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A FIXED POINT ITERATION PROCESS FOR HAMMERSTEIN EQUATIONS INVOLVING ANGLE-BOUNDED OPERATORS*

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1. Introduction

Let X be a real Banach space and $T : X \to X^*$ where X^* is the dual of X. Then T is said to be monotone if

(1) $$\langle Tx - Ty, x - y\rangle \geq 0$$

which coincides with the monotonic condition in the sense of Browder [4] and Minty [20] (see [8]). The mapping T is said to be *strongly monotone* if for each $x, y \in D(T)$ and for some constant $\alpha > 0$

(2) $$\langle Tx - Ty, x - y\rangle \geq \alpha \| x - y \|^2$$

and T is said to be *hemicontinuous* if $T(x + t_n y) \xrightarrow{w} Tx$ as $t_n \to 0^+$ for each pair $x, y \in D(T)$ where $\xrightarrow{w}$ denotes weak convergence.

A linear monotone operator $L : X \to X^*$ is called *angle-bounded* with constant $\sigma > 0$ if for all $x, y \in D(L)$,

(3) $$|\langle Lx, y\rangle - \langle Ly, x\rangle| \leq 2\sigma \langle Lx, x\rangle^{1/2} \langle Ly, y\rangle^{1/2};$$

L is called *symmetric* if it is angle-bounded with constant $\sigma = 0$.

2. Preliminaries

In this paper, we are concerned with operator equations of the form

(4) $$u + KNu = f;$$

which are called Hammerstein equations in the light of [16]. Equations of the form (4) have been studied by several authors (see e. g., [1, 5-6, 10-13, 15-17, 21-24]) and it is known that several problems occurring in Differential Equations can be put in the form (4) (see e. g., [23]). It is also known (see e. g., [13], Chapter IV) that Hammerstein operators i. e., operators of the form $I + AB$, play a crucial role in the study of feedback control systems.

Angle bounded operators turn out to play an important role in the theory of Hammerstein equations (see e. g., [1, 6, 11, 12, 15, 23]). An important result on the solvability of Hammerstein equations involving angle-bounded operators is the following:

THEOREM (BG1). (Browder-Gupta, [6]). *Let X be a real Banach space and X^* its dual. Let $K : X \to X^*$ be an angle-bounded (with constant $\sigma > 0$)*

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bounded (in the sense that for some $\beta > 0$ and for all $x \in X$, $\| Kx \| \leq \beta \| x \|$) linear monotone operator and let $N : X^* \to X$ be a hemicontinuous mapping such that for some constant $\alpha \geq 0$

$$(5) \qquad \langle Nu - Nw, u - w \rangle \geq -\alpha \| u - w \|^2; \qquad u, w \in X^*,$$

with $\alpha(1 + \sigma^2)\beta < 1$. Then the Hammerstein equation (4) is uniquely solvable in X^*.

In 1975, Brézis-Browder [1], established the strong convergence of suitably defined Galerkin approximations to a solution of a Hammerstein equation involving angle-bounded nonlinear operators. It is the aim in this paper to establish the norm convergence of a suitably defined Mann iteration process (see e. g., [19]) to such a solution.

An essential tool in the study of angle-bounded operators which we shall use in the sequel is the spliting of linear maps. We shall, therefore, find the following results useful in what follows.

THEOREM (BG2). (Browder-Gupta, [6]). *Let $K : X \in X^*$ be a linear monotone angle-bounded (with constant $\sigma \geq 0$) operator. Then there exists a Hilbert space H, a continuous linear map S of X into H with S^* (the adjoint of S) injective, and a skew-symmetric linear map B of H into H such that $K = S^*(I + B)S$, where I denotes the identity map on H, with the following inequalities holding:* (i) $\| B \| \leq \sigma$, *and* (ii) $\| S \|^2 \leq \beta \leftrightarrow \forall x \in X$, $\langle Kx, x \rangle \leq \beta \| x \|^2$.

Moreover, $(I + B)^{-1} : H \to H$ is a bounded linear isomorphism such that $\langle (I + B)^{-1}x, x \rangle \geq (1 + \sigma^2)^{-1} \| x \|^2, \forall x \in H$.

