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SHIFT GROUP FOR A WEAKLY HARMONIZABLE PROCESS ON A LOCALLY COMPACT ABELIAN GROUP*

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1. Preliminary

In contrast with the second order continuous stationary processes, the existence of the shift group for a weakly harmonizable process is not always guaranteed. In this work we deal with the problem of the existence and the boundedness of the shift group for a weakly harmonizable process defined on a locally compact Abelian (l.c.a.) group. We investigate three kinds of shift groups: shift groups without boundedness hypothesis about the shift operators (called here general shift groups), shift groups with bounded shift operators, and uniformly bounded shift groups.

An extensive body of knowledge has been developed for weakly harmonizable processes on $\mathbf{Z}$ or $\mathbf{R}$ with a uniformly bounded shift group [2,8,12]. In [1], Abreu and Fetter studied the strongly harmonizable processes on $\mathbf{Z}$ with a general shift group. Here we present some improvements and some generalizations of the results of these papers, to the weakly harmonizable processes defined on an l.c.a. group.

A brief synopsis of the paper is as follows. In Section 2, we characterize the spectral Hilbert measure of a weakly harmonizable process on an l.c.a. group $\mathbf{G}$ which admits a general shift group (Lemma 2.1 and Corollary 2.2) or a uniformly bounded shift group (Theorem 2.6) (for $\mathbf{G} = \mathbf{Z}$ see [1,2]). Unfortunately, we do not succeed in the characterization for a shift group with bounded shift operators (Lemma 2.4). Afterwards in Section 3 (Theorems 3.1 and 3.2) we deduce an extension to our general setting, of the factorization properties of the shift groups stated in [1] and in [12] .

Notations (1.1). From now on, consider $\mathbf{G}$ an l.c.a. group, $\hat{\mathbf{G}}$ its dual, and $\mathcal{B}(\hat{\mathbf{G}})$ the Borel σ-algebra of $\hat{\mathbf{G}}$. Let $\mathbf{P}(\hat{\mathbf{G}})$ be the set of trigonometric functions on $\hat{\mathbf{G}}$, that is the set of functions $P : \hat{\mathbf{G}} \to \mathbf{C}$ defined by $P(\gamma) = \sum a_k < t_k, \gamma >$ for some $a_1, \dots, a_n$ in $\mathbf{C}$ and some $t_1, \dots, t_n$ in $\mathbf{G}$, where $< t, \gamma >$ designates the action of γ in $\hat{\mathbf{G}}$ at t in $\mathbf{G}$.

Consider a Hilbert space $\mathbf{H}$, for instance $\mathbf{H} = L^2_{\mathbf{C}}(\Omega, \mathcal{A}, \mathrm{P})$ for some probability space $(\Omega, \mathcal{A}, \mathrm{P})$. For a nonnull process $X : \mathbf{G} \to \mathbf{H}$, let $l.s.X$ be the linear space spanned by the set $\{X(t) \,:\, t \in \mathbf{G}\}$, $\mathbf{H}(X)$ be its closure in the Hilbert space $\mathbf{H}$.

Shift group. A nonnull process $X : \mathbf{G} \to \mathbf{H}$ is said to admit a shift group whenever there exists a group $S = \{S_t \,:\, t \in \mathbf{G}\}$ of (linear) operators $S_t : l.s.X \to l.s.X$ such that $S_t X(s) = X(s+t)$ for all s, t in $\mathbf{G}$. The shift operators are densely defined in $\mathbf{H}(X)$, and in this paper no hypothesis is assumed about

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their adjoint operators in contrast with the assumptions in [6] and in [12]. If a shift operator S_t is bounded, we still denote by S_t its unique extension as a bounded operator on $\mathbf{H}(X)$. Following Tjöstheim and Thomas [12], whenever a process $X : \mathbf{G} \to \mathbf{H}$ has a uniformly bounded shift group, we say that the process X is uniformly bounded linear stationary (UBLS).

Weakly harmonizable process. Finally, we recall that for any weakly harmonizable process $X : \mathbf{G} \to \mathbf{H}$, there exists a unique regular σ-additive $\mathbf{H}$-valued set function $\nu_X : \mathcal{B}(\hat{\mathbf{G}}) \to \mathbf{H}$, called the spectral Hilbert measure of X, such that $X(t) = \int_{\hat{\mathbf{G}}} < t, . > d\nu_X$, for $t \in \mathbf{G}$ [5,9,10].

In the following example, we point out various possibilities about the existence and the boundedness of the shift groups.

