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ON SEQUENTIAL BARRELLEDNESS WITHOUT LOCAL CONVEXITY CONDITIONS

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1. Introduction

Throughout this paper the word *space* will satnd for any Hausdorff topological vector space. Let us recall some classes of locally convex spaces with sequential barrelledness conditions: a space is $\aleph_0$-barrelled, [3], if every $\aleph_0$-barrel, i.e. every barrel which is the intersection of a sequence of closed absolutely convex neighborhoods of the origin, is a neighborhood of the origin; a space $E(\tau)$ has property (L), [4], if for every increasing sequence of absolutely convex sets $\{A_n : n \in \mathbb{N}\}$ in E such that each $x \in E$ is absorbed by some A_n, τ is the finest locally convex topology on E agreeing with the topology induced on each A_n;and a space E is dual locally complete, [5], if $E'(\sigma(E', E))$ is locally complete, that is if every $\sigma(E', E)$-bounded closed absolutely convex subset of E' is a Banach disk, holding

$$\text{barrelled} \Rightarrow \aleph_0\text{-barrelled} \Rightarrow \text{property } (L) \Rightarrow \text{dual locally complete.}$$

Clearly, if a space E has property (L) the E is the strict inductive limit of any increasing sequence of subspaces covering E and this is also true when E is a Mackey dual locally complete space [6]. This property, shared by these classes of locally convex spaces, suggested in [2] the definition of weakly barrelled spaces as those characterized by enjoying it.

The aim of this paper is to study the class of topological vector spaces E, without local convexity conditions, characterized by being the strict $\mathcal{L}$-inductive limit of any increasing sequence of subspaces that cover E.

Our notation follows [1]. A sequence $(U_n)_{n=1}^{\infty}$ of absorbing balanced subsets in a space $E(\tau)$ is called a string if $U_{n+1} + U_{n+1} \subset U_n \ \ \forall n \in \mathbb{N}$, and if each U_n is closed (a neighborhood of the origin) in $E(\tau)$ then the string is said to be closed (topological). A space is $\mathcal{L}$-barrelled if each closed string is topological, and a space is countably $\mathcal{L}$-barrelled if each string that is the intersection of a sequence of closed topological strings is topological. Clearly each $\mathcal{L}$-barrelled space is countably $\mathcal{L}$-barrelled space. Given a sequence of balanced subsets $\sigma := (B_n)_{n=1}^{\infty}$ in a space $E(\tau)$ such that $B_n + B_n \subset B_{n+1} \ \forall n \in \mathbb{N}$ and $E = \bigcup\{B_n : n \in \mathbb{N}\}$, τ_σ will denote the finest linear topology on E inducing on each B_n the same topology as τ.

Given a family of spaces $\{E_i : i \in \mathbb{I}\}$ and a subset J of I, $E(J)$ will denote the subspace of $E := \prod\{E_i : i \in \mathbb{I}\}$, formed by all those elements whose coordinates positions indexed by $\mathbb{I}\backslash J$ are null.

2. Weakly $\mathcal{L}$-barrelled spaces

Definition. We shall say a space E is weakly $\mathcal{L}$-barrelled if E is the strict $\mathcal{L}$-inductive limit of any increasing sequence of subspaces that covers E.

Each countably $\mathcal{L}$-barrelled space is weakly $\mathcal{L}$-barrelled, [1, §16.(8)]. The $\mathcal{L}$-inductive limit of countably many locally convex spaces is locally convex, [1, §4.(6)], furthermore a locally convex space E is weakly $\mathcal{L}$-barrelled if and only if E is weakly barrelled. So, every weakly barrelled space is weakly $\mathcal{L}$-barrelled. However, any non locally convex $\mathcal{L}$-barrelled space provides an example of a weakly $\mathcal{L}$-barrelled space which is not weakly barrelled.

It is easy to check that if $\omega^{(\mathbb{N})}$ denotes the topological direct sum of at most countably many copies of ω, E is a weakly $\mathcal{L}$-barrelled space and f a linear mapping of E into Φ or $\omega^{(\mathbb{N})}$ with closed graph, then f is continuous.