THEOREM (DG). (De Figueiredo-Gupta, [11]). *Let $K : D(K) \subseteq X \to X^*$ be a linear symmetric monotonic densely defined operator in X. Then there exists a Hilbert space H and a linear map $S : D(K) \subseteq H \to H$ such that $K = S^*S$.*

Also, the following result shall be usefull in the sequel.

LEMMA (D). (Dunn, [14]). *Let $\{\alpha_n\}$ be a recursively generated by*

$$\alpha_{n+1} = (1 - \delta_n)\alpha_n + \sigma_n^2$$

with $n \geq 1$, $\alpha_1 \geq 0$, $\{\delta_n\} \subset [0, 1]$, and

$$\sum_{n \geq 1} \sigma_n^2 < \infty, \qquad \sum_{n \geq 1} \delta_n = +\infty,$$

then $\lim_{n \to \infty} \alpha_n = 0$.

3. Main results

THEOREM (1). *Let X, K, N, H, S and B be as in theorem* (BG2)*with $\alpha(1 + \sigma^2)\beta < 1$. Let $M = (I + B)^{-1} + SNS^*$ and let C be an appropriate bounded closed convex nonempty subset of H. Define $T : C \to H$ by $Tx = h + x - Mx$*

where $h = (S^*(I+B))^{-1}f$. *Let* $\{c_n\}_{n\geq 0}$ *be a real sequence satisfying* $0 \leq c_n < 1$ *for all* $n \geq 0$, $\sum c_n = \infty$ *and* $\sum c_n^2 < \infty$. *Then, the sequence* $\{P_n\} \subset H$ *recursively generated from arbitrary* $x_0 \in C$ *by*

$$P_{n+1} = (1-c_n)x_n + c_n T x_n, \qquad n \geq 0; \tag{6}$$

where $\{x_n\} \subset C$ *is such that*

$$\| P_{n-1} - x_n \| = \inf_{x \in C} \| P_{n-1} - x \| \tag{7}$$

converges strongly to the unique solution $q \in H$ *to* $Mx = h$ *so that* S^*q *is the unique solution to the Hammerstein equation*(4).

Proof. Existence and uniqueness of a solution to the Hammerstein equation (4) follows from Theorem (BG1). Since $K = S^*(I+B)S$ (see Theorem (BG2)), (4) now becomes

$$w + S^*(I+B)SNw = f$$

where $w \in X^*$ is the unique solution to (4). By the injectiveness of S^*, there exists a unique $x \in H$ such that $S^*x = w$. We, therefore, have

$$S^*x + S^*(I+B)SNS^*x = f, \tag{8}$$

$S^*(I+B)$ is single-valued and invertible, hence (8) reduces to

$$(I+B)^{-1}x + SNS^*x = (S^*(I+B))^{-1}f = h; \tag{9}$$

which is an equation in H. Now, $M = (I+B)^{-1} + SNS^*$ is strongly monotone since

$$\begin{aligned}
\langle Mx - My, x - y\rangle &= \langle (I+B)^{-1}(x-y), x-y\rangle + \langle SNS^*x - SNS^*y, x-y\rangle \\
&\geq (1+\sigma^2)^{-1} \| x-y \|^2 + \langle NS^*x - NS^*y, S^*(x-y)\rangle \\
&\geq (1+\sigma^2)^{-1} \| x-y \|^2 + \alpha \| S^*(x-y) \|^2 \\
&\geq [(1+\sigma^2)^{-1} - \alpha\beta] \| x-y \|^2 \\
&= \lambda \| x-y \|^2
\end{aligned}$$

where $\lambda = ((1+\sigma^2)^{-1} - \alpha\beta) \in (0,1)$. Also, M is hemicontinuous. Thus, $M: H \to H$ is surjective and hence the equation $Mx = h$ is uniquely solvable. Let $q \in H$ denote such a solution. Then q is also a fixed point of T. Moreover,

$$\langle Tx - Ty, x - y\rangle \leq (1-\lambda) \| x - y \|^2$$

Let $R : H \to C$ be the proximity map, i.e., the map which assigns the unique element of C nearest to $x \in H$, $\forall x$. Then R is nonexpansive (see e.g., [7]). Then $x_{n+1} = R(P_n)$. Now,

