Boletín de la Sociedad Matemática Mexicana Vol. 36, 1991

AN APPLICATION OF SARD'S THEOREM

BY M. R. R. HOOLE

Introduction

Sard's theorem states that : if U $\subseteq R^m$ and f: U $\rightarrow R^n$ is a smooth function (C^∞), then the image of the singular points of U (= the singular values of R^n) has measure zero [1].

This theorem together with the inverse function theorem provides an interesting link between Analysis and Algebraic Topology. For instance Brouwer's fixed point theorem can by proved using techniques from analysis [2]. We consider the following result from degree theory which is proved using relative homology theory (see [3]):

Let O be an open subset of R^m with compact closure $\overline{O}$ and B the boundary of $\overline{O}$. $F : (\overline{O}, B) \times [0,1] \rightarrow (\overline{O}; B)$, $f : (\overline{O}; B) \rightarrow (\overline{O}, B)$ if F, f are continuous and F(x,t), f(x) $\in B$ for x $\in B$ and t $\in [0,1]$. F is a homotopy of f if F_o = f ($F_t(x) = F(x,t)$). f is said to be null homotopic if there is a homotopy F of f with $F_1[O]$ consisting of just one point y $\in$ B.

THEOREM. *If $f : (\overline{O}; B) \rightarrow (\overline{O}; B)$ is C^1 in a neighbourhood of $f^{-1}(D)$ and $2n+1$ to 1 at any regular value x of f inside D, then f is not null homotopic.*

What our analytic proof does is to give an insight into the intermediate process and thus clarify the geometric content of the algebraic proof. Thus we arrive at a clearer understanding of the interplay between analysis and algebraic topology.

Proof. Let $f : f(\overline{O}; B) \rightarrow (\overline{O}; B)$ as in the statment of the theorem with b $\in O$ a regular value of f with $f^{-1}\{b\}$ having exactly the elements $a_1, ..., a_{2n+1}$.

Then there are neighbourhoods $B(b,\epsilon), B(a_1,\delta_1), ...B(a_{2n+1},\delta_{2n+1}), \epsilon, \delta_1, ..., \delta_{2n+1} > 0$, such that

$f : B(a_i, \delta_i) \rightarrow B(b,\epsilon)$ is a diffeomorphism into its range, i = 1,...,2n+1.

Let $F : (\overline{O}; B) \times [0,1] \rightarrow (\overline{O}; B)$ be a homotopy with $F_o = f$. We shall prove that b $\epsilon F_1[0]$ so that f is not null homotopic. Extend F to a homotopy G on [-1,2] such that

$$G_t = F_o, \; - \leq t \leq 0; \; G_t = F_t, \; 0 \leq t \leq 1; \; G_t = F_t, \; 1 \leq t \leq 2.$$

Then G is C^1 in a neighbourhood of - $\frac{1}{2}$.

Let $\alpha : R^{m+1} \rightarrow |0,\infty)$ be a C^∞ convolution kernel with small suport radius $r_\alpha (\int_{R^{m+1}} \alpha = 1, \alpha(x) = 0$ for $|x| \leq r_\alpha)$, and the convolution transform G_α of G given by $G_\alpha(x,t) = \int_{R^{m+1}} G(y)\alpha(y-(x,t))dy$

It is easily shown that G_α is C^∞ and given any $\eta_1, \eta_2 > 0$ we can take the support radius r_α so small that $| \, G_\alpha(x,t) - G(x,t)| < \eta_1$, $|DG_\alpha - DG| < \eta_2$; for

$|g_\alpha(z) - g(z)| \leq \sup_{|y-z|<r_\alpha} |g(z) - g(y)|$.

$|\frac{\partial g_\alpha(z)}{\partial s} - \frac{\partial g(z)}{\partial s}| \leq \sup_{|y-z|<r_\alpha} |\frac{\partial g(z)}{\partial s} - \frac{\partial g(y)}{\partial s}|$. for any C^1 function g.

Put $h_\alpha(x) = (G_\alpha)_{\frac{-1}{2}}(x) = G_\alpha(x, \frac{-1}{2})$, and consider $f : B(a_i, \delta_i) \to B(b, c)$ Let $T = Df(a_i)$. T can be represented as a non-singular $m \times m$ matrix.

Let $|T| = \sup_{|x|=1} |T|$.