PROPOSITION (1). *A space $E(\tau)$ is weakly $\mathcal{L}$-barrelled if and only if E is the $\mathcal{L}$-inductive limit of any increasing sequence of closed subspaces that covers E.*

Proof. We just have to show this proposition in one way. Let $\sigma := \{E_n : n \in \mathbb{N}\}$ be an increasing sequence of subspaces that covers E and let A be a linear mapping of $E(\tau)$ into an (F)-space F, such that each of its restrictions A_n to E_n is continuous. If we show that A is continuous then, by [1,§16.(2)], $\tau = \tau_\sigma$ and the proof will be concluded. Let v_n be the unique continuous extensions of A_n to $\overline{E_n}$. Clearly, $v_{n+1}|_{\overline{E_n}} = v_n$. In fact, v_{n+1} and v_n are continuous on $\overline{E_n}$ and coincide on E_n since $v_{n+1}|_{E_n} = v_n|_{E_n} = A_n$. Let $v : E \to F$ be the linear mapping such that $v|_{\overline{E_n}} = v_n$. Now the restriction of v to each $\overline{E_n}$ is continuous. If $\overline{\sigma}$ denotes the sequence $\{\overline{E_n} : n \in \mathbb{N}\}$, then $\tau = \tau_{\overline{\sigma}}$ and v is continuous on E. Since if $x \in A$, there exists $p \in \mathbb{N}$ such that $x \in E_p$, so $v(x) = v_p(x) = A_p(x) = A(x)$ and $v = A$. Therefore A is continuous on E. □

PROPOSITION (2). *Let F be a dense subspace of $E(\tau)$. If F is weakly $\mathcal{L}$-barrelled then E is weakly $\mathcal{L}$-barrelled.*

Proof. Let $\sigma := \{E_n : n \in \mathbb{N}\}$ be an increasing sequence of closed subspaces that covers E. Then $\sigma^* := \{G_n := E_n \cap F, n \in \mathbb{N}\}$ is an increasing sequence of closed subspaces covering F. Let $\hat{\tau}$ be the topology induced by τ on F, we have $\hat{\tau} = \hat{\tau}_{\sigma^*}$. By [1, §16.(11)], $E = \bigcup\{\overline{G_n}^\tau : n \in \mathbb{N}\}$ and if $\overline{\sigma^*} := \{\overline{G_n}^\tau : n \in \mathbb{N}\}$, then $\tau = \tau_{\overline{\sigma^*}}$. Furthermore, since $\overline{G_n}^\tau = \overline{E_n \cap F}^\tau \subset E_n$, $\tau_\sigma \subset \tau_{\overline{\sigma^*}}$, i.e. $\tau \subset \tau_\sigma \subset \tau_{\overline{\sigma^*}} = \tau$. Therefore $\tau = \tau\sigma$ and $E(\tau)$ is weakly $\mathcal{L}$-barrelled. □

The following result generalizes the one obtained in [6] for Mackey dual locally complete spaces, being the version without convexity conditions of [2, Proposition 3].

PROPOSITION (3). *Let F be a subspace of E of countable codimension. If $E(\tau)$ is weakly $\mathcal{L}$-barrelled then F is weakly $\mathcal{L}$-barrelled.*

Proof. Let $\sigma := \{F_n : n \in \mathbb{N}\}$ be an increasing sequence of subspaces of F that covers F and let $B := \{x_i : i \in \mathbb{I}\}$ be a cobasis of $\bigcup\{\overline{F_n}^E : n \in \mathbb{N}\}$, where the cardinal of I satisfies $^*\mathbb{I} \leq \aleph_0$. Let $\{L_n : n \in \mathbb{N}\}$ be an increasing sequence of subspaces of E, where each L_n is the linear span of n elements of B (if $\#\mathbb{I} = m < \aleph_0$, take $L_n = [B]$ $\forall n \geq m$), and define $L := \bigcup\{L_n : n \in \mathbb{N}\}$.

