is finite valued.*
(e) *Every continuous map of $X$ into $\mathbb{N}$ is finite valued.*

PROPOSITION (1.2). *Let $S$ be a dense subgroup of a compact totally disconnected topological group $G$. Then the following conditions are equivalent.*

(a) *$S$ is pseudocompact*
(b) *Every continuous map of $S$ into a discrete space $D$ is extendable to a continuous map of $G$ into $D$.*

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*Proof*. (a)⇒(b). Let $f : S \to D$ be a continuous map. Then $f(S)$ is a pseudocompact subset of the discrete metric space $D$ and hence is compact and hence finite. Let $a_1, \dots, a_n$ be the values of $f$. Define $g : \{a_1, \dots, a_n\} \to \mathbb{R}$ by $g(a_i) = i$. Now $g \circ f$ is a continuous map of $S$ into $\mathbb{R}$. Since $S$ is pseudocompact by Theorem 1.5 of [2] $g \circ f$ is uniformly continuous on $S$ and since $\mathbb{R}$ is complete, $g \circ f$ extends to a continuous map $h$ of $G$ into $\mathbb{R}$. Now $(g \circ f)(S) = \{1, \dots, n\} = h(S)$ is a closed set in $\mathbb{R}$, $S$ is dense in $G$ and so $h(G) = \{1, \dots, n\}$. Now it easily follows by considering the map $g^{-1} : \{1, \dots, n\} \to f(S)$, $i \to a_i$ that $f$ extends to a continuous map $(g^{-1} \circ h)$ from $G$ into $D$.

(b)⇒(a). Since $G$ is a compact totally disconnected group it is Zero-dimensional and hence $S$ is a zero-dimensional Hausdorff space. If $S$ is not pseudocompact, by (e) of Lemma (1.1) there is a continuous map $f : S \to \mathbb{N}$ which is not finite valued. This extends to a continuous map $h$ of $G$ into $\mathbb{N}$. Since $G$ is compact, $h(G)$ is compact in $\mathbb{N}$ and so $h(G)$ is a finite subset. But $f(S) \subset h(G)$. This contradicts that $f$ is not finite valued.

Hence (b)⇒(a).

*Definition*. (1.3). Let $X$ be a topological space. $G$ a topological group under multiplication. We say $G$ weakly acts on $X$ if there exists a map $G \times X \to X$, $(g, x) \mapsto gx$ such that

(1) $(g_1 g_2)x = g_1(g_2 x)$ for all $g_1 g_2 \in G$, $x \in X$.
(2) $1x = x$ for 1 identity of $G$ and $x \in X$.
(3) For each $x \in X$, $g \mapsto gx$ is a continuous map of $G$ into $X$ and for each $g \in G$, $x \mapsto gx$ is a continuous map of $X$ into $X$.

If $(A, +)$ is an Abelian topological group, we say $G$ weakly acts on $A$ if further for each $g \in G$, the map $a \mapsto ga$ is an automorphism of $(A, +)$.

*Remark* (1.4). If $G$ is a topological group and weakly acts on a discrete space $X$ then the map $G \times X \to X$ is jointly continuous. Let $(s, x) \in G \times X$. Since $g \mapsto gx$ is a continuous function of $G$ into $X$, there exists a neighbourhood $U$ of $s$ such that $gx = sx$ for all $g \in U$. Now $U \times \{x\}$ maps into $\{sx\}$ and hence it follows that $G \times X \to X$ is continuous.

THEOREM (1.5). *Let $G$ be a compact totally disconnected topological group and $S$ a dense subgroup of $G$. Then the following conditions are equivalent*:

a) *$S$ is pseudocompact.*
b) *If $G$ weakly acts on a discrete space $X$, $B \subset X$ is a countable subset, $\sigma : B \to X$ is a map such that if $b_1, \dots, b_n \in B$ there exists a $g \in G$ such that $\sigma(b_i) = gb_i$ for $i = 1, \dots, n$ then there exists an $s \in S$ such that $s|_B = \sigma$.*

*Proof*. (a)⇒(b): For each $b \in B$ let $G_b = \{s \in G | sb = b\}$. Since $G$ acts on the discrete space $X$, $G_b$ is a clopen subgroup of $G$. Further let $s_b \in G$ be such that $\sigma(b) = s_b b$. Now the collection $\{s_b G_b\}$ is a countable family of nonempty clopen sets in $G$ with finite intersection property. Since $S$ is dense $\{S \cap s_b G_b\}$ is a countable family of nonempty clopen sets in $S$ with finite intersection property. Since $S$ is pseudocompact we get easily from (1.1) (b) that $\bigcap_{b \in B}(S \cap s_b G_b) \neq \emptyset$.

