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STABILITY OF MAPPINGS BETWEEN FOLIATED MANIFOLDS

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The problem of stability of mappings between manifolds equipped with geometric structures of the same type was first posed by V. Poenaru in [6]. Namely, the stability of equivariant maps between compact G-manifolds is studied, G being a compact Lie group.

The definition of a concept of stability for the mappings between foliated manifolds is given by L. A. Favaro in [1]. Namely, given smooth regular foliations $\mathscr{F}_M$, $\mathscr{F}_N$ on the manifolds M, N respectively, a mapping $F \in \mathscr{C}^\infty(M, N)$ is called *stable in tangential sense* if there is a neighbourhood $V_F \subseteq \mathscr{C}^\infty(M,N)$ of F such that for each $G \in V_F$ satisfying the condition $G(x)$ and $F(x)$ belong to the same leaf of $\mathscr{F}_N$ for each $x \in M$, there are the diffeomorphisms $H \in \mathrm{Diff}^\infty(M)$, $K \in \mathrm{Diff}^\infty(N)$ taking each leaf of $\mathscr{F}_M$, respectively $\mathscr{F}_N$, onto itself such that $G = K \circ F \circ H$.

Favaro gives a proper concept of infinitesimal stability and proves the implication: infinitesimal stability implies stability (both in the tangential sense).

Later, in the paper [4], S. Izumiya shows that foliation preserving mappings $F \in \mathscr{C}^\infty_{\mathscr{F}}(M, N)$ are the natural objects for which Favaro's notation of stability in the tangential sense is defined and for a such mapping F the equivalence: F stable $\iff$ F infinitesimally stable is proved.

Denoting by $\mathscr{C}^\infty(M, N, F) = \{G \in \mathscr{C}^\infty(M,N), F(x)$ and $G(x)$ belong to the same leaf of $\mathscr{F}_N$ for each $x \in M\}$, S. Izumiya observes in the same paper that Favaro's stability concept for $F \in \mathscr{C}^\infty(M, N)$ is a stability in the space $\mathscr{C}^\infty(M, N, F)$. If $F \in \mathscr{C}^\infty_{\mathscr{F}}(M, N)$, then $\mathscr{C}^\infty(M,N,F) \subseteq \mathscr{C}^\infty_{\mathscr{F}}(M, N)$. In this case, denoting by $f\colon M|_{\mathscr{F}_M} \to N|_{\mathscr{F}_N}$ the mapping between the spaces of leaves defined by F, it is obvious that each mapping $G \in \mathscr{C}^\infty(M, N, F)$ induces *the same* mapping f between the spaces of leaves.

Thus the Favaro-Izumiya stability concept is the stability of (F, f) in the category of commutative diagrams of the following type:

$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ \downarrow{\scriptstyle P_M} & & \downarrow{\scriptstyle P_N} \\ M|\mathscr{F}_M & \xrightarrow{f} & N|\mathscr{F}_N \end{array}$$

Now we propose a more general concept of stability for a mapping $F \in \mathscr{C}^\infty_{\mathscr{F}}(M, N)$, namely the stability in $\mathscr{C}^\infty_{\mathscr{F}}(M, N)$.

Definition 1. Let $(M, \mathscr{F}_M)$, $(N, \mathscr{F}_N)$ be $\mathscr{C}^\infty$-foliated manifolds and let $F \in \mathscr{C}^\infty_{\mathscr{F}}(M, N)$. We say that F is *$\mathscr{C}^\infty$-$\mathscr{F}$-stable* if there is a neighbourhood $V_F \subseteq \mathscr{C}^\infty_{\mathscr{F}}(M, N)$ of F in the $\mathscr{C}^\infty$-fine topology such that for every $G \in V_F$ there is a $\mathscr{C}^\infty$-diffeomorphism $H \in \mathrm{Diff}^\infty(M) \cap \mathscr{C}^\infty_{\mathscr{F}}(M, M) = \mathrm{Diff}^\infty_{\mathscr{F}}(M)$ and a $\mathscr{C}^\infty$-diffeomorphism $K \in \mathrm{Diff}^\infty(N) \cap \mathscr{C}^\infty_{\mathscr{F}}(N,N) = \mathrm{Diff}^\infty_{\mathscr{F}}(N)$ such that $G = K \circ F \circ H$.

