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ORTHOGONAL PROJECTIONS OF CONVEX BODIES AND CENTRAL SYMMETRY

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Let K be a solid body in $\mathbb{R}^n$. Suppose that we want to recognize K just by using information that comes from some of its orthogonal projections. When can we conclude that K has some property just form the fact that a sufficiently big collection of its orthogonal projection have the same property? More precisely, suppose that we have a collection of orthogonal projections of K which are centrally symmetric, have constant width or are solid spheres. We would like to conclude that, if this collection is big enough, then K has the same property. In particular, we would be interested in a set of orthogonal projections generated by a collection $\mathscr{H}$ of linear subspaces of $\mathbb{R}^n$ with the property that for every 1-dimensional linear subspace L of $\mathbb{R}^n$ there is $H \in \mathscr{H}$ such that $L \subset H$. Several aplications of these results will be given. In particular, we prove that if all orthogonal projections of a solid body K are similar then K is a solid sphere. This result is closely related to Gromov's Conjecture concerning affinely equivalent sections of solid bodies.

1. Preliminaries

We shall start stating without proof some elementary results which will be very useful in the sequel.

Let H_1 and H_2 be linear subspaces of $\mathbb{R}^n$ and let $H_i^\perp$, $i = 1, 2$, be the corresponding orthogonal subspaces. Finally, let $b_1, b_2, b_3 \in \mathbb{R}^n$.

(1.1) $\dim H_1 + \dim H_2 = \dim(H_1 + H_2) + \dim(H_1 \cap H_2)$.
In particular, if Γ_i are α_i-planes in $\mathbb{R}^n$, $i = 0, 1, 2$, and $\Gamma_1 \subset \Gamma_0$, $\Gamma_2 \subset \Gamma_0$, then $\dim(\Gamma_1 \cap \Gamma_2) \geq \alpha_1 + \alpha_2 - \alpha_0$.

(1.2) Either $(b_1 + H_1) \cap (b_2 + H_2) = \phi$ or there exists $b_3 \in \mathbb{R}^n$ such that

$$(b_1 + H_1) \cap (b_2 + H_2) = b_3 + (H_1 \cap H_2).$$

(1.3) $H_1^\perp \cap H_2^\perp = (H_1 + H_2)^\perp$
$H_1^\perp + H_2^\perp = (H_1 \cap H_2)^\perp$

(1.4) Let $H_1 \subset H_2$ be subspaces of $\mathbb{R}^n$. Let us denote by $P_{H_1}^{H_2} : H_2 \to H_1$ the orthogonal projection. Note that $P_{H_1}^{H_2} P_{H_2} = P_{H_1}$, where P_H simply denotes the projection $P_H^{\mathbb{R}^n}$.

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(1.5) If $K \subset \mathbb{R}^n$ is a convex body which is centrally symmetric with respect to b, then $P_H(K)$ is centrally symmetric with respect to $P_H(b)$.

Let K be a strictly convex body in $\mathbb{R}^n$, that is, every support hyperplane of K meets K in exactly one point. Let L be a line through the origin and let Π_1 and Π_2 be the support hyperplanes of K orthogonal to L. We will denote by D_L, the *diametral chord* of K with respect to L, the chord of K with extreme points $\Pi_1 \cap K$ and $\Pi_2 \cap K$. Note that $P_L(K) = P_L(D_L)$.

(1.6) Let K be a strictly convex body in $\mathbb{R}^n$ which is centrally symmetric with respect to the point b_o, then the midpoint of every diametral chord of K is b_o.

2. Some lemmas

We start this section with the following technical lemma.

LEMMA (2.1). *Let H_1 and H_2 be linear subspaces of $\mathbb{R}^n$ and let $c_1 \in H_1$, $c_2 \in H_2$ such that*

$$P^{H_1}_{H_1 \cap H_2}(c_1) = P^{H_2}_{H_1 \cap H_2}(c_2).$$

Then $P^{-1}_{H_1}(c_1) \cap P^{-1}_{H_2}(c_2)$ is an α-plane parallel to $(H_1 + H_2)^{\perp}$, where $\alpha = n - \dim H_1 - \dim H_2 + \dim(H_1 \cap H_2)$.