Let $s \in \cap(S \cap s_b G_b)$. Then for each $b \in B$ we get $\sigma(b) = s_b(b) = s(b)$. This establishes (b).

(b)⇒(a): Let $\{U_i\}$ be a sequence of open sets in $G$ such that $\bigcap_1^\infty U_i \neq \emptyset$. We assert that $\bigcap_1^\infty U_i \cap S \neq \emptyset$. Let $\sigma \in \cap U_i$. Since $G$ is a compact totally disconnected group it has a basis at the identity consisting of compact open normal subgroups of finite index [11, 2.5, p. 56]. Hence for each $i$ we get a compact open normal subgroup $N_i$ of finite index such that $\sigma N_i \subset U_i$. Now $G$ weakly acts on the finite discrete space $G/N_i = \{N_i, b_{i1}N_i, \ldots, b_{in_i}N_i\}$ transitively. Let $X = \{\ldots, N_i, b_{i1}N_i, \ldots, b_{in_i}N_i, \ldots\}$ with discrete topology. If $s \in G$ we define $s(b_{ij}N_i) = b_{ik}N_i$ if $sb_{ij}N_i = b_{ik}N_i$ and $s(N_i) = (sN_i)$. This yields easily that $G$ weakly acts on $X$. Let $B = \{b_{11}N_1, b_{21}N_2, b_{31}N_3, \ldots\}$. Consider the map of $B$ into $X$ given by $(b_{i1}N_i) \to \sigma(b_{i1}N_i)$. Since $\sigma \in G$, the map of $B$ into $X$ easily satisfies the condition in (b). Hence by (b) there exists an $s \in S$ such that $\sigma(b_{i1}N_i) = s(b_{i1}N_i)$ for $i = 1, 2, \ldots$. Hence $\sigma^{-1}s(b_{i1}N_i) = (b_{i1}N_i)$ for all $i$. Now if $t \in G$ and $t(b_{i1}N_i) = (b_{i1}N_i)$ we claim $t \in N_i$. For $tb_{i1} = b_{i1}x$, $x \in N_i$. So $t \in b_{i1}N_i b_{i1}^{-1} = N_i$. Hence $\sigma^{-1}s \in N_i$ for all $i$; i.e. $s \in \sigma N_i$ for all $i$. So $s \in \bigcap_1^\infty U_i$.

Now $S$ is pseudocompact follows straight from (1.5) of [2] or theorem (4.2) of [5].

*Definition* (1.6). Let $G$ be a topological group weakly acting on an Abelian topological group $(A, +)$. If $n \in \mathbb{N}$ we define $C^0(G, A) = A$ and $C^n(G, A) = \{f : G^n \to A | f$ is continuous and $f(x_1, \ldots, x_n) = 0$ whenever any one of the $x_i$ is $1\}$. $C^n(G, A)$ is an Abelian group under $+$. If $f \in C^n(G, A)$ we define $df$ on $G^{n+1}$ by

$$df(x_1, \ldots, x_{n+1}) = x_1 f(x_2, \ldots, x_{n+1}) + \sum_1^n (-1)^i f(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1}) + (-1)^{n+1} f(x_1, \ldots, x_n).$$

$d$ is a map from $C^n(G, A)$ into $A^{G^{n+1}}$ such that $d$ is a homomorphism and $d^2 = 0$. For each $n \geq 0$ we define $Z^n(G, A) = \{f \in C^n(G, A) | df = 0\}$. Elements of $Z^n(G, A)$ are called $n$-cocycles. We define $B^0(G, A) = 0$ and if $n \geq 1$, $B^n(G, A) = \{f \in C^n(G, A) | f = dg$ for some $g \in C^{n-1}(G, A)\}$. Elements of $B^n(G, A)$ are called $n$-coboundaries. $Z^n(G, A)$, $B^n(G, A)$ are subgroups of $C^n(G, A)$ and $B^n(G, A) \subset Z^n(G, A)$. We define $H^n(G, A) = Z^n(G, A)/B^n(G, A)$ and call it the $n^{\text{th}}$ cohomology group of $G$ with coefficients in $A$.

*Remark* (1.7). (a) Let $G$ be a topological group acting on an Abelian topological group $(A, +)$ such that $(g, a) \to ga$ is a continuous map from $G \times A$ into $A$. Then $d$ maps $C^n(G, A)$ into $C^{n+1}(G, A)$. This is easy.