REMARKS

1. If, for a mapping $F \in \mathscr{C}^\infty_{\mathscr{F}}(M,N)$ denoted by a capital letter, we denote by the corresponding small letter the associated mapping between the spaces of leaves, the condition from the previous definition implies that for every $G \in V_F$ we have $g = k \circ f \circ h$.
2. If we require in Definition 1 that $h = \mathrm{id}_{M|\mathscr{F}_M}$; i.e. H takes each leaf of $\mathscr{F}_M$ onto itself, we obtain the application of Favaro's $(\mathscr{F}_M, \mathscr{F}_N)$-stability ([1]) for a foliation preserving mapping F.
3. Denoting by Φ the following natural action of the group $\mathrm{Diff}^\infty_{\mathscr{F}}(M) \times \mathrm{Diff}^\infty_{\mathscr{F}}(M)$ on the space $\mathscr{C}^\infty_{\mathscr{F}}(M,N)$:
$$\Phi\colon \mathrm{Diff}^\infty_{\mathscr{F}}(M) \times \mathrm{Diff}^\infty_{\mathscr{F}}(N) \times \mathscr{C}^\infty_{\mathscr{F}}(M,N) \to \mathscr{C}^\infty_{\mathscr{F}}(M,N),$$
$$((H,K),F) \mapsto K \circ F \circ H,$$
we obtain that $F \in \mathscr{C}^\infty_{\mathscr{F}}(M,N)$ is a $\mathscr{C}^\infty$-$\mathscr{F}$-stable mapping iff the orbit of F under the action Φ is an open subset of $\mathscr{C}^\infty_{\mathscr{F}}(M,N)$.
4. If $\mathscr{F}_M$ and $\mathscr{F}_N$ are simple foliations, the previous concept of stability leads to an adequate concept of stability in the category of fibre bundle morphisms.

In order to give a corresponding infinitesimal concept of stability, let $T\mathscr{F}_M$ be the involutive subbundle of vectors tangent to the foliation $\mathscr{F}_M$ on M and let Q_M be the normal bundle, the quotient defined by the short exact bundle sequence:

$$O_M \longrightarrow T\mathscr{F}_M \longrightarrow TM \xrightarrow{\Pi_M} Q_M \longrightarrow O_M. \tag{1}$$

We denote by $\mathscr{C}^\infty_{\mathscr{F}}(TM)$ the following set of all $\mathscr{C}^\infty$ infinitesimal automorphisms of $\mathscr{F}_M$:

$$\mathscr{C}^\infty_{\mathscr{F}}(TM) = \{Y \in \mathscr{C}^\infty(TM), \Pi_M([X,Y]) = 0 \text{ for each } X \in \mathscr{C}^\infty(T\mathscr{F}_M)\}.$$

Let $\mathscr{C}^\infty_{\mathscr{F}}(F^*TN) = \{\widetilde{X} \in \mathscr{C}^\infty(F^*TN), \Pi_N \circ \widetilde{X}$ is locally constant along the leaves of $\mathscr{F}_M\}$.

DEFINITION 2. Let $(M,\mathscr{F}_M)$ and $(N,\mathscr{F}_N)$ be $\mathscr{C}^\infty$-foliated manifolds and let $F \in \mathscr{C}^\infty_{\mathscr{F}}(M,N)$. We say that F is *$\mathscr{C}^\infty$-$\mathscr{F}$-infinitesimally stable* if

$$tF(\mathscr{C}^\infty_{\mathscr{F}}(TM)) + \omega F(\mathscr{C}^\infty_{\mathscr{F}}(TN)) = \mathscr{C}^\infty_{\mathscr{F}}(F^*TN)$$

where the mappings

$$tF: \mathscr{C}^{\infty}_{\mathscr{F}}(TM) \to \mathscr{C}^{\infty}_{\mathscr{F}}(F^*TN),$$
$$\omega F: \mathscr{C}^{\infty}_{\mathscr{F}}(TN) \to \mathscr{C}^{\infty}_{\mathscr{F}}(F^*TN),$$

are given by $tF(\xi) = TF \circ \xi$, $\omega F(\eta) = \eta \circ F$.

Our main result is the following:

THEOREM 1. *Let $(M, \mathscr{F}_M)$ and $(N, \mathscr{F}_N)$ be foliated manifolds with M compact and $F \in \mathscr{C}^{\infty}_{\mathscr{F}}(M, N)$. Then F $\mathscr{C}^{\infty}$-$\mathscr{F}$-stable implies F is a $\mathscr{C}^{\infty}$-$\mathscr{F}$-infinitesimally stable mapping.*

Proof. We will use the same terminology and notation of [2] and [5].

The proof uses several lemmas.

