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AN EXTENSION RESULT FOR FUNCTIONS DEFINED ON A STRAIGHT LAYER

BY JOSÉ A. CANAVATI

Introduction

Let ω be an open subset of $\mathbb{R}^n$ such that the boundary of ω satisfies some kind of regularity condition. The problem of constructing a continuous extension operator between Sobolev spaces $E: W^{k,p}(\omega) \to W^{k,p}(\mathbb{R}^n)$, originates, among others, in the works of Deny and Lions [4], for the case $p = 2$; Calderón [3], for the case $1 < p < \infty$; Adams, Aroszajn and Smith [2], for the case $p = 2$; and Stein [7], for the case $1 \leq p \leq \infty$. This last author has constructed a"universal" extension operator $E: W^{k,p}(\omega) \to W^{k,p}(\mathbb{R}^n)$ for $1 \leq p \leq \infty$ and all non-negative integral k, when the boundary of the domain ω is"minimally smooth" (cf. [7; p. 189]). This extension operator satisfies

$$\|Eu\|_{W^{k,p}(\mathbb{R}^n)} \leq C(k,n,\omega)\|u\|_{W^{k,p}(\omega)} \quad \text{if} \quad u \in W^{k,p}(\omega),$$

and is universal in the sense that E is defined for all the Sobolev spaces and the constant $C(k,n,\omega)$ does not depend on p for $1 \leq p \leq \infty$.

Now, the problem of giving explicit bounds on the norm of the operator E in terms of the"shape" of ω, is of a more difficult nature. Here we study the following particular case:

Let $\delta > 0$ be fixed, and consider the open subsets of $\mathbb{R}^n$:

$$\omega = \mathbb{R}^{n-1} \times]-\delta, 0[\quad \text{and} \quad \Omega = \mathbb{R}^{n-1} \times]-\delta, +\infty[.$$

In this note we prove that is possible to define an extension operator between the usual Sobolev spaces

$$E: W^{k,2}(\omega) \to W^{k,2}(\Omega)$$

such that its norm satisfies an estimate of the type $\|E\| \leq C\sqrt{1 + 1/\delta}$, and C is a constant depending only on n and k. We also show that this extension operator is optimal in a sense made precise in Theorem 3.5.

1. Notation and basic terminology

A general point of $\mathbb{R}^n$ will be denoted by $x = (x_1, \ldots, x_n)$. Frecuently we will write $x = (x', x_n)$, $x' = (x_1, \ldots, x_{n-1})$. If Ω is an open set in $\mathbb{R}^n$ and $0 \leq k < \infty$ is an integer, we denote by $W^{k,2}(\Omega)$ the Hilbert space consisting of all measurable functions u on Ω such that $\partial^\alpha u \in L^2(\Omega)$, $| \alpha | \leq k$, where the derivatives are taken in the sense of distributions. The inner product and norm on $W^{k,2}(\Omega)$ are given respectively by

$$(1.1)\qquad (u,v)_{k,\Omega} = \sum_{|\alpha|\le k} \binom{k}{\alpha} \int_\Omega \partial^\alpha u.\partial^\alpha \bar{v}\, dx,$$

and

$$(1.2)\qquad \|u\|_{k,\Omega} = \left\{ \sum_{|\alpha|\le k} \binom{k}{\alpha} \int_\Omega |\, \partial^\alpha u \,|^2\, dx \right\}^{1/2},$$

where

$$\binom{k}{\alpha} = \frac{k!}{(k-|\,\alpha\,|)!\alpha!}.$$

We let $W_0^{k,2}(\Omega)$ be the closure in $W^{k,2}(\Omega)$ of $D(\Omega)$, the space of all C^∞ functions on Ω with compact support contained in Ω. Define $D(\bar{\Omega})$ as the space consisting of the restrictions to Ω of functions in $D(\mathbb{R}^n)$. It is well known (cf. [5], p. 248), that if the boundary $\partial\Omega$ of Ω is of class C, then $D(\bar{\Omega})$ is dense in $W^{k,2}(\Omega)$.

