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SOME SURGERIES IN COMPACT REDUCIBLE 3-MANIFOLDS WHICH PRODUCE IRREDUCIBLE MANIFOLDS

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1. It is proved that, given M_1 and M_2 3-manifolds which are closed, connected, compact, orientable, irreducible and different from S^3, then the manifold obtained by doing non integral surgery in $M_1 \# M_2$ along certain closed curves is irreducible. These curves are obtained by glueing two arcs properly embedded in M_1 and M_2 punctured, whose extensions to closed loops, using the boundary spheres, are not contained in 3-balls. We will work in the piecewise linear category.

This result may have some consequences in knot theory. For instance, there is a conjeture which says that the knot obtained by applying a full twist to the connected sum of two prime knots, represented by braids, cannot be composite. Using two-fold branched covers, the above result may give some information about this conjecture. I would like to thank José María Montesinos for suggesting the original problem and Francisco González Acuña, who has acquainted me in many ways with this subject.

2. In the following, M_1 and M_2 will be as above.

By M_j^0, $j = 1, 2$, we will mean $\overline{M_j - B_j^3}$, where B_j^3 denote 3-balls embedded in M_1 and M_2 respectively.

To perform Dehn surgery along a curve γ in a manifold M we will assume that:

a) A regular neighbourhood $N(\gamma)$ of γ is chosen.

b) A homeomorphism

$$H : T^2 \to N(\gamma)$$

is chosen, where T^2 is a solid torus; thereby one has a natural selection for a meridian and a longitude.

Hence, $\frac{p}{q}$ surgery along γ in M will mean to attach a 2-handle along a "thickened" curve in $\partial\{\overline{M - N(\gamma)}\}$, which runs p times meridinally and q times longitudinally, and then cap the resulting manifold. Our results will apply to all such framings.

We will use a result of González Acuña called the Six Lemma. First we need a definition:

Definition. Let X, Y be topological spaces. A function $f : X \to Y$ is called π_1-injective, if given a loop α in X such that $f(\alpha)$ is contractible in Y, one has that α is contractible in X.

LEMMA (González Acuña). *Let W_1, W_2 be n-submanifolds of W^n such that $W = W_1 \cup W_2$ and $W_1 \cap W_2$ is a submanifold of both ∂W_1 and ∂W_2. Suppose also that*

a) *The following inclusion maps are π_1-injective*

$$\partial W_1 \xrightarrow{i_1} W_1, \qquad W_1 \cap W_2 \xrightarrow{i_2} W_2$$

and
$$\partial W_2 - W_1 \xrightarrow{i_3} W_2.$$

b) *A loop in $\partial(W_1 \cap W_2)$ is simultaneously contractible or non contractible in $W_1 \cap W_2$, $\overline{\partial W_1 - W_2}$ and $\overline{\partial W_2 - W_1}$.*

Then the natural inclusion map $\partial W \to W$ is π_1-injective.

A reference can be found in [3].

3. Before proving the main result we need a lemma.

LEMMA 1. *Let $M^3 = T_1 \cup T_2$ be a 3-manifold obtained by glueing two solid tori T_1, T_2 along an annulus A, which is an essential submanifold of both ∂T_1 and ∂T_2. Suppose also that the homomorphisms*

$$i_{j\#} : \Pi_1(A) \to \Pi_1(T_j), \qquad j = 1, 2$$

induced by the natural inclusions $i_j : A \mapsto T_j$, $j = 1, 2$ are neither zero nor epimorphisms. Then M^3 has incompressible boundary.

Proof. If ∂M^3 is compressible, by Dehn's lemma, there is a properly embedded 2-disk D^2 in M^3 such that ∂D^2 is essential in ∂M^3, hence one may construct a homeomorphism

$$H : (\partial M^3 \times I) \cup \{\text{2-handle}\} \to T^0,$$

where T^0 denotes a punctured solid torus and I an interval. This can be achieved by sending $\partial D^2 \times I$ onto $\{\text{meridian}\} \times I$. Furthermore, we may extend H to a homeomorphism between M^3 and a solid torus, being the former an irreducible manifold (union of two solid tori glued along an incompressible annulus in both tori). However, M^3 can not be homeomorphic to a solid torus, since $\Pi_1(M^3)$ is not abelian, being a proper free product with amalgamation.

Given a closed three manifold M and a properly embedded arc γ in M^0, we denote by $\overline{\gamma}$ any closed loop in M^0 obtained from γ using the boundary sphere, ∂M^0. Now we can state the main result:

THEOREM. *Let M_1, M_2 be two 3-manifolds which are closed, connected, compact, orientable, irreducible and different from S^3, α and β properly embedded arcs in M_1^0, M_2^0 respectively, such that $\overline{\alpha}$ and $\overline{\beta}$ are not contained in 3-balls. It is also assumed that $\partial\alpha = \partial\beta$ in $M_1 \# M_2$. Then*

$$\mathfrak{M} = \{M_1 \# M_2,\ \alpha \cup \beta,\ \frac{p}{q} \in \mathbb{Q} - \mathbb{Z}\}$$

is irreducible, where $\mathfrak{M}$ is the manifold obtained by doing $\frac{p}{q}$ surgery, $q \neq 1$, along $\alpha \cup \beta$, in $M_1 \# M_2$.

