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SOME RESULTS ON WALLMAN COMPACTIFICATIONS

BY ADALBERTO GARCÍA-MÁYNEZ

1. Introduction

We shall be concerned with normal Wallman basis and their associated compactifications (see [GT], 3.44-3.48). All spaces we consider will be completely regular and Hausdorff and all maps will be continuous. Our main results will deal with rim compact spaces and their Freudenthal compactifications. We give an upper bound for the weight of the Freudenthal compactification of a rim compact space and prove a set theoretic lemma which has some useful applications to extend a map to certain Wallman compactifications of its domain and codomain.

2. Rim compact spaces

Recall a space X is *rim compact* if its topology has as a basis the family $\mathcal{B}$ consisting of all open sets with compact boundary. In this case, $\mathcal{B}$ happens to be a normal Wallman basis of X and its associated compactification, denoted by FX, is the so called *Freudenthal compactification* of X. For any space (X, τ), the family of clopen subsets of X is a basis for a topology τ_0 of X and, of course, $\tau_0 \subset \tau$. The *quasi-component space* $Q(X)$ (see [Is]) consists of all quasi-components of X and a subset $\mathcal{G}$ of $Q(X)$ is defined to be open in $Q(X)$ if $\cup\{G|G \in \mathcal{G}\} \in \tau_0$, *i.e.*, if $\cup\{G|G \in \mathcal{G}\}$ is a union of clopen subsets of X. Clearly $Q(X)$ is 0-dimensional and T_1 (and hence, it is completely regular and Hausdorff) and there is an onto map between βX and $\beta Q(X)$ which extends the map h sending each $x \in X$ to the quasi-component of X which contains it.

Recall the *weight* (resp., *net weight*) of a space X, ωX (resp., $n\omega X$), is the minimum cardinality of a basis (resp., a net) of X.

We give some other definitions:

(2.1). a) If X is any space, zX and $z'X$ denote, respectively, the cardinalities of the families $\{H|H \text{ is a zero set in } X\}$ and $\{H|H \text{ is clopen in } X\}$. Also, ∇LX denotes the least infinite cardinal α such that every open cover of X has a subcover consisting of less than α elements. Thus, X is compact if and only if $\nabla LX = \chi_0$ and X is Lindelöf if and only if $\nabla LX \leq \chi_1$.

b) A space X is *Q-compact* (resp., *Q-Lindelöf*) if $Q(X)$ is compact (resp., Lindelöf).

c) If α, β are infinite cardinal numbers, we define

$$\alpha^{<\beta} = \sum_{i<\beta} \alpha^i$$

In [DC] it is proved that if X is rim compact, then $\omega FX = \omega X \cdot z'X$. We shall prove the following inequality:

(2.2). $z'X \leq \omega Q(X)^{<\nabla LQ(X)}$ for any rim compact space X.

Proof. For any space X, it is easy to prove that $z'X = z'Q(X)$ and $z'X \leq \omega X^{<\nabla LX}$. Hence, $z'X = z'Q(X) \leq \omega Q(X)^{<\nabla LQ(X)}$.

Combining (2.2) with the equality $\omega FX = \omega X \cdot z'X$ we obtain the inequalities:

(2.3). Let X be a rim compact space. Then:

a) $\omega FX \leq \omega X \cdot \omega Q(X)^{<\nabla LQ(X)}$. In particular, if X is Q-compact, we have $\omega FX = \omega X$.

b) $\omega FX \leq \omega X^{<\nabla LX}$.

Definition. (see [Sk]). Let $X \subset Z$ be dense. Z is a *perfect extension* of X if given two closed sets $A, B \subset X$ with $X = A \bigcup B$, we have $C_z(A \cap B) = C_z(A) \cap C_z(B)$.

We know βX is a perfect compactification of X for any space X and FX is a perfect compactification of X for any rim compact space X. We prove now that if X is a rim compact Lindelöf space and FX is first countable, then X is Q-compact. More generally:

(2.4). Let X be a Q-Lindelöf space and let Z be a perfect, Q-compact, first countable extension of X. Then X is Q-compact.

