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AVERAGE OPTIMALITY IN SEMI-MARKOV CONTROL MODELS ON BOREL SPACES: UNBOUNDED COST AND CONTROLS

BY OSCAR VEGA-AMAYA

1. Introduction

We deal with semi-Markov control models (SMCMs) with Borel state and control spaces, allowing unbounded one-stage cost functions and non-compact constraint sets. The problem we are concerned with is the existence of optimal stationary policies for the (long-run) average cost (AC) criterion. Most of literature related to this problem is concentrated on the countable state case under restrictive recurrence/ergodicity assumptions and/or continuity/compactness requirements; see, e.g. [1,2,5,7,8,9,10]. However, recents works by Hernández-Lerma [3] and Hernández-Lerma and Lasserre [4] on Markov control processes provide weak conditions that ensure the existence of AC optimal sationary policies. In both works, variants of the so-called "vanishing discount factor" approach are used.

In this paper we extend the assumptions in Hernández-Lerma [3] to the context of SMCMs. We show the existence of AC optimal stationary policies under three kinds of hypothesis: a) the first one guarantees that the processes are "regular" (Assumption (2.3)); b) the second one is associated to the discounted expected cost criterion (Assumption (4.1)); c) the third one is concerned with the AC criterion itself (Assumption (5.1)).

The remainder of the paper is organized as follows. In Section 2 we introduce the SMCM and the regularity assumption; the perfomance criteria are introduced in Section 3, together with a 'Tauberian Theorem' which relates these criteria. In Section 4 we present some preliminary results on the α- discounted expected cost criterion, and Section 5 contains our main result. Finally, Section 6 contains the proof of the purely technical results stated in previous sections.

2. The semi-Markov control model

We will use the following notation. Given a Borel space S (i.e., a Borel subset of a complete and separable metric space), its Borel sigma-algebra is denoted by $\beta(S)$ and $M^+(S)$ stands for the space of real - valued nonnegative measurable functions on S.

Definition (2.1). A semi-Markov control model (SMCM) is specified by the following objects:

a. a state space X, which is assumed to be a nonempty Borel space;

b. an action (or control) space A, a nonempty Borel space;

c. a collection $\{A(x) : x \in X\}$ of nonempty Borel subsets of A. For each $x \in X$, $A(x)$ is the set of admissible actions (or controls) in the state $x \in X$. Moreover, we assume that the set $K := \{(x, a) : x \in X, a \in A(x)\}$ is a Borel subset of $X \times A$ and contains the graph of a measurable map from X to A. The set of all such maps will be denoted by F, i.e., F denotes the class of all measurable functions $f: X \to A$ such that $f(x) \in A(x)$ for all $x \in X$;

d. a transition law $Q(\ |\ )$, which is a stochastic kernel on X given K;

e. a distribution function $F(t|x, a, y)$ for every $(x, a, y) \in K \times X$, which we assume to be jointly measurable in (x, a, y) for each $t \in R$;

f. $D, d \in M^+(K)$, are the so-called cost functions.

The SMCM is interpreted as representing a controlled stochastic system for which, whenever the current state is $x \in X$ and an action $a \in A(x)$ is chosen, the following things happens: a cost $D(x, a)$ is incurred instantaneously; the next state $y \in X$ is chosen according to the probability measure $Q(\ |x, a)$; conditionally on the next state being $y \in X$, the time δ until the transition into that state occurs has the distribution function $t \to F(t|x, a, y,)$; and finally, $\delta d(x, a)$ is the cost incurred during the sojourn time in x. When the transition to the new state $y \in X$ occurs, a new action $a' \in A(y)$ is chosen and the process continues in the same way indefinitely.

Let x_n, a_n, δ_{n+1} be the state of the system after the n^{th} transition, the action chosen in that state and the corresponding sojourn (or holding) time, respectively.

Definition (2.2).

a. A control policy is a sequence $\pi = (\pi_n)$ such that, for each $n = 0, 1, \ldots, \pi_n$ is a conditional probability on $\beta(A)$ given the history $h_n = (x_0, a_0, \ldots, x_{n-1}, a_{n-1}, x_n)$ and which satisfies $\pi_n \left(A(x_n)|h_n\right) = 1$. The class of all policies is denoted by Π;

b. a policy $\pi = (\pi_n) \in \Pi$ is said to be stationary if there exist $f \in F$ such that $\pi_n(\ |h_n)$ is concentrated at $f(x_n)$ for all $n = 0, 1, \ldots$; in this case we identify π with f, and refer to F as the set of stationary policies.