Let $G_n := \overline{F_n}^E \oplus_t L_n$ $\forall n \in \mathbb{N}$, then $E = \bigcup\{G_n : n \in \mathbb{N}\}$ and, if $\sigma^* := (G_n)_{n=1}^{\infty}$, $\tau = \tau_{\sigma^*}$ since E is weakly $\mathcal{L}$-barrelled. On the other hand $F(\hat{\tau}_{\sigma^*})$ has a base of neighborhoods of the origin formed by $\hat{\tau}$-closed subsets, [1, §16.(4)]. Thus let $\mathcal{U} = (U_m)_{m=1}^{\infty}$ be a $\hat{\tau}$-closed string in F such that $(U_m \cap F_n)_{m=1}^{\infty}$ is a topological string in F_n $\forall n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, $(\overline{U_m \cap F_n}^E)_{m=1}^{\infty}$ is a topological string in $\overline{F_n}^E$ and, since $\overline{U_m \cap F_n}^E \subset \overline{U_m}^E \cap \overline{F_n}^E$, $(\overline{U_m}^E \cap \overline{F_n}^E)_{m=1}^{\infty}$ is a topological string in $\overline{F_n}^E$ and $(\overline{U_m}^E \cap \overline{F_n}^E + L_n)_{m=1}^{\infty}$ is a topological string in G_n. Therefore, $(\overline{U_m}^E + L)_{m=1}^{\infty}$ is a topological string in $E(\tau_{\sigma^*}) = E(\tau)$. Then $(\overline{U_m}^E \cap F = U_m)_{m=1}^{\infty}$ is a topological string in $F(\hat{\tau})$, that is $\hat{\tau} = \hat{\tau}_\sigma$ and F is weakly $\mathcal{L}$-barrelled. □

PROPOSITION (4). *Let $E(\tau)$ be the $\mathcal{L}$-inductive limit of the spaces $\{E_i(\tau_i) : i \in \mathbb{I}\}$, with respect to $\{A_i : i \in \mathbb{I}\}$. If each $E_i(\tau_i)$, $i \in \mathbb{I}$ is weakly $\mathcal{L}$-barrelled, then E is weakly $\mathcal{L}$-barrelled.*

Proof. Let $\sigma := \{F_n : n \in \mathbb{N}\}$ be an increasing sequence of subspaces of E that covers E and let $G_n^i := A_i^{-1}(F_n)$ for each $n \in \mathbb{N}, i \in \mathbb{I}$. Now $\sigma^i := \{G_n^i : n \in \mathbb{N}\}$ is an increasing sequence of subspaces of $E_i(\tau_i)$ that covers $E_i(\tau_i)$ for each $i \in \mathbb{I}$ and, therefore, $\tau_i = (\tau_i)_{\sigma^i}$ $\forall i \in \mathbb{I}$ since each $E_i(\tau_i)$ is weakly $\mathcal{L}$-barrelled.

Let now $\mathcal{U} = (U_m)_{m=1}^{\infty}$ be a topological string in $E(\tau_\sigma)$. If $A_{i,n}$ denotes the restriction of A_i to G_n^i, τ_n the topology induced by τ on F_n and $\tau_{i,n}$ the topology induced by τ_i on G_n^i $\forall n \in \mathbb{N}, \forall i \in \mathbb{I}$ then $(U_m \cap F_n)_{m=1}^{\infty}$ is a topological string $F_n(\tau_n)$ for each $n \in \mathbb{N}$ and, since $A_i^{-1}(U_m) \cap G_n^i = A_i^{-1}(U_m) \cap A_i^{-1}(F_n) \supset A_{i,n}^{-1}(U_m \cap F_n)$ $\forall n, m \in \mathbb{N}, \forall i \in \mathbb{I}$, then $(A_i^{-1}(U_m) \cap G_n^i)_{m=1}^{\infty}$ happens to be a topological string in $G_n^i(\tau_{i,n})$ $\forall n \in \mathbb{N}, \forall i \in \mathbb{I}$. Finally, since $E_i(\tau_i)$ is weakly $\mathcal{L}$-barrelled, $(A_i^{-1}(U_m))_{m=1}^{\infty}$ is a topological string in $E_i(\tau_i)$ $\forall i \in \mathbb{I}$. Therefore $\mathcal{U}$ is a topological string in $E(\tau)$, that is $\tau = \tau_\sigma$ and E is weakly $\mathcal{L}$-barrelled. □

COROLLARY (1). *Let $\{E_i(\tau_i) : i \in \mathbb{I}\}$ be a non empty family of weakly $\mathcal{L}$-barrelled spaces. Then $E := \bigoplus\{E_i(\tau_i) : i \in \mathbb{I}\}$ is weakly $\mathcal{L}$-barrelled.*

COROLLARY (2). *The topological product of finitely many weakly $\mathcal{L}$-barrelled spaces is weakly $\mathcal{L}$-barrelled.*

COROLLARY (3). *Let F be a closed subspace of E. If E is weakly $\mathcal{L}$-barrelled then E/F is weakly $\mathcal{L}$-barrelled.*

In order to show that any topological product of weakly $\mathcal{L}$-barrelled spaces is weakly $\mathcal{L}$-barrelled, we shall prove the following Lemma suggested by [7 Ch. 1§2.1(15)]. Let us recall a subset of the vector space E is semibalanced if $\lambda x \in M$ for each $x \in M$, $\lambda \in [0, 1]$.