Proof. Suppose, on the contrary, that X is not Q-compact. Since X is Q-Lindelöf but not Q-compact, there exists a sequence $L_1, L_2, \ldots$ of clopen, non-empty, mutually disjoint subsets of X such that $X = \bigcup_{m=1}^{\infty} L_m$. Since Z is a perfect extension of X, for each $m = 1, 2, \ldots$ there exists a clopen subset L^*_m of Z such that $L_m = L^*_m \bigcap X$ (in fact, we may take $L^*_m = C_z L_m$). The density of X in Z implies that $L^*_i \cap L^*_j = \Phi$ for $i \neq j$. Since Z is Q-compact, we must have $Z \neq \bigcup_{m=1}^{\infty} L^*_m$. Choose a point $p \in Z - \bigcup_{m=1}^{\infty} L^*_m$. Since Z is first countable, there exists a subsequence $L^*_{m_1}, L^*_{m_2}, \ldots$ of $L^*_1, L^*_2, \ldots$, such that $p \in \liminf L^*_{m_i}$. Let H be the union of the L_{m_i} with i an odd integer and let J be the union of the L_{m_i} with i an even integer. We let also $K = X - (H \cup J)$, $J^* = C_z J$ and $H^* = Z - J^*$. Then H^*, J^* are disjoint clopen subsets of Z with union Z and $J^* \cap X = J$, $H^* \cap X = H \cup K$. However, if $p \in H^*$, then H^* is an open set in Z containing p and disjoint from each $L^*_{m_i}$ for even i, and if $p \in J^*$, J^* is an open set in Z containing p and disjoint from each $L^*_{m_i}$ for odd i. In both cases we contradict the fact that $p \in \liminf L^*_{m_i}$. Therefore, X must be Q-compact.

COROLLARY (1). *Let X be a rim compact Lindelöf space. If FX is first countable, then X is Q-compact and therefore $\omega FX = \omega X$.*

COROLLARY (2). (see [Is]) *The Freudenthal compactification of a rim compact separable metrizable space X is metrizable if and only if X is Q-compact.*

Extensions of maps to Wallman compactifications

The universal property of the Stone-Cech compactification of a space X may be stated as follows:

(3.1). If $\varphi: X \to Y$ is any map, φ has an extension to map $\varphi_1: \beta X \to \beta Y$.

Considering βX as the Wallman compactification of X associated to the basis of X consisting of all its cozero sets, (3.1) has the following generalization (see [GT], 3.48)

(3.2). Let $\varphi: X \to Y$ be any map and let $\mathcal{B}_x, \mathcal{B}_y$ be normal Wallman basis of X, Y, respectively. Let X^*, Y^* be the corresponding Wallman compactifications. Then φ has an extension to a map $\varphi_1: X^* \to Y^*$ if and only if for every pair H, K of disjoint elements of $C(\mathcal{B}_Y)$, there exist disjoint elements L, M of $C(\mathcal{B}_X)$ such that $\varphi^{-1}(H) \subset L$ and $\varphi^{-1}(K) \subset M$.

The following definition is apparently unrelated to (3.2). However, we shall exhibit a connection.

Definition. Let X be a set, $\mathcal{U}$ a cover of X and $A \subset X$. A is *subordinated* to $\mathcal{U}$ if every member of $\mathcal{U}$ contains, at most, a finite number of elements of A.

(3.3). *Remark:* a) If $\mathcal{U}$ is an open cover of a space X and $A \subset X$ is subordinated to $\mathcal{U}$ then A is closed and discrete.

b) If $\mathcal{U}$ is a normal cover of a space X and $A \subset X$ is subordinated to $\mathcal{U}$ then A is C-discrete (see $[G]_2$).

The following set theoretic lemma has several topological applications:

(3.4). Let $f: X \to Y$ be a function, α, β cardinal numbers with β regular and $\omega \leq \beta \leq \alpha$. Let $\mathcal{U}$ be a cover of X such that $|\{U \in \mathcal{U} | x \in U\}| < \beta$ for each $x \in X$. Then at least one of the following properties holds:

a) There exist $\mathcal{U}_0 \subset \mathcal{U}$ and $Y_0 \subset Y$ such that $|\mathcal{U}_0| < \alpha$, $|Y_0| < \alpha$ and $X = \cup\, \mathcal{U}_0 \cup f^{-1}(Y_0)$.

b) There exists $A \subset X$ such that $|A| = \alpha$, A is subordinated to $\mathcal{U}$ and $f|A$ is $1-1$.