Given $x \in X$ and $\pi \in \Pi$, there exists a probability space (Ω, F, P_x^π) such that

i. $P_x^\pi [x_o = x] = 1$;

ii. $P_x^\pi \left[x_{n+1} \in B|h_n, a_n\right] = Q(B|x_n, a_n)$ for all $B \in \beta(X)$, h_n and $a_n \in A(x_n)$, $n = 0, 1 \ldots$;

iii. $P_x^\pi \left[a_n \in C|h_n\right] = \pi_n(C|h_n)$ for all $C \in \beta(A)$ and h_n, $n = 0, 1, \ldots$;

iv. $P_x^\pi\left[\delta_n \le t | h_{n+1}\right] = F(t|x_n, a_n, x_{n+1})$ for all $t \in R$, h_{n+1} and $a_n \in A(x_n)$, $n = 0, 1, \ldots$;

v. The random variables $\delta_1, \delta_2, \delta_3, \ldots$ are conditionally independent given the processes $(x_o, a_o, ..., x_n, a_n, ..)$.

The expectation with respect to P_x^π is denoted by E_x^π.

To ensure that the process is regular, i.e., it has only finitely many transitions during any finite time interval, we need to impose some condition. To do this, we introduce the following notation:

$$H(t|x,a) := \int_X F\left(t \mid x, a, y\right) Q\left(dy|x,a\right) \tag{2.1}$$

and

$$\tau\left(x,a\right) := \int_0^\infty tH(dt|x,a) \tag{2.2}$$

denote the **distribution function** and the **mean holding time** in the state $x \in X$ when the action $a \in A(x)$ is chosen. We also introduce the auxiliary functions

$$\Delta_\alpha(x,a) := \int_0^\infty \exp(-\alpha t) H(dt|x,a) \tag{2.3}$$

$$\tau_\alpha(x,a) := \left[1 - \Delta_\alpha\left(x,a\right)\right]/\alpha \tag{2.4}$$

for $\alpha \in (0,1)$ and $(x,a) \in K$.

Assumption (2.3). There exist $\varepsilon > 0$ and $\theta > 0$ such that $1 - H\left(\theta|x,a\right) \ge \varepsilon$ for all $(x,a) \in K$.

Assumption (2.3) yields the regularity condition in part (c) of the following proposition, which is proved in the Appendix (Section 6).

PROPOSITION (2.4). *If Assumption (2.3) holds, then*

a. $\inf_K \tau(x,a) \ge \varepsilon\theta$;

b. $\Delta_\alpha < 1$, *where* $\Delta_\alpha := \sup_K \Delta_\alpha(x,a)$;

c. $P_x^\pi\left[\infty \sum_{n=1}^\infty \delta_n = \infty\right] = 1$ *for all* $x \in X$ *and* $\pi \in \Pi$.

3. Perfomance criteria

We define the "one-stage cost" functions as

$$C_\alpha(x,a) := D(x,a) + \tau_\alpha(x,a)d(x,a) \tag{3.1}$$

$$C(x,a) := D(x,a) + \tau(x,a)d(x,a) \tag{3.2}$$

for $\alpha \in (0,1)$ and $(x,a) \in K$, and let

$$Z_n := \sum_{k=0}^{n-1} C(x_k, a_k) \tag{3.3}$$

$$T_n := T_{n-1} + \delta_n \text{ for } n = 1,2,\dots \text{ and } T_o \equiv 0 \tag{3.4}$$

for any policy $\pi \in \Pi$ and initial state $x_o = x \in X$, let

$$\phi(\pi,x) := \limsup_n \left[E_x^\pi T_n\right]^{-1} E_x^\pi Z_n \tag{3.5}$$

be the **expected average cost,** and

$$V_\alpha(\pi,x) := E_x^\pi \sum_{n=0}^{\infty} \exp(-\alpha T_n) C_\alpha(x_n, a_n) \tag{3.6}$$

the α-**discounted expected cost,** $0 < \alpha < 1$.