Proof. We start expresing $P^{-1}_{H_i}(c_i)$ as $c_i + H_i^{\perp}$. $P_{H_1 \cap H_2}(c_i + H_i^{\perp}) = P^{H_i}_{H_1 \cap H_2} P_{H_i}(c_i + H_i^{\perp}) = P^{H_i}_{H_1 \cap H_2}(c_i) = c_3$ and consequently $c_i + H_i^{\perp} \subset P^{-1}_{H_1 \cap H_2}(c_3) = c_3 + (H_1 \cap H_2)^{\perp}$. By (1.3) and the above, $(c_1 - c_3) + H_i^{\perp} \subset H_1^{\perp} + H_2^{\perp}$ and hence, $((c_1 - c_3) + H_1^{\perp}) \cap ((c_2 - c_3) + H_2^{\perp})$ is not empty, which implies, by (1.2) and (1.3), that there is $b \in \mathbb{R}^n$ such that $((c_1 - c_3) + H_1^{\perp}) \cap ((c_2 - c_3) + H_2^{\perp}) = b + (H_1 + H_2)^{\perp}$. Therefore, $P^{-1}_{H_1}(c_1) \cap P^{-1}_{H_2}(c_2) = (c_1 + H_1^{\perp}) \cap (c_2 + H_2^{\perp}) = (b + c_3) + (H_1 + H_2)^{\perp}$, which is a plane parallel to $(H_1 + H_2)^{\perp}$ ∎

The following simple lemma characterizes central symmetry in terms of 1-dimensional orthogonal projections.

LEMMA (2.2). *Let $K \subset \mathbb{R}^n$ be a convex body and let $a \in \mathbb{R}^n$ such that for every 1-dimensional linear subspace L, $P_L(a)$ is the midpoint of $P_L(K)$. Then K is centrally symmetric with respect to a.*

Proof. Let us assume that a is the origin and K is not centrally symmetric with respect the origin. Then there is a point $x \in K$ such that $-x \notin K$. Let H be a support hyperplane of K which separates $-x$ from K, and let L be the 1-dimensional subspace orthogonal to H. Now its is clear that the origin is not the midpoint of $P_L(K)$, which is a contradiction. ∎

Definitions Let $\mathscr{X}$ be a collection of linear subspaces of $\mathbb{R}^n$. We say that $\mathscr{X}$ is *linear* if for every 1-dimensional linear subspace L of $\mathbb{R}^n$ there is $H \in \mathscr{X}$ such that $L \subset H$. Furthermore, if K is a convex body, we say that $\mathscr{X}$ is *K-symmetric* if for every $H \in \mathscr{X}$, the convex body $P_H(K)$ is centrally symmetric.

The following lemma studies the central symmetry of some of the orthogonal projections of a convex body in terms of the central symmetry of some others of its orthogonal projections.

LEMMA (2.3). *Let $K \subset \mathbb{R}^n$ be a convex body and let $\mathscr{X}$ be a linear K-symmetric collection of subspaces of $\mathbb{R}^n$. Suppose that given any three members H_0, H_1 and H_2 of $\mathscr{X}$*

$$P_{H_0}^{-1}(c_{H_0}) \cap P_{H_1}^{-1}(c_{H_1}) \cap P_{H_2}^{-1}(c_{H_2}) \neq \phi,$$

where, for every $H \in \mathscr{X}$, c_H is the centre of symmetry of $P_H(K)$. Then for every two members Γ and H of $\mathscr{X}$, $P_{\Gamma+H}(K)$ is centrally symmetric.