From the formula of integration by parts it follows immediately that if $u \in W^{k,2}(\Omega)$ and $\phi \in D(\Omega)$, then

$$(u,\phi)_{k,\Omega} = \int_\Omega u(1-\triangle)^k \bar{\phi} dx,$$

where $\triangle = \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_n^2$ is the Laplace operator.

If $u \in L^2(\mathbb{R}^n)$, then its Fourier transform will be denoted by

$$(1.3)\qquad Fu(\xi) = \hat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi.x} u(x) dx.$$

Sometimes we will have to take the Fourier transform of u but only with respect to the first $n-1$ variables $x' = (x_1, \ldots, x_{n-1})$, and it will be written as

$$(1.4)\qquad F'u(\xi', x_n) = \frac{1}{(2\pi)^{(n-1)/2}} \int_{\mathbb{R}^{n-1}} e^{-i\xi'.x'} u(x', x_n) dx'.$$

If $s > 0$, we denote by $H^s(\mathbb{R}^n)$ the Hilbert space consisting of all functions $u \in L^2(\mathbb{R}^n)$ such that $(1+|\xi|^2)^{s/2}\hat{u} \in L^2(\mathbb{R}^n)$, together with the inner product

$$(1.5)\qquad (u,v)_{s,n} = \int_{\mathbb{R}^n} (1+|\xi|^2)^s \hat{u}(\xi)\overline{\hat{v}(\xi)}\, d\xi,$$

and associated norm

$$(1.6)\qquad |\, u\,|_{s,n} = \left\{ \int_{\mathbb{R}^n} (1+|\,\xi\,|^2)^s \,|\, \hat{u}(\xi)\,|^2\, d\xi \right\}^{1/2}.$$

As is well known, (cf. [1], [5]), $D(\mathbb{R}^n)$ is a dense subspace of $W^{k,2}(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n)$ respectively. Also, if $s = k$ an integer, then

$$W^{k,2}(\mathbb{R}^n) = H^k(\mathbb{R}^n),$$

and the norms (1.2) and (1.6) are equivalent.

2. Trace operators on half-space

Consider the Hilbert space

$$T^k = H^{k-1/2}(\mathbb{R}^{n-1}) \times H^{k-3/2}(\mathbb{R}^{n-1}) \times \ldots \times H^{1/2}(\mathbb{R}^{n-1}),$$

together with the inner product

$$(\vec{f}, \vec{g})_{T^k} = \sum_{j=0}^{k-1} (f_j, g_j)_{k-j-1/2,n-1}$$

$\vec{f} = (f_0, \ldots, f_{k-1})$, $\vec{g} = (g_0, \ldots, g_{k-1}) \in T^k$, and asociated norm

$$|\vec{f}|_{T^k} = \left\{ \sum_{j=0}^{k-1} |f_j|^2_{k-j-1/2,n-1} \right\}^{1/2}.$$

Let $\mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^n : x_n > 0\}$. Then we have the following well known result [6, p.84]: There exists a bounded linear operator

$$\vec{\gamma}: W^{k,2}(\mathbb{R}^n_+) \to T^k,$$
$$\vec{\gamma}u = (\gamma_0 u, \gamma_1 u, \ldots, \gamma_{k-1} u), u \in W^{k,2}(\mathbb{R}^n_+),$$

such that

$$\gamma_j u = \partial_n{}^j u|_{x_n=0} \ (j = 0,\ 1,\ \ldots,\ k-1), \text{ for all } u \in D(\overline{\mathbb{R}^n_+})$$

It is clear that $\vec{\gamma}\phi = \vec{0}$ for every $\phi \in D(\mathbb{R}^n_+)$, and if we use the fact that $D(\mathbb{R}^n_+)$ is dense in $W_0^{k,2}(\mathbb{R}^n_+)$, we see that $\vec{\gamma}u = \vec{0}$ for all $u \in W_0^{k,2}(\mathbb{R}^n_+)$. In fact, it can be shown (cf. [6], p. 90) that

$$W_0^{k,2}(\mathbb{R}^n_+) = \{u \in W^{k,2}(\mathbb{R}^n_+) : \gamma_j u = 0 \qquad (j = 0,\ 1,\ \ldots,\ k-1)\}.$$