The functions

$$\phi(x) := \inf_\pi \phi(\pi,x) \text{ and } V_\alpha(x) := \inf_\pi V_\alpha(\pi,x) \tag{3.7}$$

are the **optimal average cost** and α-**discounted cost,** respectively, when the initial state is $x \in X$. A policy $\pi \in \Pi$ is said to be **average cost optimal** (ACO) if $\phi(x) = \phi(\pi,x)$ for all $x \in X$, and similary for the α-discounted case.

The two perfomance criteria are related by the following "Tauberian Theorem", which is proved in the Appendix (Section 6).

LEMMA (3.1). *Let $\{c_n : n = 0,1,\dots\}$ be a sequence of nonnegative numbers and $\{b_n : n = 0,1,\dots\}$ a sequence of positive numbers such that*

$$0 < \limsup_n n^{-s} b_n < \infty \tag{3.8}$$

for some real number $s \geq 1$. Then

$$\limsup_{\beta \uparrow 1} (1-\beta) \sum_{n=0}^{\infty} \beta^n c_n \leq \limsup_n b_n^{-1} S_n \tag{3.9}$$

where $S_n := \sum_{k=0}^{n-1} c_k$, $n = 1, 2, \ldots,$ *and* $S_o = 0$.

Remark (3.2).

a. Since $\Delta_\alpha(x,a) \uparrow 1$, uniformly in $(x,a) \in K$ when $\alpha \downarrow 0$, then: i. $\Delta_\alpha := \sup_K \Delta_\alpha(x,a) \uparrow 1$; ii) the functions $C_\alpha(x,a)$ and $\tau_\alpha(x,a)$ converge increasingly and uniformly to $C(x,a)$ and $\tau(x,a)$, respectively, as $\alpha \downarrow 0$;

b. Observe that

$$V_\alpha(\pi,x) = E_x^\pi\left\{C_\alpha(x_o,a_o) + \sum_{n=1}^{\infty} \Delta_\alpha(x_o,a_o)\ldots\Delta_\alpha(x_{n-1},a_{n-1})C_\alpha(x_n,a_n)\right\}$$

for all $\pi \in \Pi$ and $x \in X$. Thus, from Remark (3.2)(a),

$$\limsup_{\alpha\downarrow 0}(1-\Delta_\alpha)V_\alpha(\pi,x) \le \limsup_{\alpha\downarrow 0}(1-\Delta_\alpha)\sum_{n=0}^{\infty}\Delta_\alpha^n E_x^\pi C(x_n,a_n);$$

hence:

c. Taking $c_n = E_x^\pi C(x_n,a_n)$ and $b_n = E_x^\pi T_n$ in Lemma (3.1), we see that

$$\limsup_{\alpha\downarrow 0}(1-\Delta_\alpha)V_\alpha(\pi,x) \le \phi(\pi,x) \tag{3.10}$$

if the condition in (3.8) holds for some real $s \ge 1$.

d. On the other hand, note that

$$b_n = E_x^\pi T_n = E_x^\pi \sum_{k=0}^{n-1} \tau(x_k,a_k)$$

for all $x \in X$ and $\pi \in \Pi$, $n = 1, 2, \ldots$ Thus, if there exists $M > 0$ such that $\tau(x,a) \le M$ for all $(x,a) \in K$, from Proposition (2.4)(a), then $\varepsilon\theta \le n^{-1}b_n \le M$. Hence, (3.8) holds with $s = 1$.

4. The discounted case

In this section we consider an arbitrary but fixed discount factor $\alpha \in (0,1)$ and provide conditons under which there exists an optimal stationary policy for the α-discounted expected cost criterion. To guarantee that F contains suitable "minimizers", we require the following (semi) continuity and compactness assumption, which is of commonly use in the related stochastic control literature.

Assumption (4.1).

a. $D(x,a)$ and $d(x,a)$ are nonnegative, lower semi-continuous (1.s.c) functions in $a \in A(x)$ for each $x \in X$;

b. $\int v(y)Q(dy|x,a)$ is l.s.c in $a \in A(x)$ for every $x \in X$ and every bounded function $v \in M^+(X)$;

c. $F(t|x,a,y)$ is continuous in $a \in A(x)$ for every $x, y \in X$ and $t \in R$;

d. the set $A(r,x) := \{a \in A(x) : D(x,a) \leq r\}$ is compact for every $r \in R$ and $x \in X$.