Proof. Let H_1 and H_2 be any two members of $\mathscr{X}$. By Lemma (2.1), (1.4) and (1.5), $P_{H_1}^{-1}(c_{H_1}) \cap P_{H_2}^{-1}(c_{H_2})$ is a plane perpendicular to $H_1 + H_2$. Let $a \in H_1 \cap H_2$ be such that $P_{H_1}^{-1}(c_{H_1}) \cap P_{H_2}^{-1}(c_{H_2}) \cap (H_1 + H_2) = a$. We want to prove that $P_{H_1+H_2}(K)$ is centrally symmetric with respect to a. By Lemma (2.2), it would be enough to prove that for every 1-dimensional subspace L of $H_1 + H_2$ the point $P_L^{H_1+H_2}(a)$ is the midpoint of $P_L^{H_1+H_2}(P_{H_1+H_2}(K)) = P_L(K)$.

Since $\mathscr{X}$ is linear, then there is $H_0 \in \mathscr{X}$ such that $L \subset H_0$. By hypothesis, there is $b \in P_{H_0}^{-1}(c_{H_0}) \cap P_{H_1}^{-1}(c_{H_1}) \cap P_{H_2}^{-1}(c_{H_2})$. Note that $P_{H_1+H_2}(b) = a$, because $P_{H_1+H_2}(b) \in P_{H_1}^{-1}(c_{H_1}) \cap P_{H_2}^{-1}(c_{H_2}) \cap (H_1 + H_2)$, and $P_{H_0}(b) = c_{H_0}$. Therefore, by the above and (1.4), $P_L^{H_1+H_2}(a) = P_L^{H_1+H_2}(P_{H_1+H_2}(b)) = P_L(b) = P_L^{H_0}(P_{H_0}(b)) = P_L^{H_0}(c_{H_0})$. Finally, note that by (1.5), $P_L^{H_0}(c_{H_0})$ is the midpoint of $P_L^{H_0}(P_{H_0}(K)) = P_L(K)$. This concludes the proof of Lemma (2.3). ■

LEMMA (2.4). *Let $K \subset \mathbb{R}^n$ and H_0, H_1, H_2 be subspaces of $\mathbb{R}^n$ such that $P_{H_i}(K)$ is centrally symmetric with respect to $c_i \in H_i$, $i = 0, 1, 2$. Suppose that $H_0 \cap (H_1 + H_2) = (H_0 \cap H_1) + (H_0 \cap H_2)$. Then*

$$P_{H_0}^{-1}(c_0) \cap P_{H_1}^{-1}(c_1) \cap P_{H_2}^{-1}(c_2) \neq \phi.$$

Proof. First note that $P_{H_1 \cap H_2}^{H_1}(c_1) = P_{H_1 \cap H_2}^{H_2}(c_2)$, because by (1.5), the centre of symmetry of $P_{H_1 \cap H_2}^{H_1}(P_{H_1}(K))$ is $P_{H_1 \cap H_2}^{H_1}(c_1)$, the centre of symmetry of $P_{H_1 \cap H_2}^{H_2}(P_{H_2}(K))$ is $P_{H_1 \cap H_2}^{H_2}(c_2)$, and $P_{H_1 \cap H_2}^{H_1} P_{H_1}(K) = P_{H_1 \cap H_2}(K) = P_{H_1 \cap H_2}^{H_2} P_{H_2}(K)$. Thus, by Lemma (2.1), $P_{H_1}^{-1}(c_1) \cap P_{H_2}^{-1}(c_2) = a + (H_1 + H_2)^{\perp}$.

Let $\Gamma = H_0 \cap (H_1 + H_2)$ and let c be the centre of symmetry of $P_\Gamma(K)$. Then $P_\Gamma^{H_0}(c_0) = c$. In order to prove our lemma, by Lemma (2.1), it is enough to prove that $P_\Gamma^{H_1+H_2}(a) = c$, because $P_{H_1+H_2}^{-1}(a) = P_{H_1}^{-1}(c_1) \cap P_{H_2}^{-1}(c_2)$.