Example (1.2). Harmonizable white noise. In this paper a white noise on $\mathbf{Z}$ will be a process $\epsilon : \mathbf{Z} \to \mathbf{H}$, such that $< \epsilon(n), \epsilon(p) >_H = 0$ for $n \neq p$, and $\sup\{\|\epsilon(n)\|_H : n \in \mathbf{Z}\} = K(\epsilon) < \infty$ for some constant $K(\epsilon)$. The classical notion of stationary white noise is recovered whenever $\|\epsilon(n)\|_H = K(\epsilon) > 0$, for any $n \in \mathbf{Z}$. It is known that such a white noise ϵ is weakly harmonizable, and we easily obtain the following equivalences.

1. $\forall n \in \mathbf{Z}, \epsilon(n) \neq 0 \Leftrightarrow \epsilon$ admits a general shift group.
2. $\forall n \in \mathbf{Z}, \epsilon(n) \neq 0$ and $\sup\{\|\epsilon(n+1)\|_H / \|\epsilon(n)\|_H : n \in \mathbf{Z}\} < \infty \Leftrightarrow \forall n \geq 0, S_n$ is bounded.
3. $\forall n \in \mathbf{Z}, \epsilon(n) \neq 0$ and $\inf\{\|\epsilon(n+1)\|_H / \|\epsilon(n)\|_H : n \in \mathbf{Z}\} > 0 \Leftrightarrow \forall n \leq 0, S_n$ is bounded.
4. $\inf\{\|\epsilon(n)\|_H : n \in \mathbf{Z}\} > 0 \Leftrightarrow \epsilon$ admits a uniformly bounded shift group.

Furthermore, let $\eta : \mathbf{Z} \to \mathbf{H}$ be a stationary white noise. The white noise ϵ_1 defined on $\mathbf{Z}$ by $\epsilon_1(n) = 2^{-|n|}\eta(n)$, admits a shift group which is not uniformly bounded, but each shift mapping $S_{1,n}$, $n \in \mathbf{Z}$, is bounded. On the other hand, the white noise ϵ_2 defined on $\mathbf{Z}$ by $\epsilon_2(2n) = 2^{-|n|}\eta(2n)$ and $\epsilon_2(2n+1) = (2n+1)^{-1}\eta(2n+1)$ admits a shift group with the shift mappings $S_{2,2n+1}$, $n \in \mathbf{Z}$, which are unbounded and with the shift mappings $S_{2,2n}$, $n \in \mathbf{Z}$ which are bounded but not uniformly bounded.

2. Spectral Hilbert measures and shift groups

From the uniqueness of the spectral Hilbert measure of a weakly harmonizable process, we state a characterization of the spectral Hilbert measure of a weakly harmonizable with a general shift group.

LEMMA (2.1). *A nonnull weakly harmonizable process X has a general shift group if and only if for any P in $\mathbf{P}(\hat{\mathbf{G}})$ we have*

$$\int_{\hat{\mathbf{G}}} P d\nu_X = 0 \Rightarrow P(\gamma) = 0 \ \nu_X - a.e.$$

Proof. Assume S exists and let $P \in \mathbf{P}(\hat{\mathbf{G}})$. We easily obtain that for any t, $S_t \int_{\hat{\mathbf{G}}} P\, d\nu_X = \int_{\hat{\mathbf{G}}} < t, . > P\, d\nu_X$. Thus we have $\int_{\hat{\mathbf{G}}} P\, d\nu_X = 0 \Rightarrow \forall t \in \mathbf{G}$,

$\int_{\hat{G}} < t,. > P\, d\nu_X = 0 \Rightarrow \forall A \in \mathcal{B}(\hat{G}), \int_A P\, d\nu_X = 0 \Rightarrow P(\gamma) = 0\ \nu_X - a.e,$ since the $\mathbf{H}$-valued measure $A \to \int_A P\, d\nu_X$ is regular. The converse is evident. Q.E.D.

Hence, for $\mathbf{G} = \mathbf{Z}$ or $\mathbf{R}$, we immediately deduce the following equivalence.

COROLLARY (2.2). *Whenever $\mathbf{G} = \mathbf{Z}$ or $\mathbf{R}$, a nonnull weakly harmonizable process X admits a general shift group if and only if its spectral Hilbert measure satisfies one of the following conditions.*

i) $\int_{\hat{G}} P\, d\nu_X \neq 0$ *, for any nonnull* $P \in \mathbf{P}(\hat{G})$.

ii) ν_X *is a discrete* $\mathbf{H}$*-valued measure, its support is locally finite and if* $supp(\nu_X) = \{\gamma_1, \ldots, \gamma_j, \ldots\}$ *then for any* $P \in \mathbf{P}(\hat{G})$*, we have*

$$\int_{\hat{G}} P\, d\nu_X = \sum P(\gamma_j)\nu_X(\gamma_j) = 0 \Rightarrow P(\gamma_j) = 0, \quad \textit{for any } j.$$

Condition (i) is equivalent to: every finite subfamily of $\{X(t) : t \in \mathbf{G}\}$ is linearly independent. An interpretation of condition (ii) is given for $\mathbf{G} = \mathbf{Z}$ in the next corollary.