The next proposition is proved in the Appendix (Section 6).

PROPOSITION (4.2). *If Assumption (4.1) holds, then for every $x \in X$:*

a. *$H(t|x,a)$ is continuous in $a \in A(x)$ for all $t \in R$.*

b. *the functions $\tau_\alpha(x,a)$, $\Delta_\alpha(x,a)$ and $\tau(x,a)$ are continuous in $a \in A(x)$;*

c. *$C_\alpha(x,a)$ and $C(x,a)$ are l.s.c in $a \in A(x)$;*

d. *for any two sequences $\{a_n\} \subset A(x)$ such that $a_n \to a \in A(x)$ and $\{\alpha_n\} \subset (0,1)$ such that $\alpha_n \downarrow 0$ as $n \to \infty$,*

$$\liminf_n C_{\alpha_n}(x,a_n) \geq C(x,a).$$

Remark (4.3). Note that Proposition (4.2) implies that the following sets

$$\left\{a \in A(x) : \ C_\alpha(x,a) + \Delta_\alpha(x,a)\int v(y)Q(dy|x,a) \leq r\right\}$$

$$\left\{a \in A(x) : C(x,a) + \int v(y)Q(dy \mid x,a) \leq r\right\}$$

are both compact for all $x \in X$, $r \in R$, $v \in M^+(X)$ and $\alpha \in (0,1)$.

The following Measurable Selection Theorem will be repeatedly used below. For a proof see [3] and references therein.

LEMMA (4.4). *If Assumption (4.1) holds, then the functions on X defined by*

$$v^*(x) := \inf_{a\in A(x)} \left\{C_\alpha(x,a) + \Delta_\alpha(x,a)\int v(y)Q(dy|x,a)\right\}$$

$$u^*(x) := \inf_{a\in A(x)} \left\{C(x,a) + \int u(y)Q(dy|x,a)\right\}$$

are measurable, and there exist f, $f^* \in F$ such that

$$v*(x) = C_\alpha(x,f(x)) + \Delta_\alpha(x,f(x))\int v(y)Q)dy|x,f(x))$$

$$u^*(x) = C(x,f^*(x))\int u(y)Q(dy|x,f^*(x))$$

for all $x \in X$.

The next theorem provides a solution to the α-discounted problem in (3.6) and (3.7).

THEOREM (4.5). *Suppose that Assumptions (2.3) and (4.1) hold. If* $V_\alpha(x) < \infty$ *for every* $x \in X$ *then*

a. $V_\alpha(\,)$ *satisfies the equation*

$$V_\alpha(x) = \min_{a \in A(x)} \left\{ C_\alpha(x,a) + \Delta_\alpha(x,a) \int V_\alpha(y) Q(dy|x,a) \right\}, \quad x \in X;$$

b. *there exists* $f \in F$ *such that*

$$V_\alpha(x) = C_\alpha\left(x, f(x)\right) + \Delta_\alpha\left(x, f(x)\right) \int V_\alpha(y) Q\left(dy|x, f(x)\right)$$

for all $x \in X$;

c. *any stationary policy* f *as in b) is* α*-discounted optimal.*

Proof. The proof of this theorem can be obtained using the some arguments provided in [3,4] or [6]. ∎

5. The average case

In this section we state our main result. The following notation and Assumption (5.1) are adapted from [3] to our present context. (Related papers are referred to in [3]).

For each $x \in X$ and $\alpha \varepsilon (0,1)$ let

$$m_\alpha := \inf_{x \in X} V_\alpha(x) \tag{5.1}$$

$$g^L := \liminf_{\alpha \downarrow 0} (1 - \Delta_\alpha) m_\alpha \qquad g^U := \limsup_{\alpha \downarrow 0} (1 - \Delta_\alpha) m_\alpha \tag{5.2}$$

$$g_\alpha(x) := V_\alpha(x) - (1 - \Delta_\alpha) m_\alpha \tag{5.3}$$

Note that $g_\alpha(\,)$ is nonnegative.

Assumption (5.1).

a. There exists $M > 0$ such that $\tau(x,a) \leq M$ for all $(x,a) \in K$;

b. there exists $\rho > 0$ such that $\sup\{g_\alpha(x) : 0 < \alpha < \rho\} < \infty$ for every $x \in X$;

c. there exist x^* and π^* such that $\phi(\pi^*, x^*) < \infty$.