$$\begin{aligned} \| P_n - q \|^2 &= (1-c_n)^2 \| x_n - q \|^2 + c_n^2 \| Tx_n - Tq \|^2 \\ &\quad + 2c_n(1-c_n)\langle Tx_n - Tq, x_n - q\rangle \\ &\le \Big\{(1-c_n)^2 + 2(1-\lambda)c_n(1-c_n)\Big\} \| x_n - q \|^2 \\ &\quad + c_n^2 \| Tx_n - Tq \|^2 \\ &\le (1-\lambda c_n)^2 \| P_{n-1} - q \|^2 + c_n^2 d^2 \end{aligned}$$

on adding $(1-\lambda)^2 c_n^2 \| x_n - q \|^2$ to the RHS, setting $d = \sup_{n\ge 0} \| Tx_n - Tq \|$ and using the nonexpansiveness of R. Observe that $d < \infty$ since $D(T) \subset C$ and $R(N)$ is bounded. Routine argument, using lemma (D), now shows that $x_n \to q$, in norm, as $n \to \infty$ (see e.g., [9,14]). However, for completeness, we present the details.

Let $1 - r_n = (1-\lambda c_n)^2 > 0$ so that $r_n = \lambda c_n(2-\lambda c_n)$ and set $\varphi_{n+1} = \| P_n - q \|^2$ to obtain

$$\varphi_{n+1} \le (1-r_n)\varphi_n + c_n^2 d^2; \tag{10}$$

A simple induction on (10) easily yields

$$0 \le \varphi_n \le A^2 \mu_n, \quad \text{for all } n \ge 1; \tag{11}$$

where $\mu_n \ge 0$ is recursively generated by

$$\mu_{n+1} = (1-r_n)\mu_n + c_n^2, \quad \mu_1 = 1; \tag{12a}$$

and

$$A^2 = \max\{\varphi_1, d^2\}, \tag{12b}$$

Conditions on the real sequence now yield

$$\sum r_n = 2\lambda \sum c_n - \lambda^2 \sum c_n^2 = +\infty \tag{13}$$

From $\sum c_n^2 < \infty$ we have $\lim c_n = 0$. Hence, we can choose an integer $n_0 > 0$ sufficiently large such that for all $n \ge n_0$, $r_n \in [0,1]$. Now, for $j \ge 1$, set $\alpha_j = \mu_{n_0+j}$, $\delta_j = r_{n_0+j}$, $\sigma_j = c_{n_0+j}$. So that, from $\alpha_1 = \mu_{n_0+1} \ge 0$, Lemma (D) applies to show that $\mu_n \to 0$ as $n \to \infty$. The inequality (11) now implies that $\varphi_n \to 0$ as $n \to \infty$ so that $\{P_n\}$ converges strongly to q.

Now from $Mq = h$, or equivalently

$$(I+B)^{-1}q + SNS^*q = (S^*(I+B))^{-1}f$$

we have

$$S^*q + S^*(I+B)SNS^*q = f$$

so that

$$S^*q + KNS^*q = f$$

and S^*q is the unique solution to (4) and completes the proof.

Remark. If N is monotone, the hypothesis on K can be significantly weakened. We have the following result.

COROLLARY (1). *Let X, X^*, C, M, T, $\{x_n\}$ and $\{c_n\}$ be as in Theorem* (1). *Suppose $K : X \to X^*$ is an angle-bounded (with constant $\sigma > 0$) linear monotone operator, and $N : X^* \to X$ is a hemicontinuous monotone operator. Then the conclusions of Theorem* (1) *reamain valid.*

Proof. The Corollary follows immediately from Theorem (1) on setting $\alpha = 0$ so that $\lambda = (1+\sigma^2)^{-1} \in (0,1)$.

Remark. (i) Theorem (1) remains valid if the requirement that $R(N)$ be bounded is replaced with the condition that N be a bounded map or more generally, that $N(C)$ be a bounded set.