COROLLARY (2.3). *Whenever $\mathbf{G} = \mathbf{Z}$, conditions (i) and (ii) of Corollary 2.2 are disjoint. Moreover condition (ii) is equivalent to the following condition.*

ii') $card(supp(\nu_X)) = p < \infty$*, and if* $supp(\nu_X) = \{\gamma_1, \ldots, \gamma_p\}$*, then the vectors* $\nu_X(\gamma_1), \ldots, \nu_X(\gamma_p)$ *are linearly independent.*

In fact in this case, the process X is strongly harmonizable and admits a uniformly bounded shift group. The families $\{\nu_X(\gamma_1), \ldots, \nu_X(\gamma_p)\}$ and $\{X(n), \ldots, X(n+p-1)\}$, for any $n \in \mathbf{Z}$, are bases of $\mathbf{H}(X) = l.s.X$.

Proof. Since $\hat{G} = \mathbf{R}/2\pi$ is compact, conditions (i) and (ii) are disjoint. The implication (ii')$\Rightarrow$(ii) is evident. Conversely, assume that condition (ii) is fulfilled. Then $supp(\nu_X)$ is finite, say with p elements $\{\gamma_1, \ldots, \gamma_p\}$. Consider $Q_X \in \mathbf{P}(\mathbf{R}/2\pi)$ defined by $Q_X(\gamma) = \Pi_{k=1,\ldots,p}(\exp(i\gamma) - exp(i\gamma_k)) = \sum_{k=0,\ldots,p} a_k \exp(ik\gamma)$. Then for any $n \in \mathbf{Z}$, $\sum a_k X(k+n) = 0$, and since $a_p = 1$, we deduce that $X(p+n) = \sum_{k=0,\ldots,p-1} -a_k X(k+n)$. Moreover, whenever $R(\gamma) = \sum b_k \exp(ik\gamma)$ is another element of $\mathbf{P}(\mathbf{R}/2\pi)$ such that $\sum b_k X(k) = 0$, then from condition (ii), the set $\{\gamma \in \mathbf{R}/2\pi : R(\gamma) = 0\}$ contains $supp(\nu_X)$ and R is divisible by Q_X, that is, there exists $P \in \mathbf{P}(\mathbf{R}/2\pi)$ such that $R = PQ_X$. Hence, with dimensional arguments we easily deduce that for any $n \in \mathbf{Z}$, the family $\{X(n), \ldots, X(n+p-1)\}$ forms a basis of $l.s.X$. Then we readily complete the proof. Q.E.D.

For bounded shift operators S_t, $t \in \mathbf{G}$, we only get a necessary condition.

LEMMA (2.4). *Let X be a nonnull weakly harmonizable process which admits a shift group with bounded shift operators. Then for any ν_X-integrable function $f : \hat{G} \to \mathbf{C}$ we have $S_t \int_{\hat{G}} f\, d\nu_X = \int_{\hat{G}} < t,. > f\, d\nu_X$ for any t in $\mathbf{G}$, and $\int_{\hat{G}} f\, d\nu_X = 0 \Rightarrow f(\gamma) = 0,\ \nu_X - a.e.$*

Proof. It is a direct consequence of the uniqueness property of the spectral Hilbert measure of a weakly harmonizable process. Q.E.D.

As for the uniformly bounded shift groups, from Theorem 4.2 in [2] and Theorem 3.2 in Section 3, we can state a topological characterization. We refer to [2] and [3] for the definition of Schauder basic measures.

THEOREM (2.5). *A nonnull weakly harmonizable process X on an l.c.a. group is UBLS (that is, admits a uniformly bounded shift group) if and only if its spectral Hilbert measure ν_X is a Schauder basic measure.*

3. Dilations and shift groups

In [7], Niemi stated that for any weakly harmonizable process $X : \mathbf{G} \to \mathbf{H}$, there exist a Hilbert space $\mathbf{H}'$, a continuous stationary process $Y : \mathbf{G} \to \mathbf{H}'$ with orthogonally scattered spectral Hilbert measure ν_Y, and a contraction $\Pi : \mathbf{H}'(Y) \to \mathbf{H}(X)$ such that $\Pi Y(t) = X(t)$ for any $t \in \mathbf{G}$, and consequently $\Pi\nu_Y = \nu_X$. Such a pair (Y, Π) is said to be a stationary dilation of the process X. Theorem 3.1 ensures a geometric characterization of the weakly harmonizable processes with shift groups, which generalizes Abreu and Fetter's results [1, Theorems 2.1 and 2.2].