Remark (5.2).

a. From Assumption (5.1)(b), we see that $V_\alpha(\,) < \infty$ for all $\alpha \in (0, \rho)$;

b. From Assumption (5.1)(a), (3.10) and Remark (3.2)(d), the following chain of inequalities holds for all $x \in X$ and $\pi \in \Pi$,

$$g^L \leq \limsup_{\alpha \downarrow 0} (1 - \Delta_\alpha) V_\alpha (\pi, x) \leq \phi (\pi, x) \tag{5.4}$$

and therefore:

c. From Assumption (5.1)(c) we see that

$$g^L \leq g^U \leq j^* < \infty \tag{5.5}$$

where $j^* := \inf_\pi \phi (\pi, x)$.

We state next our main theorem.

THEOREM (5.3). *Suppose that Assumption (2.3), (4.1) and (5.1) hold. Then*

a. *there exists $g \in M^+(X)$ such that*

$$g(x) \geq \min_{a \in A(x)} \left\{ C(x,a) + \int g(y) Q(dy|x,a) - g^L \tau(x,a) \right\} \forall x \in X; \tag{5.6}$$

b. *there exists $f^* \in F$ such that*

$$g(x) \geq C(x, f^*(x)) + \int g(y) Q(dy|x, f^*(x)) - g^L \tau(x, f^*(x)) \qquad \forall x \in X; \tag{5.7}$$

c. $g^L \geq \phi(f^*, x)$; *hence, from (5.5),* $\phi(f^*, x) = g^L = g^U = j^* \quad \forall x \in X$.

In order to prove this theorem we need a preliminary results:

LEMMA (5.4). *For each $x \in X$ sequence $\alpha_n \downarrow 0$,*

a. *there exists a sequence $\{a_n\} \subset A(x)$ such that*

$$V_{\alpha_n}(x) = C_{\alpha_n}(x, a_n) + \Delta_{\alpha_n}(x, a_n) \int V_{\alpha_n}(y) Q(dy|x, a_n) \qquad \forall n \geq 0 \tag{5.8}$$

b. *if $\{a_n\} \subset A(x)$ satisfies (5.8), then it has an accumulation point $a \in A(x)$.*

Proof. The property in (a) is an immediate consequence of Lemma (4.4) and Theorem (4.5). To prove (b) note that the equation (5.8) can also be written as

$$[1 - \Delta_{\alpha_n}(x, a_n)][1 - \Delta_{\alpha_n}] m_{\alpha_n} + g_{\alpha_n}(x) = C_{\alpha_n}(x, a_n) + \Delta_{\alpha_n}(x, a_n) \int g_{\alpha_n}(y) Q(dy|x, a_n) \tag{5.9}$$

for each $n = 0, 1, 2, \ldots$. Since $\alpha_n < 1$

$$(5.10)\quad [1 - \Delta_{\alpha_n}(x, a_n)] \le [1 - \Delta_{\alpha_n}(x, a_n)]/\alpha_n = \tau_{\alpha_n}(x, a_n) \le \tau(x, a_n) \le M,$$

we obtain

$$(5.11)\quad C_{\alpha_n}(x, a_n) + \Delta_{a_n}(x, a_n) \int g_{\alpha_n}(y) Q(dy|x, a_n) \le [1 - \Delta_{\alpha_n}] m_{\alpha_n} M + g_{\alpha_n}(x)$$

for each $n = 0, 1, \ldots$. Define

$$(5.12)\qquad h^* := \liminf_n (1 - \Delta_{\alpha_n}) m_{\alpha_n} \text{ and } h(y) := \liminf_n g_{\alpha_n}(y), \quad y \in X.$$

Then given $\varepsilon > 0$ there exist N and a sequence $\{n_i\}$ such that

$$(5.13)\qquad (1 - \Delta_{\alpha_{n_i}}) m_{\alpha_{n_i}} + g_{a_{n_i}}(x) \le h(x) + h^* M + \varepsilon$$

for all $n_i \ge N$.