LEMMA (1). *Let $\{E_i(\tau_i) : i \in \mathbb{I}\}$ be a non empty family of spaces, $E(\tau) = \prod\{E_i(\tau_i) : i \in \mathbb{I}\}$ and let $\mathcal{A} = \{B_n : n \in \mathbb{N}\}$ be a family of closed semibalanced subsets of E that covers E, then there exists a finite subset $J \subset I$ and a positive integer k such that $B_k \supset E(I \backslash J)$.*

Proof. Given $M \subset I$ and $B \in \mathcal{A}$, if $\bigoplus\{E(\{i\}) : i \in M\} \subset B$ then $E(M) \subset B$ since $\bigoplus\{E(\{i\}) : i \in M\}$ is dense in $E(M)$ and B is closed. Assume the Lemma is false, then $\bigoplus\{E(\{i\}) : i \in I\} \not\subset B_1$, and there is a finite subset $J_{1,1} \subset \mathbb{I}$ such that $\bigoplus\{E(\{i\}) : i \in J_{1,1}\} \not\subset B_1$ and $\bigoplus\{E(\{i\}) : i \in \mathbb{I}\backslash J_{1,1}\} \not\subset B_1$. So there is a finite subset $J_{1,2} \subset \mathbb{I}\backslash J_{1,1}$ such that $\bigoplus\{E(\{i\}) : i \in J_{1,2}\} \not\subset B_1$ and $\bigoplus\{E(\{i\}) : i \in \mathbb{I}\backslash(J_{1,1} \cup J_{1,2}\} \not\subset B_1$. In this way we can find a sequence $\{J_{1,p} : p \in \mathbb{N}\}$ of pairwise disjoint finite subsets of $\mathbb{I}$ such that $\bigoplus\{E(\{i\}) : i \in J_{1,p}\} \not\subset B_1$. By recurrence, given the sequences $\{J_{r,p} : p \in \mathbb{N}\}$ of pairwise disjoint finite subsets of $\mathbb{I}$ such that $\bigoplus\{E(\{i\}) : i \in J_{r,p}\} \not\subset B_r$, for $r \in \{1, 2, \ldots, n\}$, $p \in \mathbb{N}$, we can find a sequence $\{J_{n+1,p} : p \in \mathbb{N}\}$ of pairwise disjoint finite subsets of $\mathbb{I}$ such that $J_{n+1,p} \cap J_{r,q} = \emptyset$ for $r, q \in \{1, 2, \ldots, n\}$ and $\bigoplus\{E(\{i\}) : i \in J_{n+1,p}\} \not\subset B_{n+1}$, $\forall p \in \mathbb{N}$. For each $n, p \in \mathbb{N}$, take $x_{n,p} \in \bigoplus\{E(\{i\}) : i \in J_{n,p}\}\backslash B_n$, set $L_{n,p} := [x_{n,p}]$ and L the closure of $[\{x_{n,p} : n, p \in \mathbb{N}\}]$, endowed with the topology induced by E. L is an (F)-space since the $E(\{i\})$ that contain some non null coordinate of some $x_{n,p}$ are countably many, and each one of them contains at most finitely many $x_{n,p}$. Then for each $i \in \mathbb{I}$, the linear span of the $x_{n,p}$ contained in $E(\{i\})$ is an (F)-space. So L is topologically isomorphic to $\prod\{L_{n,p} : n, p \in \mathbb{N}\}$, which is an (F)-space. Therefore, since $L = \bigcup\{B_m \cap L : m \in \mathbb{N}\}$, there is some $q \in \mathbb{N}$ such that $B_q \cap L$ has an interior point x in L. Hence there is some finite subset $p \subset \mathbb{I}$ and a neighborhood of the origin U_i in E_i for each $i \in P$; thus if $U_i := E_j$ $\forall i \in \mathbb{I}\backslash P$ then $(x + \prod\{U_i : i \in \mathbb{I}\}) \cap L \subset B_q$. Now, if from $\{J_{q,p} : p \in \mathbb{N}\}$ we take some $J_{q,m}$ disjoint with P then $x + nx_{q,m} \in B_q$ $\forall n \in \mathbb{N}$ and, since B_q is semibalanced $(x/n) + x_{q,m} \in B_q$ $\forall n \in \mathbb{N}$. Hence $x_{q,m} \in B_q$ since the set is closed which contradicts the selection of the $x_{n,p}$. □