(ii) If N is assumed to be Lipschitzean (with constant $L_N > 0$) in Theorem (1), we obtain the additional information of an explicit error estimate. Moreover, in this case, T defined on the whole of H is Lip (L) with $L = 1+\delta+\beta L_N$. Observe that if $C = H$, then (6) reduces to

$$x_{n+1} = (1-c_n)x_n + c_n T x_n, \quad n \geq 0.$$

COROLLARY (2). *In theorem* (1) *(as well Corollary (1)), let $T : H \to H$ and let N be additionally Lipschitzean. Further, let $\{c_n\}$ be a real sequence satisfying (i) $0 \leq c_n \leq \lambda(L^2+2\lambda-1)^{-1}$, for all $n \geq 0$, and (ii) $\sum c_n = +\infty$. Then the conclusions of theorem* (1) *remain valid. Moreover, if $c_n = \lambda(L^2+2-1)^{-1}$ for all $n \geq 0$, then*

$$\| x_n - q \| \leq (1-\mu)^{n/2} \| x_0 - q \|$$

with $\mu = \lambda^2(L^2+2\lambda-1)^{-1} \in (0,1)$.

Proof. $\| Tx - Ty \| \leq \| x-y \| + \delta \| x-y \| + \beta L_N \| x-y \|$

$\| Tx - Ty \| = L \| x-y \|$; $L = 1+\delta+\beta L_N$

so that we obtain

$$\begin{aligned} \| x_{n+1} - q \|^2 &\leq [(1-c_n)^2 + 2(1-\lambda)c_n(1-c_n) + L^2 c_n^2] \| x_n - q \|^2 \\ &\leq (1-\lambda c_n) \| x_n - q \|^2 \\ &\quad \exp(-\lambda c_n) \| x_n - q \|^2 \end{aligned} \tag{14}$$

Thus, iterating from o to N, using (iii), yields

$$\| x_{N+1} - q \|^2 \leq \exp\left(-\lambda \sum_{n=0}^{N} c_n\right) \| x_0 - q \|^2 \to 0 \text{ as } N \to \infty.$$

If $c_n \equiv \lambda(L^2 + 2\lambda - 1)^{-1}$, inequality (14) reduces to

$$\| x_{n+1} - q \|^2 \geq [1 - \lambda^2(L^2 + 2\lambda - 1)^{-1}] \| X_n - q \|^2$$

Routine induction now yields the desired error estimate.
This completes the proof.

In the case K is not everywhere defined on X, some alternative hypotheses on K yield the same result as the foregoing ones.

THEOREM (2). *Suppose $K : D(K) \subset X \to X^*$ is a densely-defined linear symmetric (or self-adjoint) monotonic operator such that $\forall x \in D(K)$ and some $\beta > 0$, $\langle Kx, x\rangle \leq \beta \| x \|^2$, and $N : X^* \to X$ is a hemicontinuous bounded below operator with constant $-\alpha$, $\alpha \in \mathbb{R}$ (i.e., N satisfies* (5)*) with $\alpha\beta < 1$. Let $M = I + SNS^*$ and define $T : C \to H$ by $Tx = g + x - Mx$ where $g = (S^*)^{-1}f$ and C is an appropriate bounded closed convex nonempty subset of the Hilbert space H. Then the conclusions of Theorem* (1) *remain valid.*

Proof. By theorem (DG), $K = S^*S$. Let $w \in X^*$ be the unique solution to (4). Then we have

$$w + S^*SNw = f$$

so that the injectiveness of S^* yields the existence of a unique $x \in H$ such that

$$S^*x + S^*SNS^*x = f$$

and hence by the unique invertibility of S^*,

$$x + SNS^*x = (S^*)^{-1}f = g$$

which is an equation in H. Now,

$$\langle Mx - My, x - y\rangle \geq \lambda \| x - y \|^2; \quad \lambda = (1 - \alpha\beta) \in (0, 1)$$

The rest of the argument now follows as in theorem (1) and the proof is complete.

COROLLARY (3). *In theorem* (2)*, let $N : X^* \to X$ be a hemicontinuous monotone map and let $K : X \to X^*$ be a linear densely-defined symmetric monotone map. Then the conclusions of theorem* (2) *reamin valid.*

Proof. Set $\alpha = 0$ in theorem (2) so that $\lambda = 1$. Now, in the proof of theorem (1), set $\lambda = 1$ and the corollary follows.