THEOREM (3.1). *A weakly harmonizable process $X : \mathbf{G} \to \mathbf{H}$ has a general shift group if and only if there exists a stationary dilation (Y, Π) of X such that Π is one to one from $l.s.Y$ onto $l.s.X$, that is, Π admits an inverse Π^{-1} from $l.s.X$ onto $l.s.Y$.*

In this case, there exists a unitary operator group $\{U_t : t \in \mathbf{G}\}$ on a Hilbert space $\mathbf{H}'$ such that $S_t = \Pi U_t \Pi^{-1}$ on $l.s.X$ for any $t \in \mathbf{G}$. Of course if $\Pi^{-1} : l.s.X \to l.s.Y$ is bounded, the shift group S is uniformly bounded.

Proof. Let (Y, Π) be a stationary dilation of X. Consider the continuous stationary process Z with spectral Hilbert measure $\nu_Z : \mathcal{B}(\hat{\mathbf{G}}) \to \mathbf{H}'$ defined by $\nu_Z(A) = \nu_Y(A \bigcap supp(\nu_X))$. We note that $supp(\nu_Z) \subset supp(\nu_X)$. Although, in general, the processes Y and Z do not coincide, the boundedness of the mapping Π and the fact that $\Pi\nu_Y(A) = \nu_X(A) = \nu_X(A \bigcap supp(\nu_X)) = \Pi\nu_Z(A)$ for any A, ensure that $\Pi Y(t) = X(t) = \Pi Z(t)$ for any $t \in \mathbf{G}$. Hence the restriction Π' of Π to $\mathbf{H}'(Z)$ is a contraction $\Pi' : \mathbf{H}'(Z) \to \mathbf{H}(X)$ such that $\Pi' Z(t) = X(t)$ for any $t \in \mathbf{G}$.

Whenever the shift group of X exists, for $t_1, \ldots, t_n$ in $\mathbf{G}$ and $a_1, \ldots, a_n$ in $\mathbf{C}$ such that $\sum a_k X(t_k) = 0$, Lemma 2.1 ensures that $\sum a_k Z(t_k) = 0$. Thus the contraction Π' is one to one from $l.s.Y$ onto $l.s.X$. Furthermore for the unitary shift group $U = \{U_t : t \in \mathbf{G}\}$ associated with the stationary process Z, we have $S_t = \Pi' U_t \Pi'^{-1}$ on $l.s.X$ for any t. The converse implication is immediate. Q.E.D.

From the Sz. Nagy factorization theorem for the uniformly bounded families of operators $\{S_t : t \in \mathbf{G}\}$ [4, Lemma XV.6.1], we readily generalize

Tjöstheim and Thomas' characterization of the UBLS processes [12, Theorems 1 and 2].

THEOREM (3.2). *A process $X : \mathbb{G} \to \mathbb{H}$ is UBLS if and only if there exist a stationary process $Y : \mathbb{G} \to \mathbb{H}'$ and a bounded isomorphism $\Pi : \mathbb{H}'(Y) \to \mathbb{H}(X)$ with a bounded inverse Π^{-1}, and such that $\Pi Y(t) = X(t)$ for any t, that is, the process X admits a stationary similitude in the sense of Niemi [8]. If $\{U_t : t \in \mathbb{G}\}$ designates the unitary shift group of the stationary process Y, then $S_t = \Pi U_t \Pi^{-1}$ for any t.*

Hence, any continuous UBLS process on an l.c.a. group is weakly harmonizable.

Example (3.3). Let ϵ be a white noise which admits a shift group. Consider the stationary white noise η defined on $\mathbb{Z}$ by $\eta(n) = (\sup\{\|\epsilon(p)\|_H : p \in \mathbb{Z}\}/\|\epsilon(n)\|_H)\ \epsilon(n)$. The mapping Π defined on $l.s.\eta = l.s.\epsilon$ by $\Pi\eta(n) = \epsilon(n)$ for any $n \in \mathbb{Z}$, is continuous and one to one from $l.s.\eta$ onto $l.s.\epsilon$. Furthermore, the mapping Π^{-1} is continuous on $l.s.\epsilon$ if and only if $\inf\{\|\epsilon(n)\|_H : n \in \mathbb{Z}\} > 0$, that is, if and only if the white noise ϵ is UBLS.

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