The following is a well known existence, uniqueness, and continuity of the solutions (with respect to the data) result, for strong solutions of the Dirichlet problem in $\mathbb{R}^n_+$ (cf. [6]): Given $\vec{g} = (g_0, g_1, \ldots, g_{k-1}) \in T^k$, there exists a unique $u \in W^{k,2}(\mathbb{R}^n_+) \cap C^\infty(\mathbb{R}^n_+)$ such that

$$\begin{cases} (1-\triangle)^k u = 0 & \text{in } \mathbb{R}^n_+ \\ \gamma_j u = g_j & (j = 0,\ 1,\ \ldots, k-1). \end{cases} \tag{2.1}$$

Furthermore, if we let $u = Z\vec{g}$, then $Z\colon T^k \to W^{k,2}(\mathbb{R}^n_+)$ is a bounded linear operator, i,e., there is a constant $C = C(n,k)$ such that

$$\|Z\vec{g}\|_{k,\mathbb{R}^n_+} \le C|\vec{g}|_{T^k}, \quad \text{for all} \quad \vec{g} \in T^k. \tag{2.2}$$

3. An extension operator on a straight layer

First we need a lemma.

LEMMA (3.1). *Let $G = \mathbb{R}^{n-1}\times]a,b[$, $-\infty \le a < b \le \infty$. If $u,\ v \in W^{k,2}(G)$, then*

$$(u,v)_{k,G} = \int_{\mathbb{R}^{n-1}} \left\{ \int_a^b \sum_{j=0}^{k} \binom{k}{j} \sigma^{2k-2j} \partial_n{}^j F'u . \partial_n{}^j \overline{F'v}\, dx_n \right\} d\xi', \tag{3.1}$$

where $\sigma = (1 + |\ \xi'\ |^2)^{1/2}$.

Proof. Let $\alpha = (\alpha', \alpha_n)$. We have from Parseval's identity

$$\begin{aligned}(u,v)_{k,G} &= \int_{\mathbb{R}^{n-1}} \left\{ \int_a^b \sum_{|\alpha|\le k} \binom{k}{\alpha} \partial^\alpha u . \partial^\alpha \bar{v}\, dx_n \right\} dx' \\ &= \int_{\mathbb{R}^{n-1}} \left\{ \int_a^b \sum_{|\alpha|\le k} \binom{k}{\alpha} (i\xi')^{\alpha'} \partial_n^{\alpha_n} F'u . (-i\xi')^{\alpha'} \partial_n^{\alpha_n} \overline{F'v}\, dx_n \right\} d\xi'.\end{aligned}$$

Noticing that

$$\binom{k}{\alpha} = \binom{k}{\alpha_n}\binom{k-\alpha_n}{\alpha'}, \quad \alpha = (\alpha', \alpha_n),$$

and

$$\begin{aligned}\sum_{|\alpha'|\le k-\alpha_n} \binom{k-\alpha_n}{\alpha'} (\xi')^{2\alpha'} &= \sum_{l=0}^{k-\alpha_n} \binom{k-\alpha_n}{l} \sum_{|\alpha'|=l} \frac{l!}{\alpha'!} (\xi')^{2\alpha'} \\ &= \sum_{l=0}^{k-\alpha_n} \binom{k-\alpha_n}{l} (\xi_1^2 + \ldots + \xi_{n-1}^2)^l \\ &= (1 + |\xi'|^2)^{k-\alpha_n}\end{aligned}$$

we obtain

$$\sum_{|\alpha|\leq k}\binom{k}{\alpha}(\xi')^{2\alpha'}\partial_n^{\alpha_n}F'u.\partial_n^{\alpha_n}\overline{F'v}=\sum_{\alpha_n=0}^{k}\binom{k}{\alpha_n}\partial_n^{\alpha_n}F'u.\partial_n^{\alpha_n}\overline{F'v}(1+|\xi'|^2)^{k-\alpha_n},$$

and from this result follows.