First we give a generalized Malgrange Preparation theorem for the local ring of germs of basic functions on a foliated manifold.

Let $(M, \mathscr{F}_M)$ be a smooth, connected, n-dimensional foliated manifold where $\mathscr{F}_M$ is a smooth l-codimensional foliation on M. A smooth real-valued function f on M is called basic (foliated) related to $\mathscr{F}_M$ (or, shortly, $\mathscr{F}_M$-basic) if $Xf = 0$ for all $X \in \mathscr{C}^{\infty}(T\mathscr{F}_M)$. We denote by $\mathscr{C}^{\infty}_{\mathscr{F}}(M)$ the ring of these functions. For an arbitrary point $p \in M$ and $f \in \mathscr{C}^{\infty}_{p,\mathscr{F}}(M)$, let $[f]_p$ denote the germ of f at p, that is the equivalence class of f in the germ equivalence relation on $\mathscr{C}^{\infty}_{\mathscr{F}}(M)$ in p. Let $\mathscr{C}^{\infty}_{p,\mathscr{F}}(M)$ denote the set of all germs of smooth, $\mathscr{F}_M$-basic, real-valued functions defined on a neighbourhood of p.

LEMMA 1. *$\mathscr{C}^{\infty}_{p,\mathscr{F}}(M)$ is structured as a local ring by the usual addition and multiplication of functions.*

Proof. Let $\mathscr{M}_{p,\mathscr{F}}(M) = \{[f]_p \in \mathscr{C}^{\infty}_{p,\mathscr{F}}(M), f(p) = 0\}$. It is easy to see that $\mathscr{M}_{p,\mathscr{F}}(M)$ is an ideal of $\mathscr{C}^{\infty}_{p,\mathscr{F}}(M)$. Let $\mathscr{M}$ be another ideal in $\mathscr{C}^{\infty}_{p,\mathscr{F}}(M)$, and let $\mathscr{M} \supset \mathscr{M}_{p,\mathscr{F}}(M)$ and $[f]_p \in \mathscr{M} - \mathscr{M}_{p,\mathscr{F}}(M)$. Then $[\frac{1}{f}]_p$ is defined and $[\frac{1}{f}]_p \in \mathscr{C}^{\infty}_{p,\mathscr{F}}(M)$ because for every $X \in \mathscr{C}^{\infty}(T\mathscr{F}_M)$ we have locally $X(\frac{1}{f}) = -\frac{1}{f^2}X(f) = 0$. Therefore $[\frac{1}{f}]_p \cdot [f]_p = [1]_p \in \mathscr{M}$ so that $\mathscr{M} = \mathscr{C}^{\infty}_{p,\mathscr{F}}(M)$. Thus $\mathscr{M}_{p,\mathscr{F}}(M)$ is the unique maximal ideal of $\mathscr{C}^{\infty}_{p,\mathscr{F}}(M)$.

LEMMA 2. *Let $F \in \mathscr{C}^{\infty}_{\mathscr{F}}(M, N)$, $p \in M$, $q = F(p)$. Then F induces a ring homomorphism $F^*_{p,q}: \mathscr{C}^{\infty}_{q,\mathscr{F}}(N) \to \mathscr{C}^{\infty}_{p,\mathscr{F}}(N)$ given by $[f]_q \mapsto [f \circ F]_p$. Moreover, if F is locally (near p) a leaf-preserving diffeomorphism, then $F^*_{p,q}$ is an isomorphism.*

Proof. To see that $F^*_{p,q}$ is well defined, let $[f]_q \in \mathscr{C}^{\infty}_{q,\mathscr{F}}(N)$. We prove that $[f \circ F]_p \in \mathscr{C}^{\infty}_{p,\mathscr{F}}(M)$; that is, $X(f \circ F) = 0$ for every $X \in \mathscr{C}^{\infty}(T\mathscr{F}_M)$ defined on a neighbourhood of p.

But $[f]_q \in \mathscr{C}^\infty_{q,\mathscr{F}}(N)$ is equivalent to the condition $Yf = 0$, for every $Y \in \mathscr{C}^\infty(T\mathscr{F}_M)$ defined on a neighbourhood V of q in M. This last condition is equivalent to the fact that $f|_V$ is constant along the leaves. Thus it follows that $(f \circ F)|_{F^{-1}(V)}$ is constant along leaves, which is equivalent to the fact that $X(f \circ F) = 0$ for every $X \in \mathscr{C}^\infty(T\mathscr{F}_M)$ defined on $F^{-1}(V)$.