Proof. Suppose a) does not hold. Choose any point $x_0 \in X$. Let $\mathcal{U}_0 = \{V \in \mathcal{U} | x_0 \in V\}$. By hypothesis, there exists a point $x_1 \in X - [\cup\, \mathcal{U}_0 \cup f^{-1}f(x_0)]$. Inductively, let i be an ordinal number $< \alpha$ and suppose $x_j \in X$ has already been defined for each $j < i$. Let $\mathcal{U}_j = \{V \in \mathcal{U} | x_j \in V\}$, $j < i$ and let $\mathcal{W} = \cup \{\mathcal{U}_j \,|\, j < i\}$ If $i < \beta$, the regularity of β implies that $|\mathcal{W}| \leq \sum_{j<i} |\mathcal{U}_j| < \beta \leq \alpha$. If $i \geq \beta$, then $|\mathcal{W}| \leq \sum_{j<i} |\mathcal{U}_j| \leq \beta |i| = |i| < \alpha$. By hypothesis, there exists a point $x_i \in X - [\cup\, \mathcal{W} \cup f^{-1}f(\{x_j | j < i\})]$. This completes the inductive construction of the x_i. The set $A = \{x_i | i < \alpha\}$ is subordinated to $\mathcal{U}$ because each $V \in \mathcal{U}$ contains, at most, one element of A. Besides, $f|A$ is $1-1$. Therefore, b) holds.

COROLLARY (3.4.1). (see $[G]_1$, 3.7). *Let $f: X \to Y$ be an onto $C(\alpha)$-map, where α is a regular cardinal. If Y is α-pseudocompact and for every $y \in Y$, $f^{-1}(y)$ is α-relatively pseudocompact, then X is α-pseudocompact.*

Proof. *Let $\mathcal{U}$ be an arbitrary locally finite cozero cover of X. We have to prove that $\mathcal{U}$ has a subfamily of cardinality $< \alpha$ covering X. Apply theorem taking $\beta = \alpha$. If condition a) holds, there exist $\mathcal{U}_0 \subset \mathcal{U}$ and $Y_0 \subset Y$ such that $|\mathcal{U}_0| < \alpha$, $|Y_0| < \alpha$ and $X = \cup \mathcal{U}_0 \cup f^{-1}(Y_0)$. Since each fiber of f is α-relatively pseudocompact, for each $y \in Y_0$ there exists $\mathcal{U}_y \subset \mathcal{U}$ such that $|\mathcal{U}_y| < \alpha$ and $f^{-1}(y) \subset \cup \mathcal{U}_y$. Hence, $f^{-1}(Y_0) \subset \cup \mathcal{W}$, where $\mathcal{W} = \cup \{\mathcal{U}_y | y \in Y_0\}$. The regularity of α implies that $|\mathcal{W}| < \alpha$.*

Hence, $\mathcal{U}_0 \cup \mathcal{W}$ is a subfamily of $\mathcal{U}$ of cardinality $< \alpha$ covering X. If b) holds and $A \subset X$ is as in the statement of b), then A is C-discrete (use (3.3) b)). But since f is a $C(\alpha)$-map and $f|A$ is $1-1$, $f(A)$ is a C-discrete subset of Y of cardinality α, contradicting the fact that Y is α-pseudocompact. Hence b) cannot hold, a) has to hold and the proof is complete.

COROLLARY *(3.4.2). Let $f: X \to Y$ be an EC_1-map, where Y is pseudocompact. If $\mathcal{U}$ is a locally finite cozero cover of X, there exist $U_1, U_2, \ldots, U_n \in \mathcal{U}$ and $p_1, \ldots, p_m \in Y$ such that $X = \bigcup_{i=1}^{n} U_i \cup \bigcup_{j=1}^{m} f^{-1}(p_j)$.*

Proof. We apply the theorem with $\alpha = \beta = \omega$. If condition b) held, we would have a countable subset A of X subordinated to $\mathcal{U}$, such that $f|A$ is $1-1$. A would be then C-discrete. But EC_1-maps preserve countable C-discrete sets. Therefore, $f(A)$ would be an infinite C-discrete subset of Y, contradicting the pseudocompactness of Y.

COROLLARY (3.4.3). (compare with [N], thm. 1). *Let $f: X \to Y$ be a closed map, where Y is countably compact, and let $\mathcal{U}$ be a point finite open cover of X. Then there exist $U_1, \ldots, U_n \in \mathcal{U}$ and $p_1 \ldots, p_m \in Y$ such that $X = \bigcup_{i=1}^{n} U_i \cup \bigcup_{j=1}^{m} f^{-1}(p_j)$.*

Proof. We assume again that $\alpha = \beta = \omega$. If $A \subset X$ is as in condition b), then A is a countably infinite closed discrete subset of X. Since $f|A$ is $1-1$ and f is closed, $f(A)$ is a countably infinite closed discrete subset of Y, contradicting the fact that Y is countably compact.