PROPOSITION (3.2). *There exists a bounded linear trace operator*

$$\vec{\tau}=(\tau_0,\ \tau_1,\ \ldots,\ \tau_{k-1}):W^{k,2}(\omega)\to T^k,$$

such that

$$\tau_j u=\partial_n^j u\big|_{x_n=0}(j=0,1,\ldots,k-1),\quad \textit{for all } u\in D(\overline{\omega}),$$

and

(3.2)
$$\|\vec{\tau}\|\leq\sqrt{1+1/\delta}.$$

Proof. Let $u\in D(\overline{\omega})$, and $g_j=\tau_j u$ $(j=0,1,\ldots,k-1)$ If $h_j=F'g_j$, then

$$\begin{aligned}|h_j(\xi')|^2&=\frac{1}{\delta}\left((x_n+\delta)|\partial_n^jF'u(\xi',x_n)|^2\right)_{x_n=-\delta}^{x_n=0}\\&=\frac{1}{\delta}\int_{-\delta}^{0}\frac{d}{dx_n}\left((x_n+\delta)|\partial_n^jF'u(\xi',x_n)|^2\right)dx_n\\&=\frac{1}{\delta}\int_{-\delta}^{0}\frac{d}{dx_n}\left((x_n+\delta)\partial_n^jF'u(\xi',x_n)\partial_n^j\overline{F'u(\xi',x_n)}\right)dx_n\\&=\frac{1}{\delta}\int_{-\delta}^{0}\Big(|\partial_n^jF'u|^2+(x_n+\delta)\partial_n^{j+1}F'u.\partial_n^j\overline{F'u}\\&\quad+(x_n+\delta)\partial_n^jF'u.\partial_n^{j+1}\overline{F'u}\Big)\,dx_n.\end{aligned}\tag{3.3}$$

We want to bound g_j in $H^{k-j-1/2}(\mathbb{R}^{n-1})$, and there

$$|g_j|_{k-j-1/2,n-1}=\int_{\mathbb{R}^{n-1}}\sigma^{2k-2j-1}|h_j(\xi')|^2d\xi'.$$

Consider the contribution of the second term in (3.3); since $(x_n+\delta)/\delta\leq 1$

$$\sigma^{2k-2j-1}\frac{x_n+\delta}{\delta}|\partial_n^{j+1}F'u.\partial_n^j\overline{F'u}|\leq\frac{1}{2}|\sigma^{k-j-1}\partial_n^{j+1}F'u|^2+\frac{1}{2}|\sigma^{k-j}\partial_n^jF'u\,|^2,$$

and similarly for the third term in (3.3). Thus, setting $m = j+1$ in the terms involving $\partial_n^{j+1}F'u$ we obtain

$$|\vec{\tau}u|^2_{T^k} = \sum_{j=0}^{k-1}\int_{\mathbb{R}^{n-1}} \sigma^{2k-2j-1}|h_j(\xi')|^2 d\xi'$$
$$\leq \int_{\mathbb{R}^{n-1}}\left\{\int_{-\delta}^{0}\left(\frac{1}{\delta}\sum_{j=0}^{k-1}\sigma^{2k-2j}|\partial_n^j F'u|^2 + \sum_{m=1}^{k}\sigma^{2k-2m}|\partial_n^m F'u|^2 + \right.\right.$$
$$\left.\left.\sum_{j=0}^{k-1}\sigma^{2k-2j}|\partial_n^j F'u|^2\right)dt\right\}d\xi'.$$

The last two sums combine to yield

$$\sigma^{2k}|F'u|^2 + 2\sum_{j=1}^{k-1}\sigma^{2k-2j}|\partial_n^j F'u|^2 + |\partial_n^k F'u|^2 \leq \sum_{j=0}^{k}\binom{k}{j}\sigma^{2k-2j}|\partial_n^j F'u|^2$$

and so, reference to (3.1) shows that

$$|\vec{\tau}u|^2_{T^k} \leq \left(1 + 1/\delta\right)\|u\|^2_{k,w}.$$

REMARK (3.3). There is a class of functions $\{u_\varepsilon : \varepsilon > 0\}$ in $W^{k,2}(\omega)$ for which

(3.4) $$|\,\vec{\tau}u_\varepsilon\,|_{T^k} \geq \sqrt{\delta}[1 - 0(\varepsilon)]\|\,u_\varepsilon\,\|_{k,\omega},\ \varepsilon > 0$$

Thus, the bound in (3.2) is sharp.