For $i = 1, 2$, $P^{\Gamma}_{\Gamma\cap H_i}P^{H_1+H_2}_{\Gamma}(a) = P^{H_i}_{\Gamma\cap H_i}P^{H_1+H_2}_{H_i}(a) = P^{H_i}_{\Gamma\cap H_i}(c_i)$. Moreover, by (1.6), $P^{H_i}_{\Gamma\cap H_i}(c_i)$ is the centre of symmetry of $P^{H_i}_{\Gamma\cap H_i}(P_{H_i}(K)) = P_{\Gamma\cap H_i}(K)$. Since $P^{\Gamma}_{\Gamma\cap H_i}(c)$ is also the centre of symmetry of $P_{\Gamma\cap H_i}(K)$, we have that $P^{\Gamma}_{\Gamma\cap H_i}(P^{H_1+H_2}_{\Gamma}(a)) = P^{\Gamma}_{\Gamma\cap H_i}(c)$. Note now that by hypothesis, $\Gamma = (\Gamma \cap H_1) + (\Gamma \cap H_2)$ and consequently, by Lemma (2.1), $P^{H_1+H_2}_{\Gamma}(a) = c$. This concludes the proof of Lemma (2.4). ∎

LEMMA (2.5). *Let $K \subset \mathbb{R}^n$ be a strictly convex body and let H_0, H_1, H_2 be subspaces of $\mathbb{R}^n$ such that $P_{H_i}(K)$ is centrally symmetric with respect to $c_i \in H_i$, $i = 0, 1, 2$. Suppose that* $\dim H_0 \leq \dim(H_0 \cap H_1) + \dim(H_0 \cap H_2)$. *Then,*

$$P^{-1}_{H_0}(c_0) \cap P^{-1}_{H_1}(c_1) \cap P^{-1}_{H_2}(c_2) \neq \phi.$$

Proof. Suppose first that $\dim(H_0 \cap H_1 \cap H_2) > 0$ and let L be a 1-dimensional subspace of $\mathbb{R}^n$ contained in $H_0 \cap H_1 \cap H_2$. Note that $P_L(K)$ is a closed interval with extreme points α and β. Since K is strictly convex, then there are unique points a and b of K such that $P_L(a) = \alpha$ and $P_L(b) = \beta$, respectively. Furthermore, since $P_{H_i}(K)$ is centrally symmetric with respect to c_i, then by (1.6), for $i = 0, 1, 2$, $\frac{P_{H_i}(a)+P_{H_i}(b)}{2} = c_i$, because the closed interval with extreme points $P_{H_i}(a)$ and $P_{H_i}(b)$ is a diametral chord of $P_{H_i}(K)$. Consequently, $P_{H_i}(\frac{a+b}{2}) = c_i$ and hence $\frac{a+b}{2} \in P^{-1}_{H_0}(c_0) \cap P^{-1}_{H_1}(c_1) \cap P^{-1}_{H_2}(c_2)$.

Suppose now that $\dim(H_0 \cap H_1 \cap H_2) = 0$. Hence, by hypothesis, (1.1) and the above, $\dim((H_0 \cap H_1) + (H_0 \cap H_2)) = \dim(H_0 \cap H_1) + \dim(H_0 \cap H_2) \geq \dim H_0 \geq \dim(H_0 \cap (H_1 + H_2))$. Moreover, since $(H_0 \cap H_1) + (H_0 \cap H_2) \subset H_0 \cap (H_1 + H_2)$, then $H_0 \cap (H_1 + H_2) = (H_0 \cap H_1) + (H_0 \cap H_2)$ and therefore, by Lemma (2.4), $P^{-1}_{H_0}(c_0) \cap P^{-1}_{H_1}(c_1) \cap P^{-1}_{H_2}(c_2)$ is not empty. ∎

3. The theorems

As an immediate consequence of Lemmas (2.3) and (2.5) we have the following Theorem

THEOREM (3.1). *Let $K \subset \mathbb{R}^n$ be a strictly convex body and let $\mathcal{X}$ be a collection of k-dimensional linear subspaces of $\mathbb{R}^n$, $k \geq \frac{2n}{3}$, satisfying the following two conditions*

i. *for every 1-dimensional linear subspace L of $\mathbb{R}^n$ there is $H \in \mathcal{X}$ such that $L \subset H$ (linear), and*

ii. *for every $H \in \mathcal{X}$, the convex body $P_H(K)$ is centrally symmetric (K-symmetric).*

Then K is centrally symmetric.