If F is a local foliation preserving diffeomorphism, then $(F^*_{p,q})^{-1} = (F^{-1})^*_{q,p}$; therefore $F^*_{p,q}$ is an isomorphism.

If we consider the mapping $\mathscr{C}^\infty_{p,\mathscr{F}}(M) \to \mathbb{R}$, $[f]_p \mapsto f(p)$, we obtain the isomorphisms $\mathbb{R} \simeq \mathscr{C}^\infty_{p,\mathscr{F}}(M)/\mathscr{M}_{p,\mathscr{F}}(M) \simeq \mathscr{C}^\infty_{q,\mathscr{F}}(N)/\mathscr{M}_{q,\mathscr{F}}(N)$. Therefore by [2], Corollary 3.5, Ch. IV, for every finite generated $\mathscr{C}^\infty_{p,\mathscr{F}}(M)$-module A, the quotient $A/\mathscr{M}_{p,\mathscr{F}}(M)A$ is a finite dimensional vector space over $\mathbb{R}$.

We now state the following version of the generalized Malgrange Preparation theorem for the local rings of germs of basic functions on a foliated manifold.

LEMMA 3. *Let $(M, \mathscr{F}_M)$ and $(N, \mathscr{F}_N)$ be smooth foliated manifolds, let $F \in \mathscr{C}^\infty_{\mathscr{F}}(M,N)$, $p \in M$, $q = F(p)$ and let A be a finitely generated $\mathscr{C}^\infty_{p,\mathscr{F}}(M)$-module. Then A is a finitely generated module over $\mathscr{C}^\infty_{q,\mathscr{F}}(N)$ (via $F^*_{p,q}$) iff $A/\mathscr{M}_{q,\mathscr{F}}(N)A$ is a finite dimensional $\mathbb{R}$-vector space.*

Proof. Since this is a local result, with a proper choice of F-adapted foliate coordinate systems in p and q we may assume that $M = \mathbb{R}^n$, $N = \mathbb{R}^n$, $p = 0 \in \mathbb{R}^m$, $q = 0 \in \mathbb{R}^n$, $\mathscr{F}_M$ is defined by $x^\alpha = a^\alpha$, $\forall \alpha = 1, \ldots, l$, $l = \operatorname{codim} \mathscr{F}_M$, $\mathscr{F}_N$ is defined by $y^{\alpha'} = b^{\alpha'}$, $\alpha' = 1, \ldots, l'$, $l' = \operatorname{codim} \mathscr{F}_N$, and F is locally given by $F(x) = (F^1(x), \ldots, F^n(x))$ with $F^{\alpha'}(x) = F^{\alpha'}(x^1, \ldots, x^l)$, for $\alpha' = 1, \ldots, l'$. Then locally $\mathscr{C}^\infty_{0,\mathscr{F}}(\mathbb{R}^m)$ and $\mathscr{C}^\infty_{0,\mathscr{F}}(\mathbb{R}^n)$ may be identified with $\mathscr{C}^\infty_0(\mathbb{R}^l)$ and $\mathscr{C}^\infty_0(\mathbb{R}^{l'})$ respectively, and $F^*_{p,q}$ defines the mapping $\widetilde{F}^*_{0,0}: \mathscr{C}^\infty_0(\mathbb{R}^{l'}) \to \mathscr{C}^\infty_0(\mathbb{R}^l)$ associated to $\widetilde{F}: \mathbb{R}^l \to \mathbb{R}^{l'}$ given by

$$\widetilde{F}(x^1, \ldots, x^l) = (F^1(x^1, \ldots, x^l), \ldots, F^{l'}(x^1, \ldots, x^l)).$$

Since for every $[f]_0 \in \mathscr{C}^\infty_{0,\mathscr{F}}(\mathbb{R}^m)$ the germ $[f \circ F]_0$ depends only of $\widetilde{F}$, it follows that A is a finitely generated module over $\mathscr{C}^\infty_{0,\mathscr{F}}(\mathbb{R}^n)$, via $\widetilde{F}^*_{0,0}$, iff A is a finitely generated module over $G_0(\mathbb{R}^{l'})$, via $\widetilde{F}^*_{0,0}$. Now, based on the generalized Malgrange Preparation theorem ([2], Th. 3.6, Ch IV), this last fact is equivalent to the fact that $A/\mathscr{M}_0(\mathbb{R}^{l'})A$ is a finite-dimensional $\mathbb{R}$-vector space, which in turn is equivalent to the fact that $A/\mathscr{M}_{0,\mathscr{F}}(\mathbb{R}^n)A$ is a finite dimensional $\mathbb{R}$-vector space, and the lemma is proved.