COROLLARY (4). *In corollary* (2), *let X, M, K and T be as in theorem* (2) *(with $T : H \to H$). Then the same conclusions are obtained.*

COROLLARY (5). *Let X, M, T and $\{x_n\}$ be as in corollary* (4) *and let K be as in corollary* (3). *Suppose N is Lipschitzean monotone map and $\{c_n\}$ satisfies (i) $0 \le c_n \le (L^2+1)^{-1}$ for all $n \ge 0$ and (ii) $\sum c_n = \infty$. Then the conclusions of theorem* (2) *all hold. Moreover, if $c_n = (L^2+1)^{-1}$ for all $n \ge 0$, then with $\mu = (L^2+1)^{-1}$*

$$\| x_n - q \| \le (1-\mu)^{n/2} \| x_0 - q \|$$

Further, if $c_n = (n + L^2)^{-1}$ for all $n \ge 1$, then the rate of convergence is of the order $O(n^{-1/2})$.

Proof. Clearly, $\sum c_n(1-c_n) = \sum (n+L^2-1)(n+L^2)^{-2} = \infty$. Setting $\varphi_n = \| x_n - q \|^2$ gives that $\varphi_{n+1} \le \varphi_n$ for each n and also

$$(n+L^2)^2 \varphi_{n+1} - (n+L^2-1)^2 \varphi_n \le L^2 \varphi_1; \tag{15}$$

Summing from 1 to N, observing that the LHS of (15) telescopes, yields

$$(n+L^2)^2 \varphi_{N+1} - L^4 \varphi_1 \le N L^2 \varphi_1$$

so that

$$\varphi_{N+1} \le L^2 (N + L^2)^{-1} \varphi_1.$$

This yields the stated order of convergence and completes the proof.

It turns out to be that if N is Lipschitzean, the usual Picard iterations converges. We have the following result

COROLLARY (6). *Let X, X^*, K, N, M and H be as in theorem* (1) *or* (2) *and let N be additionally Lipschitzean. Then the usual Picard iterations generated from an arbitrary $x_0 \in H$ converges strongly to the unique solution $q \in H$ to $Mx = h$. Moreover, convergence is at least as fast as a geometric progression with ratio $c = (1 - \lambda^2 L^{-2})^{1/2} \in (0,1)$.*

Proof. Define the iteration operator $T_r : H \to H$ by

$$T_r x = x - r(Mx - h); \quad x \in H \text{ and some } r > 0.$$

Now,

$$\begin{aligned} \| T_r x - T_r y \|^2 &= \| x - y \|^2 - 2r\langle Mx - My, x - y\rangle + r^2 \| Mx - My \|^2 \\ &\le (1 - 2\lambda r + r^2 L^2) \| x - y \|^2 \\ &\le c^2 \| x - y \|^2 \quad (c = 1 - \lambda^2 L^{-2}, \ r = \lambda L^{-2} > 0) \end{aligned}$$

Hence, T_r is a contraction so that the usual Picard iterations defined by $x_{n+1} = T_r x_n$ converges strongly to the unique fixed point of T_r with the stated error estimate. This completes the proof.

Remarks. The Mann process converges with convergence being (i) at least as fast as a geometric progression with ratio $k = (1 - \lambda^2(L^2 + 2\lambda - 1)^{-1})^{1/2} \in (0,1)$ provided $c_n \equiv \lambda(L^2 + 2\lambda - 1)^{-1}$ (ii) of order $O(n^{-1/2})$ if $c_n = (n + L^2)^{-1}$ for all $n \geq 0$. The usual Picard iterations converge with convergence being at least as fast as a geometric progresion with ratio $c = (1 - \lambda^2 L^{-2})^{1/2} \in (0,1)$.

Thus, if $2\lambda - 1 < 0$ then the Mann process has a faster rate of convergence than the usual Picard iterations while otherwise, the Picard iterations have a faster rate of convergence and so the Mann process is unnecessary. However, since, in general, λ can be made as small as possible, it stands that $2\lambda - 1 < 0$ so that the Mann process affords an improvement on the usual Picard iterations.