THEOREM (3.2). *Let $K \subset \mathbb{R}^n$ be a strictly convex body and let Γ be a k-dimensional subspace of $\mathbb{R}^n$. Suppose that for every $(k+1)$-dimensional subspace H of $\mathbb{R}^n$ that contains Γ, $P_H(K)$ is centrally symmetric. Then K is centrally symmetric.*

Proof. Let $\Gamma \subset H$ be a $(k+2)$-dimensional subspace of $\mathbb{R}^n$. Then the collection of all $(k+1)$-dimensional subspaces of H that contain Γ is a linear $P_H(K)$-symmetric collection of subspaces. By Theorem (3.1), since $k+1 \geq \frac{2(k+2)}{3}$, then $P_H(K)$ is centrally symmetric for every $(k+2)$-dimensional subspace H which contains Γ. Repeating the same argument inductively, we end up proving that K is centrally symmetric. ∎

THEOREM (3.3). *Let $K \subset \mathbb{R}^n$ be a convex body and let $\mathcal{X}$ be a linear collection of k-dimensional subspaces of $\mathbb{R}^n$, $k > \frac{n}{2}$, with the property that for every $H \in \mathcal{X}$ the body $P_H(K)$ has constant width. Then K has constant width.*

Proof. Let h be the diameter of K. We shall prove that for every 1-dimensional linear subspace L of $\mathbb{R}^n$ the length of $P_L(K)$ is h. Let I be a line with the property that the length of $I \cap K$ is h, L_0 the 1-dimensional subspace of $\mathbb{R}^n$ parallel to I and $H_0 \in \mathcal{X}$ such that $L_0 \subset H_0$. Note that the length of $P_{L_0}(K)$ is h and consequently, since $P_{L_0}^{H_0}(P_{H_0}(K)) = P_{L_0}(K)$, then $P_{H_0}(K)$ has constant width h. Let H_1 be any element of $\mathcal{X}$. We shall first prove that $P_{H_1}(K)$ has constant width h. Since $2k - n > 0$, then there is a line $L_1 \subset H_0 \cap H_1$. Therefore, $P_{L_1}(K)$ has length h, because $P_{H_0}(K)$ has constant width h and since $L_1 \subset H_1$, then $P_{H_1}(K)$ has constant width h. Let now L be any 1-dimensional subspace of $\mathbb{R}^n$ and by hypothesis let $H \in \mathcal{X}$ be such that $L \subset H$. By the above, $P_H(K)$ has constant width h and hence $P_L(K) = P_L^H(P_H(K))$ has length h. This concludes the proof of Theorem (3.3). ∎

As immediate consequence of Theorems (3.1) and (3.2), we have the following

THEOREM (3.4). *Let $K \subset \mathbb{R}^n$ be a convex body and let $\mathcal{X}$ be a linear collection of k-dimensional subspaces of $\mathbb{R}^n$, $k > \frac{2n}{3}$, with the property that for every $H \in \mathcal{X}$ the body $P_H(K)$ is a solid sphere. Then K is a solid sphere.*

Analogous results had been obtained in [5] for sections of bodies. In particular it was proved that the cohomology and homology of the collection of sections involved in the problem, as a subset of the corresponding Grassmanian manifold, plays an important role. As applications of our results we have the following theorems

THEOREM (3.5). *Let $K \subset \mathbb{R}^n$, $n > 2$, be a strictly convex body without corner points and suppose that for every $x \in \partial K$ there is a orthogonal projection P_H into a k-dimensional subspace H such that $P_H(x) \in \partial P_H(K)$ and $P_H(K)$ is centrally symmetric, $k > \frac{2n}{3}$ [$P_H(K)$ is a solid sphere, $k > \frac{2n}{3}$; $P_H(K)$ has* *constant width, $k > \frac{n}{2}$; respectively]. Then K is centrally symmetric [K is a solid sphere; K has constant width; respectively].*

Proof. Let $\mathscr{X}$ be a collection of k-dimensional subspaces of $\mathbb{R}^n$ with the property that for every $x \in \partial K$ there is $H \in \mathscr{X}$ such that $P_H(x) \in \partial P_H(K)$. It will be enough to prove that $\mathscr{X}$ is linear. Let L be a 1-dimensional subspace and let Γ be a support hyperplane of K through $x \in \partial K$, perpendicular to L. By hypothesis there is $H \in \mathscr{X}$ such that $P_H(x) \in \partial P_H(K)$, then $P_H(\Gamma)$ is a support hyperplane of $P_H(K)$ and therefore $H^\perp \subset L^\perp$ which implies that $L \subset H$. Consequently $\mathscr{X}$ is linear. Our theorem follows now immediately from our previous results.