Now we define inductively the sequence of ideals $\mathscr{M}^k_{p,\mathscr{F}}(M)$ in $\mathscr{C}^\infty_{p,\mathscr{F}}(M)$ by letting $\mathscr{M}^1_{p,\mathscr{F}}(M)$ be $\mathscr{M}_{p,\mathscr{F}}(M)$ and, for $k \geq 2$, letting $\mathscr{M}^k_{p,\mathscr{F}}(M)$ be the ideal generated by germs of the form $[f \cdot g]_p$ where $[f]_p \in \mathscr{M}_{p,\mathscr{F}}(M)$, $[g]_p \in \mathscr{M}^{k-1}_{p,\mathscr{F}}(M)$.

Applying the same ideas from [2] used in the proof of the theorem 3.10, ChIV, we obtain:

LEMMA 4. *Let A be a finitely generated $\mathscr{C}^{\infty}_{p,\mathscr{F}}(M)$-module, $F \in \mathscr{C}^{\infty}_{\mathscr{F}}(M, N)$, $q = F(p)$ and let $l_1, \ldots, l_k$ be the elements of A. Then $\{e_i\}$, $i = 1, \ldots, k$, generates A as a $\mathscr{C}^{\infty}_{q,\mathscr{F}}(N)$-module iff $\{\eta(e_i)\}_{i=1,\ldots,k}$ generates $A/\mathscr{M}^{k+1}_{p,\mathscr{F}}(M)A$ as a $\mathscr{C}^{\infty}_{q,\mathscr{F}}(N)$-module, where $\eta: A \to A/\mathscr{M}^{k+1}_{p.\mathscr{F}}(M)A$ is the obvious projection.*

Our first objective is to show that the concept of $\mathscr{C}^{\infty}$-$\mathscr{F}$-infinitesimal stability for a foliation-preserving mapping F is, locally, a condition of finite order; i.e., if the equations which express this type of stability can be solved locally to some finite order, they can be solved for smooth data.

In the following considerations we denote by A_p the set of the germs at $p \in M$ of all sections from a set A of smooth sections of a foliated bundle over the foliated manifold $(M, \mathscr{F}_M)$.

REMARK. Based on the considerations from [7], "any locally free $\mathscr{C}^{\infty}_{\mathscr{F}}(M)$-module of finite rank is the sheaf of germs of foliated sections of a $\mathscr{F}_M$-foliated vector bundle over M". Thus $\mathscr{C}^{\infty}_{\mathscr{F}}(TM)$, $\mathscr{C}^{\infty}_{\mathscr{F}}(TN)$, and $\mathscr{C}^{\infty}_{\mathscr{F}}(F^*TN)$, are such sheafs.

DEFINITION 3. Let $p \in M$, $F \in \mathscr{C}^{\infty}_{\mathscr{F}}(M, N)$ and $q = F(p)$. The germ $[F]_p$ is called *$\mathscr{F}$-infinitesimally stable* if for every germ $[\tau]_p \in (\mathscr{C}^{\infty}_{\mathscr{F}}(F^*TN))_p$ there exists germs of vector fields $[\xi]_p \in (\mathscr{C}^{\infty}_{\mathscr{F}}(TM))_p$ and $[\eta]_q \in (\mathscr{C}^{\infty}_{\mathscr{F}}(TN))_q$ such that

$$(*) \qquad [\tau]_p = [(TF) \circ \xi]_p + [\eta \circ F]_p$$

The mapping f is called *$\mathscr{F}$-locally infinitesimally stable* at p if the germ $[F]_p$ is $\mathscr{F}$-infinitesimally stable.

We choose now the following foliated charts: (U, ϕ), $\phi(U) = U_1 \times U_2 \subseteq \mathbb{R}^{m-l} \times \mathbb{R}^l$, $\phi(p) = (x, y)$, on $(M, \mathscr{F}_M)$, based at p, and (V, ψ), $\psi(V) = V_1 \times V_2 \subset \mathbb{R}^{n-l'} \times \mathbb{R}^{l'}$, $\psi(q) = (x', y')$, on $(N, \mathscr{F}_N)$, based at q, which are F-adapted, i.e. $F(U) \subseteq V, l = \operatorname{codim} \mathscr{F}_M, l' = \operatorname{codim} \mathscr{F}_N$. We have $(F_{\phi,\psi}(x, y) = (\psi \circ F \circ \phi^{-1})(x, y) = (F_1(x, y), F_2(y)), \forall (x, y) \in U_1 \times U_2$, where $F_1 \in \mathscr{C}^{\infty}(U_1 \times U_2, V_1)$, $F_2 \in \mathscr{C}^{\infty}(U_2, V_2)$.