Taking r_α adequately small we can find $0 < \delta < \delta_i$ such that $|x - a_i| < \delta \to |T - (Dh_\alpha)(x)| < \frac{1}{2|T^{-1}|}$ and given $\epsilon < \frac{\delta}{4|T^{-1}|}$, find α' with $r'_\alpha \leq r_\alpha$ such that $|f(a_i) - h_{\alpha'}(a_i)| < \epsilon < \frac{\delta}{4|T^{-1}|}$. It then follows from standard techniques in analysis that Dh_α, is non singular for $|x - a_i| < \delta$. Put $h_{\alpha'} = k$. By using the iteration $x_0 = a_i$, $y_{n+1} = z - k(x_n)$, $x_{n+1} = x_n + T^{-1}y_{n+1}$, we can prove that for $|z - k(x_0)| < \frac{\delta}{2|T^{-1}|}$, $k^{-1}(z)$ exists and $|k^{-1}(z) - x_0| \leq 2|T^{-1}||z - k(x_0)|$. Uniqueness is given by $|k(u) - k(v)| \geq \frac{|u-v|}{2|T^{-1}|}$. It follows that k^{-1} is defined on $B\left(b, \frac{\delta}{4|T^{-1}|}\right)$ and its range is contained in $B(a_i, \delta)$.

Let $\{\epsilon_n\}$ be a sequence of positive numbers with lim $\epsilon_n = 0$. We shall obtain a sequence $\{x_s\}$ with $x_s \epsilon 0$ and $|F_1(x_s) - b| < \epsilon_s$. By the compactness of $\overline{O}$ we may assume $lim x_s = \overline{x}$ exists and $F_1(\overline{x}) = b$ by continuity. This will prove the result.

We may take the δ above as common to all (a_i, δ_i). Let $\eta = \inf\{|f(x) - b|| |x - a_i| \geq \delta, i = 1, ..., 2n + 1\} > 0$. If r_α is so small that $|h_\alpha(x) - f(x)| < \frac{\eta}{2}$ in $\overline{O}$, then $h_\alpha(x) \neq b$ for $x \notin B(a_1, \delta) \bigcup .. \bigcup B(a_{2n+1}, \delta)$. We may assume that for all s $\epsilon_s < \frac{\eta}{2}$ and $\epsilon_s < \frac{\delta}{4|T_i^{-1}|}$, $i = 1, .., 2n + 1$ $T_i = Df(a_i)$.

Using the foregoing we can obtain G_α such that $h_\alpha^{-1}(z)$ has exactly 2n+1 points for $|z - b| < \epsilon_s$ and $|G_\alpha(x, t) - G(x, t)| < \frac{\epsilon_s}{2}$ for all (x,t). By Sard's theorem we obtain $z \in B(b, \frac{\epsilon_s}{2})$ which is a regular value of G_α, since G_α is smooth. Since $G_{\alpha,t} = h_\alpha$ in a neighbourhood of $\frac{-1}{2}$, by the inverse function theorem $G_\alpha^{-1}(z)$ consists of exactly 2n+1 lines intersecting the face $t = \frac{-1}{2}$ normally. Let $\mu = lnf|b - x| > 0$. Hence there is an open set H such that $\{a_1, ..., a_{2n+1}\} \subseteq H \subseteq 0$ and for any $(y, t) \in BdH \times [0, 1]$, $|F(y, t) - b| \to \frac{\eta}{2}$. We may thus further assume that r_α is small enough to ensure that for $(y, t) \in BdH \times [\frac{-1}{2}, 1]$, $|G_\alpha(y, t) - z| > \frac{\mu}{4} > 0$. Now since the number of points on the lines in $G_\alpha(z)$ is even, (2n+1) of these are on the face $H \times \{-\frac{1}{2}\}$ and none on $BdH \times [\frac{-1}{2}, 1]$, at least one, say (x_s, l), must be on the face $H \times \{1\}$.

Then $|F_1(x_s) - b| = |G_1(x_s) - b| \leq |G_1(x_s) - (G_\alpha)_1(x_s)| + |(G_\alpha)_1(x_s) - b| < \frac{\epsilon_s}{2} + \frac{\epsilon_s}{2} = \epsilon_s$.

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF JAFFNA

REFERENCES

[1] A. SARD, *The measure of critical points of differentiable maps*, Bull. Amer. math. Soc. 48 (1942).

[2] M. HIRSCH *Differential Topology*, Springer STM (1976).

[3] J. C. SCANLON *Fixed Points and Topological Degrees in Non Linear Analysis*, AMS, 1964.