A linear strongly monotone map K with constant $\lambda > 0$ is angle-bounded with constant $\sigma = \lambda^{-1} \| K \|$, since

$$\begin{aligned} |\langle x, Ky\rangle - \langle y, Kx\rangle| &\leq \| x \| \| Ky \| + \| y \| \| Kx \| \\ &\leq 2 \| K \| \| x \| \| y \| \\ &\leq 2\lambda^{-1} \| K \| \langle x, Kx\rangle^{1/2} \langle y, Ky\rangle^{1/2} \end{aligned}$$

If K is quasipositive with constant $\mu > 0$, i.e.,

$$\langle x, Kx\rangle \geq \mu \| Kx \|^2$$

and also satisfies a weak coercivity condition of the form

$$\| Kx \| \geq \theta \| x \|, \quad \text{for some } \theta > 0$$

then it follows that K is angle-bounded with constant $\sigma = \mu^{-1}\theta^{-1}$.

Using these facts, we obtain the following generalisations of the result in Chidume-Moore [10] and Moore [21,22].

THEOREM (3). *In Theorem* (1), *let K be a linear strongly monotone map with constant $\lambda > 0$. Then the conclusions of Theorem* (1) *remain valid.*

THEOREM (4). *In Theorem* (3), *let K be a linear quasi-positive and weakly coercive. Then the same conclusions are obtained.*

Setting $\sigma = \delta^{-1} \| K \|$ and $\beta = \| K \|$ in the first instance and $\sigma = \mu^{-1}\theta^{-1}$ and $\beta = \mu^{-1}$ in the second instance in Theorem (1) yields the assertions.

It is definitely routine now to see that the Corollaries also apply to Theorems 3 and 4.

General remarks

If X is a reflexive Banach space (in particular, if X is L_p or l_p, $1 < p < \infty$,) we then consider the Hammerstein operator $I + KN$ defined on X (instead of X^*

as done above). The same results as in Theorems (1)-(4) and their Corollaries are easily obtained in the new setting where $K : X^* \to X$ and $N : X \to X^*$.

The basic tool we have employed all through is the splitting of linear angle-bounded monotone maps; i.e., for $K : X \to X^*$ we have that either $K = S^*(I + B)S$ where $S : X \to H$, hence, $S^* : H \to X^*$, and $I + B : H \to H$ or $K = S^*S$ with $S : X \to H$ and $S^* : H \to X^*$. If however, $D(K) \subseteq X^*$, then we would have $S : X^* \to H$ and $S^* : H \to X^{**}$. We, therefore, need the reflexivity of X to indentify X^{**} with X. We have not been able to dispense with reflexivity in this regard so far. It would, therefore, be interesting to obtain results analogous to Theorems (1)-(4) and their Corollaries for the Hammerstein operator $I + KN$ defined on X, a Banach space, which need not be reflexive.

Furthermore, these results were possible because we were able to reduce the Hammerstein equation $(I + KN)u = f$ in X^* (or for a reflexive X, $(I + KN)x = f$ in X) to an equivalent equation $Mx = h$ in H a Hilbert space, which process, in turn, was made possible by the splitting of angle-bounded linear monotone maps. In a setting where such splitting is not possible, the method above fails; we require an alternate method. The methods used in Chidume-Moore [10] and Moore [21] i.e., defining T by

(i) $Tx = K^{-1}f + x - (K^{-1} + N)x, \quad (16a)$

and (ii) $Tx = f + x - (K^* + KNK^*)x, \quad (16b)$

are adequate only for $X = H$, a Hilbert space. For a real Banach space with $K : X^* \to X$ (or $K : X \to X^*$) and $N : X \to X^*$ (or $N : X^* \to X$), the definitions (16) fail to make nice sense. The alternate approach used in Chidume [8] and Moore [21,22] i.e., defining T thus

$$Tx = f - KNx$$

is again disadvantaged by the fact a product (i.e., composition) of monotone operators need not be monotone.

It would, therefore, be of considerable interest to define a mapping $T : X \to X$ or $T : X^* \to X^*$, for a real Banach space X, that will be suited for approximating solutions to the Hammerstein equations involving monotone operators using fixed point iteration processes.

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