We will compute the equation $(*)$ in these coordinates. To do this, for the fixed Riemannian metrics g_M, g_N on M and N, let σ_M, σ_N be splittings of the exact sequences of the type (1) over the foliated manifolds $(M, \mathscr{F}_M)$ and $(N, \mathscr{F}_N)$. Via these splittings, we have

$$\mathscr{C}^{\infty}(TM) = \mathscr{C}^{\infty}(T\mathscr{F}_M) \oplus \mathscr{C}^{\infty}(Q_M),$$
$$\mathscr{C}^{\infty}(TN) = \mathscr{C}^{\infty}(T\mathscr{F}_N) \oplus \mathscr{C}^{\infty}(Q_N),$$
$$\mathscr{C}^{\infty}(F^*TN) = \mathscr{C}^{\infty}(F^*T\mathscr{F}_N) \oplus \mathscr{C}^{\infty}(F^*Q_N)$$

so that we obtain:

$$\mathscr{C}^{\infty}_{\mathscr{F}}(F^*TN) = \mathscr{C}^{\infty}(F^*T\mathscr{F}_N) \oplus \mathscr{C}^{\infty}_{\mathscr{F}}(F^*Q_N),$$

$$\mathscr{C}^{\infty}_{\mathscr{F}}(TM) = \mathscr{C}^{\infty}(T\mathscr{F}_M) \oplus \mathscr{C}^{\infty}_{\mathscr{F}}(Q_M),$$
$$\mathscr{C}^{\infty}_{\mathscr{F}}(TN) = \mathscr{C}^{\infty}(T\mathscr{F}_N) \oplus \mathscr{C}^{\infty}_{\mathscr{F}}(Q_N).$$

It is obvious that, locally,

$$\mathscr{C}^{\infty}(T\mathscr{F}_M), \mathscr{C}^{\infty}(F^*T\mathscr{F}_M)$$

are finite generated modules over $\mathscr{C}^{\infty}(M)$, $\mathscr{C}^{\infty}(T\mathscr{F}_N)$ is a finitely generated module over $\mathscr{C}^{\infty}(N)$, while $\mathscr{C}^{\infty}_{\mathscr{F}}(Q_M)$, $\mathscr{C}^{\infty}_{\mathscr{F}}(F^*Q_N)$ are finitely generated modules over $\mathscr{C}^{\infty}_{\mathscr{F}}(M)$ and $\mathscr{C}^{\infty}_{\mathscr{F}}(Q_N)$ is a finitely generated module over $\mathscr{C}^{\infty}_{\mathscr{F}}(N)$.

Let $(X_\alpha)_{\alpha=(m-l+1),\dots,m'}$, $(Y_{\alpha'})_{\alpha'=(n-l'+1),\dots,n}$ be the sets of tangent vector fields on U and V respectively, such that

$$\left(\frac{\partial}{\partial x^i}, X_\alpha\right)_{\substack{i=1,\dots,m-l,\\ \alpha=(m-l+1),\dots,m}}, \left(\frac{\partial}{\partial x'^i}, Y_{\alpha'}\right)_{\substack{i'=1,\dots,n-l',\\ \alpha'=(n-l'+1),\dots n}},$$

generate locally the modules $\mathscr{C}^{\infty}(TM)$ and $\mathscr{C}^{\infty}(TN)$ respectively. For $\tau \in \mathscr{C}^{\infty}_{\mathscr{F}}(F^*TN)$, $\xi \in \mathscr{C}^{\infty}_{\mathscr{F}}(TM)$, $\eta \in \mathscr{C}^{\infty}_{\mathscr{F}}(TN)$ we have, locally, via the previous decompositions: $\tau = (\tau_1, \tau_2)$, $\xi = (\xi_1, \xi_2)$, $\eta = (\eta_1, \eta_2)$ where $\tau_1(x,y) = \tau_1^{i'}(x,y)\frac{\partial}{\partial x'^{i'}}$, $\tau_2(x,y) = \tau^{\alpha}(y)Y_{\alpha'}$, $\xi_1(x,y) = \xi^i(x,y)\frac{\partial}{\partial x^i}$, $\xi_2(x,y) = \xi^{\alpha}(y)X_\alpha$, $\eta_1(x',y') = \eta^{i'}(x',y')\frac{\partial}{\partial x'^{i'}}$, $\eta_2(x',y') = \eta^{\alpha'}(y')Y_{\alpha'}$. Then the equation $(*)$ becomes:

(1) $$[\tau_1]_{(x,y)} = [TF_1 \circ \xi]_{(x,y)} + [\eta_1 \circ F]_{(x,y)}$$

$(**)$

(2) $$[\tau_2]_y = [TF_2 \circ \xi_2]_y + [\eta_2 \circ F_2]_y$$

We obtain the following result:

LEMMA 5. *The foliation preserving mapping F is $\mathscr{F}$-locally-infinitesimally stable at p iff the equation $(*)$ can be solved to order* $\max(l', n-l') = n'$.

Proof. First, we observe that $[F]_p$ is $\mathscr{F}$-infinitesimally stable iff $\{l_{i'}\}_{i'=1,\dots,n-l'}$, generates the $\mathscr{C}^{\infty}_q(N)$-module via $F^*_{p,q}$, $M_1^p = (\mathscr{C}^{\infty}(F^*T\mathscr{F}_N))_p/A_1$ and $\{l_{\alpha'}\}_{\alpha'=(n-l'+1),\dots,n}$ generates the $\mathscr{C}^{\infty}_{q,\mathscr{F}}(N)$-module $M_2^p = (\mathscr{C}^{\infty}_{\mathscr{F}}(F^*Q_N)_p/A_2$ where $A_1 = \{[(TF_1)\circ\xi]_p, \xi \in (\mathscr{C}^{\infty}_{\mathscr{F}}(TM))_p\}$, $A_2 = \{[(TF_2)\circ\xi_2]_p, \xi_2 \in (\mathscr{C}^{\infty}_{\mathscr{F}}(Q_M))_p\}$, $l_{i'} = pr_1(F^*(\frac{\partial}{\partial x^{i'}}))$, $l_{\alpha'} = pr_2(F^*(Y_{\alpha'}))$ with $pr_1\colon (\mathscr{C}^{\infty}(F^*T\mathscr{F}_N))_p \to M_1^p$, $pr_2\colon (\mathscr{C}^{\infty}_{\mathscr{F}}(F^*Q_N))_p \to M_2^p$ the natural projections.

Applying Theorem 3.10, Ch. IV of [2] and Lemma (4) we obtain that the germ $|F|_p$ is $\mathscr{F}$-infinitesimally stable iff the module $M_1^p/\mathscr{M}_p^{n-l'+1}(M)M_1^p$ is generated over $\mathscr{C}^{\infty}_q(N)$ by the projections of $l_1, \dots, l_{n-q}$, and the module $M_2^p/\mathscr{M}_{p,\mathscr{F}}^{l'+1}(M)M_2^p$ is generated over $\mathscr{C}^{\infty}_{q,\mathscr{F}}(N)$ by the projections of $l_{n-q'-1}, \dots, l_n$. This last statement is equivalent to solving the equation $(**)(1)$ to order $(n-l')$ and $(**)(2)$ to order l'. Thus the Lemma is proved.

In the following considerations, for a locally free $\mathcal{C}^\infty_{\mathcal{F}}(M)$-module of finite rank A, that is, a sheaf of germs of foliatesections of a $\mathcal{F}_M$-foliatevector bundle over M, we denote by $J^k_S(A)$ the following set of k-jets at p: $J^k_S(A) = \{j^k_p\sigma, \sigma \in A, p \in S\}$. Let $J^k_{\{p\}}(A) = J^k_p(A)$. By using such notation we obtain the following lemma which is just a restatement of the previous lemma.

LEMMA 6. *The mapping $F \in \mathcal{C}^\infty_{\mathcal{F}}(M,N)$ is $\mathcal{F}$-locally infinitesimally stable at p iff*

$$J^{n'}_p(\mathcal{C}^\infty_{\mathcal{F}}(F^*TN)) = (TF)_p(J^{n'}_p(\mathcal{C}^\infty_{\mathcal{F}}(TM)) + F^*_{p,q}(J^{n'}_q(\mathcal{C}^\infty_{\mathcal{F}}(TN))$$

where by $(TF)_p$ and $F^*_{p,q}$ we denote the obvious mappings into $J^{n'}_p(\mathcal{C}^\infty_{\mathcal{F}}(F^*TN))$ induced by the action of tF and ωF on vector fields.