In our last theorem the hypothesis of strictly convex without corner points are necessary. otherwise it is not difficult to construct contraexamples, for instance a prism (the cartesian product of a solid triangle with a closed interval).

The hypothesis $k > \frac{n}{2}$ and $k > \frac{2n}{3}$, in our theorems, are necesary, although we wonder about to what extent they are optimal. An illuminating example is the following. Suppose that K is a solid complex ellipsoid in $\mathbb{C}^2 = \mathbb{R}^4$ and $\mathscr{X}$ is the family of all complex lines through the origin which, of course, is a linear family of 2-dimensional planes. Furthermore, note that for every $H \in \mathscr{X}$, $H \cap K$ is a complex interval, that is, a 2-solid sphere, but K may not be necesarily a 4-solid sphere or neither has constant width. In fact, if $K \subset \mathbb{R}^{2+k}$ is a solid ellipsoid, then the set of 2-dimensional planes for which the orthogonal projection of K is a 2-solid sphere is unexpectedly big because if $\mathscr{U}$ is the collection of 2-planes for which the projections of K are non spherical ellipses, then the two axes of these ellipses give rise to a decomposition of the restriction of γ to $\mathscr{U}$ into two Whitney summands, where γ is the standard vector bundle of 2-planes over the Grassmannian $G(k, 2)$. For more details see [5].

The following two theorems are closely related with the Gromov's Conjecture [2] concerning affinely equivalent sections of solid bodies. For more information about this problem see, for example, [4] and [6].

THEOREM (3.6). *Let $K \subset \mathbb{R}^n$ be a convex body and suppose that for every two k-dimensional subspaces H_1 and H_2 of $\mathbb{R}^n$ the orthogonal projections $P_{H_1}(K)$ and $P_{H_2}(K)$ are affinely equivalent, $1 < k < n$, $n > 2$. Then K is centrally symmetric.*

Proof. It is not difficult to see that it is enough to prove the theorem when $k = n-1$. First note that the collection of orthogonal projections of K into all $(n-1)$-dimensional subspaces give rise to a complete turning of $P_H(K)$ (see [1] or [3] for a definition), where H is any $(n-1)$-dimensional subspace of $\mathbb{R}^n$. Then, by Lemma 2 of [4], $P_H(K)$ is centrally symmetric and hence, K is centrally symmetric. ∎

THEOREM (3.7). *Let $K \subset \mathbb{R}^n$ be a convex body and suppose that for every two k-dimensional subspaces H_1 and H_2 of $\mathbb{R}^n$ the orthogonal projections $P_{H_1}(K)$ and $P_{H_2}(K)$ are similar, $1 < k < n$, $n > 2$. Then K is a solid sphere.*

Proof. By Theorem (3.6), K is centrally symmetric. Hence, it would be enough to prove that K has constant width. Let K^* be the unique star-shaped body with the property that for every $\overline{x} \in S^{n-1}$ the point $w(\overline{x})\overline{x}$ belongs to the boundary of K^*, where $w(\overline{x})$ is the width of K in the direction $\overline{x}$. Note that for every $(n-1)$-dimensional subspace H of $\mathbb{R}^n$, the section $H \cap K^* = (P_H(K))^*$. Consequently, by hypothesis, any two sections, through the origin of K^* are similar and hence, by Theorem 3 of [4], (note that the notion of homothecy of [4] is the clasical notion of similarity), K^* is a solid sphere centered at the origin and K is a body of constant width. This concludes the proof of Theorem (3.3). ∎

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