THEOREM (1). *Let $\{E_i(\tau_i) : i \in \mathbb{I}\}$ be a non empty family of weakly $\mathcal{L}$-barrelled spaces. Then $E(\tau) := \prod\{E_i : i \in \mathbb{I}\}$ is weakly $\mathcal{L}$-barrelled.*

Proof. Let $\sigma := \{F_n : n \in \mathbb{N}\}$ be an increasing sequence of closed subspaces of E that covers E. We have to show that $\tau_\sigma = \tau$. Assume $\mathcal{U} = (U_m)_{m=1}^\infty$ is a topologial string in $E(\tau_\sigma)$, then $(U_m \cap F_n)_{m=1}^\infty$ is a topological string in F_n for each $n \in \mathbb{N}$. By the previous Lemma, there exists some $k \in \mathbb{N}$ and some finite subset $J \subset \mathbb{I}$ such that $F_k \supset E(\mathbb{I}\backslash J)$ and, consequently, $(U_m \cap E(\mathbb{I}\backslash J))_{m=1}^\infty$ is a topological string in $E(\mathbb{I}\backslash J)$.

On the other hand, $E(J) \cong \prod\{E_i(\tau_i) : i \in J\}$ is weakly $\mathcal{L}$-barrelled since it is the product of finitely many weakly $\mathcal{L}$-barrelled spaces. Therefore, since $E(J) = \bigcup\{E(J) \cap F_n : n \in \mathbb{N}\}$ and for each $n \in \mathbb{N}(U_m \cap E(J) \cap F_n)_{m=1}^\infty$ is a topological string in $E(J) \cap F_n$, $(U_m \cap E(J))_{m=1}^\infty$ is a topological string in $E(J)$. Finally since $U_m \supset U_{m+1} + U_{m+1} \supset (U_{m+1} \cap E(J)) + (U_{m+1} \cap E(\mathbb{I}\backslash J))$, we obtain that $\mathcal{U} = (U_m)_{m=1}^\infty$ is a topological string in $E(\tau)$ and $\tau = \tau_\sigma$. □

LEMMA (2). *Let E be a weakly $\mathcal{L}$-barrelled space and F a closed subspace of countable codimension. If G is an algebraic complement of F in E, then G is a topological complement of F.*

Proof. Follow the same argument as in [2, Lemma]. □

Finally, we show the following property also holds for weakly $\mathcal{L}$-barrelled spaces.

THEOREM (2). *Let $E(\tau)$ be a weakly $\mathcal{L}$-barrelled space. If F is a closed subspace of codimension $\aleph_0$ then $E \cong F \times \Phi$.*

Proof. Let $\{x_n : n \in \mathbb{N}\}$ be a cobasis of F in E, $L_n := [\{x_1, x_2, \ldots, x_n\}]$ for each $n \in \mathbb{N}$, and $L := \bigcup\{L_n : n \in \mathbb{N}\}$. We shall prove that E is the topological direct sum of F and L, the latter being endowed with the finest locally convex topology.

Let V be an absorbing absolutely convex subset of L. Let τ^* be the topology on E which has as a base of neigborhoods of the origin in E all sets $U \cap W$, where U are knots of the topological strings in $E(\tau)$ and W are sets $F + (1/2)^m V$, $m \in \mathbb{N}$.

Clearly, $\tau \subset \tau^*$, but both topologies coincide on F and, F being of finite codimension in $G_n := F \oplus_t L_n$ $\forall n \in \mathbb{N}$, $\tau|_{G_n} = \tau^*|_{G_n}$. Thus if we define $\sigma := (G_n)_{n=1}^{\infty}$, since $E(\tau)$ is weakly $\mathcal{L}$-barrelled, then $\tau = \tau_\sigma = (\tau^*)_\sigma \supset \tau^*$ and, consequently, $\tau = \tau^*$. Therefore, there is some neigborhood of the origin in $E(\tau)$, say O, such that $O \subset F + V$, that is $O \cap L \subset V$ and V is a neighborhood of the origin in L. Hence L is endowed with the finest locally convex topology. Finally, since L is a countable dimensional algebraic complement of F, $E = F \oplus_t L$. □

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