Let $r := h(x) + h^*M + \varepsilon$ and $H_n(x) = \inf_{k \ge n} g_{\alpha_k}(x)$. Since $H_n(\quad) \le h(\quad)$ for all $n = 0, 1, \ldots$, from the inequalities (5.11) and (5.13) we get

$$(5.14)\quad C_{\alpha_{n_i}}(x, a_{n_i}) + \Delta_{a_{n_i}}(x, a_{n_i}) \int H_{\alpha_{n_i}}(y) Q(dy|x, a_{n_i}) \le r \qquad \forall n_i \ge N.$$

For i such that $n_i \ge N$ we define the following sets

$$(5.15)\quad D_i(x) := \left\{ \in A(x) : C_{\alpha_{n_i}}(x, a) + \Delta_{\alpha_{n_i}}(x, a) \int H_{\alpha_{n_i}}(y) Q(dy|x, a) \le r \right\}.$$

From Remark (4.3) and (5.14) we see that the $D_i(x)$ are nonempty and compacts sets. Since $\forall (x, a) \in K$, $C_{\alpha_{n_i}}(x, a) \uparrow C(x, a)$ and $\Delta_{\alpha_{n_i}}(x, a) H_{\alpha_{n_i}}(\quad) \uparrow g(\quad)$ as n_i tends to ∞, the sets $D_i(x)$ form a nonincreasing sequence of nonempty compacts susbsets of $A(x)$ converging to the nonempty compact set

$$D(x) := \left\{ a \in A(x) : C(x, a) + \int g(y) Q(dy|x, a) \le r \right\}$$

Therefore, there exists a $a \in A(x)$ and a subsequence of $\{n_i\}$, which we denote by $\{n_i\}$ again, such that $a_{n_i} \to a$ as n_i tends to ∞. ∎

Proof of theorem (5.3). Let first note that part (a) implies (b) and (c). Indeed, if (5.6) holds, then (b) follows from Lemma (4.4). Now if (b) holds, iteration of (5.7) yields

$$g(x) + g^L E_x^{f^*} \sum_{k=0}^{n-1} \tau(x_k, a_k) \ge E_x^{f^*} \left[\sum_{k=0}^{n-1} C(x_k, a_k) + g(x_n) \right].$$

Thus, since $g(\ )$is a nonnegative real-valued function, we obtain

$$g(x) + g^L E_x^{f^*} \sum_{k=0}^{n-1} \tau(x_k, a_k) \geq E_x^{f^*} \sum_{k=0}^{n-1} C(x_k, a_k)$$

or equivalently,

$$g(x) + g^L E_x^{f^*} T_n \geq E_x^{f^*} Z_n,$$

which implies that $g^L \geq \phi(f^*, x)$. Hence, (c) follows from (5.5).

Now we will prove part (a). Let g^L be as in (5.2), and let $\alpha_n \downarrow 0$ be such that $g^L = \lim_n (1 - \Delta_{\alpha_n}) m_{a_n}$. The Lemma (5.4) guarantees the existence of $a_x \in A(x)$ and a sequence n_i such that $a_{n_i} \to a_x$ as $n_i \to \infty$. Moreover, the equation (5.9) and the first two inequalities in (5.10), substituing n by n_i, yields (as in (5.11))

$$C_{\alpha_{n_i}}(x, a_{n_i}) + \Delta_{\alpha_{n_i}}(x, a_{n_i}) \int g_{\alpha_{n_i}}(y) Q(dy|x, a_{n_i}) \leq$$

$$(1 - \Delta_{\alpha_{n_i}}) m_{a_{n_i}} \tau(x, a_{n_i}) g_{\alpha_{n_i}}(x). \tag{5.16}$$

We define on X the following real-valued functions (see Assumption (5.1)(b))

$$G_k(x) := \inf_{n_i \geq k} g_{\alpha_{n_i}}(x) \quad \text{and} \quad g(x) := \lim_k G_k(x),$$

then, taking $\varepsilon > 0$ and using similar arguments in the proof of Lemma (5.4)(b), we get

$$C(x, a_x) + \int G_k(y) Q(dy|x, a_x) \leq g(x) + g^L \tau(x, a_x) + \varepsilon;$$

thus, by the Monotone Convergence Theorem,

$$C(x, a_x) + \int g(y) Q(dy|x, a_x) \leq g(x) + g^L \tau(x, a_x) + \varepsilon.$$

Since $x \in X$ and $\varepsilon > 0$ are arbitrary, we conclude that (5.6) holds. ■

Remark (5.5). Note that the boundedness hypothesis of $\tau(x, a)$ on Assumption (5.1)(a) has an important role in the proof of Lemma (5.4). (b), which in turn is essential for the proof of Theorem (5.3). This hypothesis can be weakened as long as the (Tauberian) Lemma (3.1) holds. For example, if either $D(x, a)$ is bounded or the sets $A(x)$, $x \in X$ are compacts, then the conclusion of Lemma (5.4) (b) is obvious. However, we are unaware of conditions easy to verify the Tauberian Lemma (3.1) when $\tau(x, a)$ is unbounded.