(b) Let $G$ be a topological group acting on an Abelian topological group $(A, +)$ such that $(g, a) \to ga$ is a continuous map from $G \times A$ into $A$. Define $C_1^n(G, A) = \{f G^n \to A | f$ continuous $\}$, define $d$ as in (1.6), define $Z_1^n$, $B_1^n$ correspondingly and define $H_1^n(G, A)$. By exactly following the proofs of section 6 in Eilenberg-Maclane [8] we can easily get $H_1^n(G, A) \simeq H^n(G, A)$ for all $n$.

THEOREM (1.8). *Let $G$ be a compact totally disconnected group and $S$ a pseudocompact dense subgroup. For any discrete abelian group $(A, +)$ on which $G$ weakly acts, $S$ also weakly on $(A, +)$ and for all $n$, $H^n(G, A) \simeq H^n(S, A)$*

*Proof*. That $S$ weakly acts on $(A, +)$ whenever $G$ weakly acts on $(A, +)$ easily follows.

If $f \in C^n(G, A)$ then $f|_{S^n}$ belongs to $C^n(S, A)$ and $f$ cocycle (coboundary) implies $f|_S$ is a cocycle (coboundary). Conversely if $g \in C^n(S, A)$ by (1.2), $g$ has a unique extension $\overline{g} \in C^n(G, A)$ (Since $S^n$ is also pseudocompact [2]) and $g$ cocycle (coboundary) implies $\overline{g}$ is a cocycle (coboundary). Hence the theorem easily follows.

*Remark* (1.9). By remark (1.4) and (1.7) (b), theorem (1.8) holds if $H^n(G, A)$, $H^n(S, A)$ are replaced by $H_1^n(G, A)$, $H_1^n(S, A)$.

*Remark* (1.10). That dense pseudocompact subgroups exist in plenty was proved by H. T. Wilcox [13,14]. They can be even chosen with stronger properties as it is shown in [3] and [4]

## 2.

LEMMA (2.1). *Let $G$ be a topological group weakly acting on a space $X$. Let $K$ be a compact set in $G$ and $(b_d)_{d \in D}$ a net in $X$ converging to $b$ in $X$. Let $W$ be an open set in $X$ such that $Kb \subset W$. Then there exists $d_0 \in D$ such that $Kb_d \subset W$ for all $d \geq d_0$.*

*Proof*. Let $a \in K$, $ab \in W$. Since $W$ is open and $G$ weakly acts on $X$ there exists an open set $U_a$ containing 1 in $G$ such that $U_a ab \subset W$. Since $G$ is a topological group there exists another open set $V_a$ containing 1 such that $V_a \cdot V_a \subset U_a$. Now $ab_d$ converges to $ab$ since $G$ weakly acts on $X$. Hence there exists $ad_a \in D$ such that $ab_d \in V_a ab$ for all $d \geq d_a$. Now $\{V_a a\}_{a \in K}$ is an open cover for $K$ and $K$ is compact. Hence there is a finite subcover $\{V_{a_1} a_1, \ldots, V_{a_n} a_n\}$ for $K$. Let $d_0 \geq d_{a_1}, \ldots, d_{a_n}$. Let now $a \in K$ and $d \geq d_0$. Consider $ab_d$. If $a \in V_{a_i} a_i$ then $ab_d \in V_{a_i} a_i b_d$ and $a_i b_d \in V_{a_i} a_i b$. Hence $ab_d \in V_{a_i} V_{a_i} a_i b \subset U_{a_i} a_i b \subset W$. Hence the lemma follows.

We now recall a proposition of J. de Vries.

PROPOSITION (2.2). (J. de Vries [6]) *Let $G$ be a topological group, $Y$ a topological space and $C_c(G, Y)$ the space of all continuous maps from $G$ into $Y$ with compact open topology. If $s \in G$ and $f \in C_c(G, Y)$ we define $sf \in C_c(G, Y)$ by $(sf)(x) = f(xs)$. Then $G$ weakly acts on $C_c(G, Y)$ with this definition.*

*Proof*. This is proposition (2.1.2) of [6].

THEOREM (2.3). Let $G$ be a topological group, $(A, +)$ an Abelian topological group on which $G$ weakly acts. With compact open topology $C^1(G, A)$ is a topological Abelian group and $G$ weakly acts on $C^1(G, A)$ if we define for $s \in G$, $f \in C^1(G, A)$, $sf$ by $(sf)(x) = f(xs) - xf(s)$.