COROLLARY (3.4.4). (see [N], thm. 2). *Let $f: X \to Y$ be a closed map of the locally compact metacompact space X onto the compact space Y. Then there exists a compact set $Z \subset X$ such that $f(X - Z)$ is finite.*

Proof. If X is compact, the result is obvious. Suppose then that X is not compact. The assumptions on X imply the existence of a point finite open cover $\mathcal{U}$ of X such that U^- is compact for each $U \in \mathcal{U}$. Applying the theorem with $\beta = \omega$ and $\alpha = |\mathcal{U}|$, we know a) or b) holds. But if b) held, there would exist a closed discrete subset $A \subset X$ such that $|A| = \alpha$ and $f|A$ is $1-1$. But since f is closed, $f(A)$ would also be a closed discrete subset of Y and this would contradict the compactness of Y. Hence a) holds and there exist $\mathcal{U}_0 \subset$

$\mathcal{U}$, $Y_0 \subset Y$, finite, such that $X = \cup\, \mathcal{U}_0 \cup f^{-1}(Y_0)$. It is enough to define $Z = \cup\{U^- | U \in \mathcal{U}_0\}$.

COROLLARY (3.4.5). *Let $f: X \to Y$ be a closed map of the locally compact and metacompact space X onto the locally compact space Y. If H, K are closed disjoint subsets of Y and both of them have compact boundaries, then there exists a closed set $L \subset Y$ such that $f^{-1}(L)$ is compact and separates $f^{-1}(H)$ and $f^{-1}(K)$.*

Proof. The quotient space $Y/\{H, K\}$ obtained by identifying H and K to single points is also locally compact and the canonical map $q: Y \to Y/\{H, K\}$ is closed. Due to these facts, we do not lose generality if we assume that H and K are different points $y_1, y_2 \in Y$. Let W be an open set in Y with compact closure such that $y_1 \in W \subset W^- \subset Y - \{y_2\}$. According to Corollary (3.4.4), there exists a finite set $\{b_1, \ldots, b_n\} \subset W$ such that $f^{-1}(y)$ is compact for each $y \in W^- - \{b_1, \ldots, b_n\}$. Let V be an open set in Y such that $y_1 \in V \subset V^- \subset W$ and such that $\{b_1, \ldots, b_n\} \cap Fr\, V = \Phi$. The set $A = f^{-1}(Fr\, V)$ is then compact and it separates $f^{-1}(y_1)$ and $f^{-1}(y_2)$.

COROLLARY (3.4.6). *Let f, X, Y, H, K be as in corollary (3.4.5). Then there exist disjoint closed sets $H_1, K_1 \subset X$, both of them with compact boundaries, such that $f^{-1}(H) \subset H_1$ and $f^{-1}(K) \subset K_1$.*

Proof. Let $L \subset Y$ be such that $f^{-1}(L)$ is compact and separates $f^{-1}(H)$ and $f^{-1}(K)$, say $X - f^{-1}(L) = U_1 \cup U_2$, where $f^{-1}(H) \subset U_1$, $f^{-1}(K) \subset U_2$ and U_1, U_2 are disjoint open sets. Define $H_1 = U_1^-$. Let T be an open set in X, with compact boundary, such that $H_1 \subset T \subset T^- \subset X - f^{-1}(K)$. Setting $K_1 = X - T$, we obtain the desired result.

COROLLARY (3.4.7). (see [N]). *Let $f: X \to Y$ be a closed map of the locally compact and metacompact space X onto the locally compact space Y. Then f can be extended to a map $f^*: FX \to FY$ between the Freudenthal compactifications of X and Y.*

Proof. Apply Corollary (3.4.6) and (3.2).

As suggested by the referee of this paper, we include one more corollary:

COROLLARY *(3.4.8). ([Ju] 2.32). Assume X is a T_1 space such that $pX \leq k$, where $pX = sup\{|A| \mid A \subset X$ closed and discrete$\}$. Then every open cover of X of point weight $\leq k$ has a subcover of cardinality $\leq k$.*

Proof. Let $\mathcal{U}$ be an open cover of X and let $\alpha = \beta = k^+$. Property (b) in (3.4) cannot hold because $pX \leq k$. By property a), there exists $\mathcal{U}_0 \subset \mathcal{U}$ and $X_0 \subset X$ such that $|\mathcal{U}_0| \leq k$ and $|X_0| \leq k$. For each $x \in X_0$, select an element $U_x \in \mathcal{U}$ such that $x \in U_x$. Hence $\mathcal{U}_0 \cup \{Ux | x \in X_0\}$ is a subcover of $\mathcal{U}$ of cardinality $\leq k$.

CENTRO DE INVESTIGACIÓN Y ESTUDIOS
AVANZADOS DEL I.P.N.
DEPARTAMENTO DE MATEMÁTICAS
MÉXICO 07000, D.F.
MEXICO

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