Appendix

Proof of proposition (2.4).

From (2.1) and (2.2),

$$\begin{aligned}\tau(x,a) &:= \int_0^\infty tH(dt|x,a)\\ &= \int_0^\infty [1-H(dt|x,a)]\,dt\\ &\geq \int_0^\theta [1-H(dt|x,a)]\,dt\\ &\geq \varepsilon\theta > 0, \qquad \text{by Assumption (2.3).}\end{aligned}$$

Hence, $\inf_k \tau(x,a) \geq \varepsilon\theta$.

b. From (2.3) and "integration by part",

$$\begin{aligned}\Delta_\alpha(x,a) &= \int_0^\infty \exp(-\alpha t)H(dt|x,a)\\ &= \alpha \int_0^\infty \exp(-\alpha t)H(t|x,a)\,dt\\ &= \alpha\left\{\int_0^\theta \exp(-\alpha t)H(t|x,a)\,dt + \int_\theta^\infty \exp(-\alpha t)H(t|x,a)\,dt\right\}\\ &\leq (1-\varepsilon)[1-\exp(-\alpha\theta)] + \exp(-\alpha\theta) \qquad \text{(by Assumption 2.3)}\\ &= 1-\varepsilon[1-\exp(-\alpha\theta)] < 1\end{aligned}$$

and then $\Delta_\alpha = \sup_K \Delta_\alpha(x,a) < 1$.

c. Let $x \in X$ and $\pi \in \Pi$ be arbitrary. From the conditional independence of sequence $\{\delta_n : n = 0, 1, \ldots,\}$ (see (v) in Section 2),

$$\begin{aligned}&E_x^\pi\,\{\exp(-\alpha \textstyle\sum_{n=1}^\infty \delta_n)|\{(x_n,a_n)\},\, n=0,1,\ldots\}\\ &\quad = \prod_{n=1}^\infty E_x^\pi\,\{\exp(-\alpha\delta_n)|\{(x_n,a_n)\} : n=0,1,\ldots\}\}\\ &\quad = \prod_{n=1}^\infty \int_0^\infty \exp(-\alpha t)H(dt/x_n,a_n)\\ &\quad = \prod_{n=1}^\infty \Delta_\alpha(x_n,a_n) = 0,\end{aligned}$$

where the last equality follows from part (b). Hence, $\sum_{n=1}^\infty \delta_n = \infty$, P_x^π a.s. ∎

Proof of lemma (3.1).

Note that if $\limsup_n b_n^{-1}S_n = \infty$ the conclusion is obvious. In order to prove this lemma suppose that $\limsup_n b_n^{-1}S_n < \infty$ and consider the following facts:

a. $\sum_{n=0}^{\infty} \beta^n c_n = (1-\beta)\sum_{n=0}^{\infty} \beta^n S_{n+1}$ for all $\beta \in (0,1)$. This follows from

$$\sum_{n=0}^{N} \beta^n c_n = \beta^N S_{N+1} + (1-\beta)\sum_{n=0}^{N-1} \beta^n S_{n+1} \text{ for all } N \geq 0,$$

and, by (3.8),

$$\beta^N S_{N+1} = \{b_{N+1}\}^{-1} S_{N+1} b_{N+1} N^{-s} \beta^N N^s \to 0 \quad \text{as} \quad N \to \infty;$$

b. the series $\sum_{n=0}^{\infty}(n+1)^s\beta^n$ is convergent; thus,

$$\sum_{n=N}^{\infty}(n+1)^s\beta^n \to 0 \qquad \text{as} \quad N \to \infty;$$

c. note that $\sum_{n=N}^{\infty} \beta^n S_{n+1} = \sum_{n=N}^{\infty}\{b_{n+1}\}^{-1}S_{n+1}b_{n+1}(n+1)^{-s}(n+1)^s\beta^n$; thus