Proof. We will consider three cases according as the boundaries of $\overline{M_1^0 - N(\alpha)}$ and $\overline{M_2^0 - N(\beta)}$ are compressible or not.

Case 1. $\overline{M_1^0 - N(\alpha)}$ and $\overline{M_2^0 - N(\beta)}$ have incompressible boundary.

González lemma implies that $\tau \cup \overline{M_1^0 - N(\alpha)}$ has incompressible boundary, where τ denotes the surgery solid torus. The hypothesis are satisfied because:

a) By assumption $\partial\{\overline{M_1^0 - N(\alpha)}\}$ is incompressible in $\overline{M_1^0 - N(\alpha)}$.

b) $\tau \cap \overline{M_1^0 - N(\alpha)}$ is an annulus which is incompressible in τ, as we are not considering the trivial surgery.

c) Also by the same reason $\partial\tau - \{\overline{M_1^0 - N(\alpha)}\}$ is an incompressible annulus in τ.

d) Condition 2 of González lemma follows, as each component of the boundary of an annulus is a strong deformation retract of such annulus.

Now it is well known that two irreducible manifolds glued along a surface which is incompressible in both of them form an irreducible manifold. Hence, as $\overline{M_1^0 - N(\alpha)}$ and $\overline{M_2^0 - N(\beta)}$ are irreducible, (since $\overline{\alpha}$ and $\overline{\beta}$ are not contained in 3-balls), one gets that $\mathfrak{M}$ is irreducible. Observe that this case is true for all surgeries except for the trivial one.

Case 2. $\overline{M_1^0 - N(\alpha)}$ has incompressible boundary but $\overline{M_2^0 - N(\beta)}$ has not.

First observe that $\overline{M_2^0 - N(\beta)}$ is a solid torus because it is an irreducible 3-manifold whose boundary is a compressible torus. One also gets that M_2 is a lens space.

Again, as the union of two irreducible manifolds glued along an incompressible surface yields an irreducible manifold, the result follows from Lemma 1. applied to $\tau \cup \overline{M_2^0 - N(\beta)}$.

To show that the hypothesis of Lemma 1. are satisfied, we write

$$T_1 = \overline{M_2^0 - N(\beta)}, \qquad T_2 = \tau \qquad \text{and} \qquad A = T_1 \cap T_2$$

The homomorphism

$$i_{1\#} : \Pi_1(A) \to \Pi_1(T_1)$$

is not zero, otherwise M_2 would be homeomorphic to $S^2 \times S^1$.

Also

$$i_{2\#} : \Pi_1(A) \to \Pi_1(T_2)$$

is not zero, as we are not doing the trivial surgery. One also has that i_{1*} is not an epimorphism, otherwise M_2 would be S^3.

Finally, i_{2*} is not an epimorphism, because this would imply that the core of A would run once along ∂T_2 in the longitudinal direction, however in that case the surgery performed is integral.

This last remark follows because if

$$\phi : \partial\tau \to \partial\{M_1 \# M_2 - N(\alpha \cup \beta)\}$$

is the surgery homeomorphism, $\partial\tau$ framed in the natural way and $\partial\{M_1\#M_2 - N(\alpha \cup \beta)\}$ as originally, one would get that ϕ_* sends curves of type $(q, 1)$ to curves isotopic to the curve $(1, 0)$ and consequently curves of the type $(1, 0)$ to curves isotopic to the curve $(a, 1)$.

Case 3. $\overline{M_1^0 - N(\alpha)}$ and $\overline{M_2^0 - N(\beta)}$ have compressible boundary.

As in Case 2, these manifolds are solid tori. We denote $\overline{M_1^0 - N(\alpha)}$ and $\overline{M_2^0 - N(\beta)}$ and τ by T_1, T_2 and T_3 respectively. With this notation $\mathfrak{M}$ is homeomorphic to

$$\bigcup_{i=1}^{3} T_i$$

glued pairwise along annuli.

Let A_{ij} be the annulus obtained by intersecting T_i with T_j, $i, j \in \{1, 2, 3\}$, $i \neq j$. For each solid torus T_i, the common boundary of A_{ij} and A_{ik} in ∂T_i, $\{i, j, k\} = \{1, 2, 3\}$ consist of two parallel curves, whose class we denote by a_i.

Since trivial surgery is not considered and M_1, M_2 are not homeomorphic to $S^2 \times S^1$, the curves a_i, $i = 1, 2, 3$ are not equivalent to the meridian class in T_i. Furthermore, the arguments used in Case 2 imply that for all $i = 1, 2, 3$, the curves a_i run along ∂T_i more than once longitudinally.

Hence, $\mathfrak{M}$ admits a Seifert fibration with exactly three exceptional fibres and S^2 as orbit surface. See [1, Theorems 3 and 4]. Finally, it is known that these manifolds are irreducible. See [2, p. 89].

Notice that in cases I and II the manifold $\mathfrak{M}$ has an incompressible torus and so, it is a Haken Manifold.

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