To see this, fix a function $\phi \in S(\mathbb{R}^{n-1})$, the Schwartz space, and let $u_\varepsilon(x', x_n) = \phi(\varepsilon x')$. Then, letting $y = \varepsilon x'$ and $\eta = \xi'/\varepsilon$ we have

$$F'u_\varepsilon(\xi', x_n) = c_{n-1}\int_{\mathbb{R}^{n-1}} e^{-ix'.\xi'}\phi(\varepsilon x')dx'$$
$$= c_{n-1}\int_{\mathbb{R}^{n-1}} e^{-iy.\eta}\phi(y)\varepsilon^{-n+1}dy = c_{n-1}\varepsilon^{-n+1}F'\phi(\eta).$$

By (3.1) with $\partial_n{}^j F'u = 0$ for $j = 1, \ldots, k$,

(3.5) $$\|u_\varepsilon\|^2_{k,\omega} = \int_{\mathbb{R}^{n-1}}\left\{\int_{-\delta}^{0}\sigma^{2k}|F'u_\varepsilon|^2 dx_n\right\}d\xi'$$
$$= c_{n-1}^2\delta\varepsilon^{-n+1}\int_{\mathbb{R}^{n-1}}\sigma^{2k}|F'\phi(\eta)|^2 d\eta.$$

where $\sigma = (1 + |\xi'|^2)^{1/2} = (1 + \varepsilon^2|\eta|^2)^{1/2}$. Also, since $\tau_j u = 0$ ($j = 1, \ldots, k-1$) and $\tau_0 u_\varepsilon(x', 0) = \phi(\varepsilon x')$,

$$|\vec{\tau} u_\varepsilon|^2_{T^k} = c^2_{n-1}\varepsilon^{-n+1} \int_{\mathbb{R}^{n-1}} \sigma^{2k-1}|F'\phi(\eta)|^2 d\eta. \tag{3.6}$$

From (3.5) and (3.6) we get

$$1 - \delta\frac{|\vec{\tau} u_\varepsilon|^2_{T^k}}{\|u_\varepsilon\|^2_{k,\omega}} = \frac{\displaystyle\int_{\mathbb{R}^{n-1}} (\sigma^{2k} - \sigma^{2k-1})|F'\phi(\eta)|^2 d\eta}{\displaystyle\int_{\mathbb{R}^{n-1}} \sigma^{2k}|F'\phi(\eta)|^2 d\eta}. \tag{3.7}$$

Now, $\frac{\partial}{\partial t}\sigma^t = \log\sigma.\sigma^t$, so that $0 \leq \sigma^{2k} - \sigma^{2k-1} \leq \log\sigma.\sigma^{2k}$ (recall that $\sigma \geq 1$.) Define $\rho = |\eta|$. For $\rho < \varepsilon^{-1/2}$,

$$\log\sigma = \frac{1}{2}\log(1 + \varepsilon^2\rho^2) \leq \frac{1}{2}\log(1 + \varepsilon) \leq \frac{\varepsilon}{2}.$$

Thus

$$\int_{\rho<\varepsilon^{-1/2}} (\sigma^{2k} - \sigma^{2k-1})|\ F'(\eta)\ |^2 d\eta \leq \frac{\varepsilon}{2}\int_{\mathbb{R}^{n-1}} \sigma^{2k}|\ F'\phi(\eta)\ |^2 d\eta.$$

Also, since $F'\phi \in S(\mathbb{R}^{n-1})$

$$\int_{\rho>\varepsilon^{-1/2}} (\sigma^{2k} - \sigma^{2k-1})|F'(\eta)|^2 d\eta = 0(\varepsilon^m) \quad \text{for all } m = 0, 1, \ldots.$$

Therefore in (3.7) the left hand side is $0(\varepsilon)$.