$$\sum_{n=N}^{\infty} \beta^n S_{n+1} \leq \left\{\sup_{n\geq N+1} n^{-s}b_n\right\}\left\{\sup_{n\geq N+1} b_n^{-1}S_n\right\}\left\{\sum_{n=N}^{\infty}(n+1)^s\beta^n\right\}$$

for all $N \geq 1$. Now, from (a) and (c) we see that the inequality

$$(1-\beta)\sum_{n=0}^{\infty}\beta^n c_n \leq (1-\beta)\times$$

$$\left[\sum_{n=0}^{N-1}\beta^n S_{n+1} + \left\{\sup_{n\geq N+1} n^{-s}b_n\right\}\left\{\sup_{n\geq N+1} b_n^{-1}S_n\right\}\left\{\sum_{n=N}^{\infty}(n+1)^s\beta^n\right\}\right]$$

holds for all $N \geq 1$. On the other hand, from (b) and (3.8), there exist N^* such that

$$\sum_{n=N}^{\infty}(n+1)^s\beta^n < \left\{(1-\beta)\limsup_n n^{-s}b_n\right\}^{-1}$$

for all $N \geq N^*$. Combining the last two inequalities, we obtain

$$(1-\beta)\sum_{n=0}^{\infty}\beta^n c_n \leq$$

$$(1-\beta)\sum_{n=0}^{N-1}\beta^n S_{n+1} + \left\{\sup_{n\geq N+1} n^{-s}b_n\right\}\left\{\sup_{n\geq N+1} b_n^{-1}s_n\right\}\left\{\limsup_n n^{-s}b_n\right\}^{-1}$$

and taking limit as $\beta \uparrow 1$ and $N \to \infty$, in this order, we see that

$$\limsup_{\beta\uparrow 1}(1-\beta)\sum_{n=0}^{\infty}\beta^n c_n \leq \sup_{n\geq N+1} b_n^{-1}s_n$$

and conclude that

$$\limsup_{\beta\uparrow 1}(1-\beta)\sum_{n=0}^{\infty}\beta^n c_n \leq \limsup_n b_n^{-1}S_n.$$

■

Proof of proposition (4.2).

a. This part is an immediate consequence of Assumption (4.1).

b. First we will prove that $\Delta_\alpha(x,a)$ is continuous on $a \in A(x)$. From (2.3),

$$\Delta_\alpha(x,a_n) = \int_0^\infty \exp(-\alpha t)H(dt|x,a_n) = \alpha\int_0^\infty \exp(-\alpha t)H(t|x,a_n)dt$$

where the last equality is obtained by "integration by parts". Since $\exp(-\alpha t)$ is integrable on $[0,\infty)$ and $\exp(-\alpha t)H(t|x,a,) \leq \exp(-\alpha t)$, the Dominated Convergence Theorem and part(a) yield

$$\begin{aligned}\lim_n \Delta_\alpha(x,a_n) &= \alpha\int_0^\infty \lim_n \exp(-\alpha t)H(t|x,a_n)\,dt\\ &= \alpha\int_0^\infty \exp(-\alpha t)H(t|x,a)dt = \Delta_\alpha(x,a).\end{aligned}$$

Finally observe that the continuity of τ_α follows from the continuity of Δ_α, whereas the continuity of τ results from the uniform convergence of τ_α to τ (Remark (3.2)(a)).

c. This follows from part (b) and Assumption (4.1)(a).

d. Since $C_{\alpha_n}(x,a) \geq C_{\alpha_k}(x,a)$ for all $n \geq k$ (see Remark (3.2)(a)) and $(x,a) \in K$, we see that

$$\liminf_n C_{\alpha_n}(x,a_n) \geq \liminf_n C_{\alpha_k}(x,a_n);$$

thus, form part (c),

$$\liminf_n C_{\alpha_n}(x,a_n) \geq \liminf_n C_{\alpha_k}(x,a);$$

and letting $k \to \infty$,

$$\liminf_n C_{\alpha_n}(x,a_n) \geq C(x,a).$$

■

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DEPARTAMENTO DE MATEMÁTICAS
UNIVERSIDAD DE SONORA
HERMOSILLO, SONORA
MÉXICO

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