*Proof*. Since $C^1(G, A) \subset C_c(G, A)$ and the latter is a topological Abelian group [1] we get $C^1(G, A)$ is a topological Abelian group. It is well known that as a group $G$ acts on $A^G$. We have only to show that if $s \in G$, $f \in C^1(G, A)$ then $sf \in C^1(G, A)$. If $t_d$ converges to $t$ in $G$ then $f(t_d s)$ converges to $f(ts)$ since $t_d s$ converges to $ts$ and $f$ is continuous. Also $G$ weakly acts on $A$. Hence $t_d f(s)$ converges to $tf(s)$. Since $A$ is a topological group $f(t_d s) - t_d f(s)$ converges to $f(ts) - tf(s)$. Hence $sf(t_d)$ convertes to $sf(t)$. Thus we get $G$ acts on $C^1(G, A)$ as groups. We now complete the proof in two steps.

Step (1): Let $s \in G$ and $f_d$ converge to $f$ in $C^1(G, A)$. We claim $sf_d$ converges to $sf$.

We now define the functions $g_d$, $h_d$, $g$, $h$ on $G$ by $g_d(t) = f_d(ts)$, $h_d(t) = tf_d(s)$, $g(t) = f(ts)$, $h(t) = tf(s)$. First of all it is easily seen that $g_d$, $h_d$, $g$, $h$ are continuous functions on $G$. By 2.2 we get $g_d$ converges to $g$ in the compact open topology. We claim now $h_d$ converges to $h$. Let $h \in (K, O)$, $K$ compact in $G$, $O$ open in $A$ and $(K, O)$ is the set of all continuous maps from $G$ into $A$ mapping $K$ into $O$. $h(k) \in O$ for all $k \in K$; i. e. $kf(s) \in O$ for all $k \in K$. Hence $Kf(s) \subset O$. By (2.1) there exists a $d_0 \in D$ such that $Kf_d(s) \subset O$ for all $d \geq d_0$, i.e. $h_d(k) \subset O$. Hence $h_d \in (K, O)$ for all $d \geq d_0$. Thus $h_d$ converges to $h$. Since $C_c(G, A)$ is a topological group $g_d - h_d$ converges to $g - h$. Hence $sf_d$ converges to $sf$ in $C^1(G, A)$.

Step (2): Let $s_d$ converges to $s$ in $G$ and $f \in C^1(G, A)$. We claim $s_d f$ converges to $sf$. We define $g_d$, $h_d$, $g$, $h$ on $G$ by $g_d(t) = f(ts_d)$, $h_d(t) = tf(s_d)$, $g(t) = f(ts)$, $h(t) = tf(s)$. Easily $g_d$, $h_d$, $g$, $h$ are continuous maps on $G$. $g_d$ converges to $g$ follows by (2.2).

Let $h \in (K, O)$ i. e. $Kf(s) \subset O$. By (2.1) there exists $d_0$ such that $Kf(s_d) \subset O$ if $d \geq d_0$. Hence $h_d \in (K, O)$. Thus $h_d$ converges to $h$. Then $g_d - h_d$ converges to $g - h$. Hence we get $G$ acts on $C^1(G, A)$ and the theorem follows.

THEOREM (2.4). *Let $G$ be a topological group weakly acting on an Abelian topological roup $(A, +)$. Let further $G^n$ be a k-space for each $n \in N$. Then $H^{n+1}(G, A) \simeq H^n(G, C^1(G, A))$ for all $n > 0$.*

*Proof*. We define a map $\sigma_n$

$$\sigma_n : C^n(G, C^1(G, A)) \to C^{n+1}(G, A)$$

by setting $(\sigma_n f)(s_1, \ldots, s_{n+1}) = (-1)^n f(s_2, \ldots, s_{n+1})(s_1)$. Since $G^{n+1}$ is a $k$-space by Corollary (3.2) in [7, p. 261], $\sigma_n$ is a bijection between $C^n(G, C^1(G, A))$ and $C^{n+1}(G, A)$. Now exactly as in [8] we get $d\sigma_n = \sigma_{n+1} d$. Hence we get easily for all $n \geq 0$, $\sigma_n$ carries cocycles (coboundaries) into cocycles (coboundaries) and cohomologous cocycles into cohomologous cocycles. Hence we get easily an isomorphism between $H^n(G, C^1(G, A))$ and $H^{n+1}(G, A)$ for all $n \geq 0$.

*Remark* (2.5). (a) If $G$ is a metric group or a locally compact Hausdorff group then $G^n$ is a $k$-sapace for each $n \in \mathbb{N}$.

(b) If $K$ is a countable field of characteristic zero and $E$ is an extension of countable trascendence degree then $G(E/K)$ wuth Krull topology is a metric group (weakly) acting on $(E, +)$ and $(E^*, \cdot)$.

(c) If $K$ is any field and $E$ is an extension of finite trascendence degree then $G(E/K)$ with Krull topology is a locally compact Hausdorff group (weakly) acting on $(E, +)$ and $(E^*, \cdot)$.

For both (b) and (c) one can consider Galois Cohomology.

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