Let $u \in W^{k,2}(\omega)$ be given. Then, the Dirichlet problem

$$\begin{cases} (1-\triangle)^k v = 0 & \text{in } \mathbb{R}^n_+ \\ \gamma_j v = \tau_j u & (j = 0,\ 1,\ \ldots,\ k-1). \end{cases} \tag{3.8}$$

has a solution $v \in W^{k,2}(\mathbb{R}^n_+) \cap C^\infty(\mathbb{R}^n_+)$. Define

$$Eu = \begin{cases} u & \text{in } \omega \\ v & \text{in } \overline{\mathbb{R}^n_+}, \end{cases}$$

then we have

THEOREM (3.4). *$E\colon W^{k,2}(\omega) \to W^{k,2}(\Omega)$ is an extension operator, and*

$$\|E\| \le C\sqrt{1+1/\delta}, \tag{3.9}$$

where C is a constant depending only on n and k.

Proof. From (2.2) we know that

$$\|Z\vec{g}\|_{k,\mathbb{R}^n_+} \le C|\vec{g}|_{T^k}, \quad \text{for all } \vec{g} \in T^k.$$

In particular, if $\vec{g} = \vec{\tau}u$, $u \in W^{k,2}(\omega)$, then

$$\|v\|_{k,\mathbb{R}^n_+} \le C|\vec{\tau}u|_{T^k}. \tag{3.10}$$

From (3.2) and (3.10) we obtain

$$\|v\|_{k,\,\mathbb{R}^n_+} \le C\sqrt{1+1/\delta}\|u\|_{k,\omega}.$$

Therefore

$$\begin{aligned}\|Eu\|_{k,\Omega} &\le \|u\|_{k,\omega} + \|v\|_{k,\mathbb{R}^n_+} \le \|u\|_{k,\omega} + C\sqrt{1+1/\delta}\|u\|_{k,\omega}\\ &\le (1+C\sqrt{1+1/\delta})\|u\|_{k,\omega} \le (1+C)\sqrt{1+1/\delta}\|u\|_{k,\omega},\end{aligned}$$

where $C+1$ is a constant depending only on n and k.

THEOREM (3.5). *The extension operator E is optimal in the following sense: Let $u \in W^{k,2}(\omega)$. Then, for any $w \in W^{k,2}(\mathbb{R}^n_+)$ such that $\vec{\gamma}w = \vec{\tau}u$ we have*

$$\|w\|_{k,\mathbb{R}^n_+} \ge \|v\|_{k,\mathbb{R}^n_+},$$

where $v \in W^{k,2}(\mathbb{R}^n_+)$ is the solution of the Dirichlet problem (3.8).

Proof. Let $w = v + f$. Then $f \in W_0^{k,2}(\mathbb{R}^n_+)$, and since $v \in C^\infty(\mathbb{R}^n_+)$, from (3.1) we obtain

$$\begin{aligned}(v,f)_{k,\mathbb{R}^n_+} &= \int_{\mathbb{R}^{n-1}}\left\{\int_0^\infty \sum_{j=0}^k \binom{k}{j}\sigma^{2k-2j}\partial_n^j F'v.\partial_n^j\overline{F'f}\,dx_n\right\}d\xi'\\ &= \int_{\mathbb{R}^{n-1}}\left\{\int_0^\infty \left(\sum_{j=0}^k \binom{k}{j}\sigma^{2k-2j}(-1)^j\partial_n^{2j}F'v\right)\overline{F'f}\,dx_n\right\}d\xi'\\ &= \int_{\mathbb{R}^{n-1}}\left\{\int_0^\infty (\sigma^2-\partial_n^2)^k F'v.\overline{F'f}\,dx_n\right\}d\xi' = 0.\end{aligned}$$

The last equality follows from applying the Fourier transform F' to the equation $(1-\triangle)^k v = 0$, yielding in this case $(\partial_n^2 - \sigma^2)^k F'v = 0$, where $\sigma = (1+|\xi'|^2)^{1/2}$, $\xi' = (\xi_1,\ldots,\xi_{n-1})$. From this result follows.

José A. Canavati
Centro de Investigación en Matemáticas
Apartado Postal 402
Guanajuato, Gto. 36000, México.

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