In order to obtain a global form of $\mathcal{F}$-infinitesimal stability we give, firstly, the following two results.

LEMMA 7. *For a fixed point $q \in N$ and a finite subset $S = \{p_1, \ldots, p_k\}$ of $F^{-1}(q)$, the mapping $F \in \mathcal{C}^\infty_{\mathcal{F}}(M,N)$ is simultaneously locally infinitesimally stable at $p_1, \ldots, p_k$ iff*

$$J^{n'}_S(\mathcal{C}^\infty_{\mathcal{F}}(F^*TN)) = (TF)(J^{n'}_S(\mathcal{C}^\infty_{\mathcal{F}}(TM)) + F^*(J^{n'}_q\mathcal{C}^\infty_{\mathcal{F}}(TN)). \tag{2}$$

Proof. For S consisting of a single point this result is just that from Lemma 6. As in the single point case, the proof for general S with more than one point uses Lemma 1.4 from [2], Ch. V and Lemma 4.

LEMMA 8. *The mapping $F \in \mathcal{C}^\infty_{\mathcal{F}}(M,N)$ is $\mathcal{F}$-infinitesimally stable iff (+) for every $q \in N$ and if for every finite subset S of $F^{-1}(q)$ with no more than $n'' + 1$ points we have the relation (2) where n'' is denots the dimension of the fibre of the foliated vector bundle over N, the sheaf of sections of which is the $\mathcal{C}^\infty_{\mathcal{F}}(N)$-module $\mathcal{C}^\infty_{\mathcal{F}}(TN)$.*

Proof. By using the previous results it is straightforward that if F is $\mathcal{F}$-infinitesimally stable, then F satisfies the condition (+). We assume that F satisfies (+). Let $T^{\mathcal{F}}_p(M)$, respectively $T^{\mathcal{F}}_q(N)$, be the $\mathbb{R}$-linear subspaces of T_pM, respectively of T_qN, defined by

$$T^{\mathcal{F}}_p(M) = \{X(p), X \in \mathcal{C}^\infty_{\mathcal{F}}(TM)\} \; (T^{\mathcal{F}}_q(N) = \{Y(q), Y \in \mathcal{C}^\infty_{\mathcal{F}}(TN)\}).$$

Let $\Sigma^{\mathcal{F}}(F) = \{p \in M, \dim(T_pF)(T^{\mathcal{F}}_p(M)) < \dim T^{\mathcal{F}}_{F(p)}(N) = n''\}$. If F satisfies (+), then for each $q \in N$, the set $(Cl_M\Sigma^{\mathcal{F}}(F)) \cap F^{-1}(q) = \Sigma^{\mathcal{F}}_q(F)$ has no more than n'' points.

Indeed, since F satisfies (+), as in [2], §1, Ch. V, it results that for every finite set $S = \{p_1, \dots, p_k\} \subset F^{-1}(q)$ the subspaces $H_i = (T_{p_i}F)(T^{\mathcal{F}}_{p_i}(M))$, $i = 1, \dots, k$ are in general position as subspaces of $T^{\mathcal{F}}_q(N)$. It is easy to see that $\Sigma^{\mathcal{F}}(F) \cap F^{-1}(q)$ has no more than n'' points. Now suppose that for some $q \in N$ there are $p_1, \dots, p_{n''+1}$ the distinct points in $Cl_M(\Sigma^{\mathcal{F}}(F)) \cap F^{-1}(q)$. For each $i \in \{1, \dots, n'' + 1\}$ let $(p_{ij})_{j\geq 1}$ be a sequence of points of $\Sigma^{\mathcal{F}}(F)$ converging to p_i and let $H_i = \{z \in T^{\mathcal{F}}_q(N),\ z = \lim_{j\to\infty}(T_{p_{ij}}F)(u_{ij}),\ u_{ij} \in (T_{p_{ij}}F)^{-1}(T^{\mathcal{F}}_{F(p_{ij})}(N)) \cap T^{\mathcal{F}}_{p_{ij}}(M))\}$. We have that H_i is a vector subspace of $T^{\mathcal{F}}_q(N)$ and $\dim H_i < n''$ because $\dim(T_{p_{ij}}F)(T^{\mathcal{F}}_{p_{ij}}(M)) \cap T^{\mathcal{F}}_{F(p_{ij})}(N) < n''$ ($p_{ij} \in \Sigma^{\mathcal{F}}(F)$). Then $\operatorname{codim} H_i \geq 1$ in $T^{\mathcal{F}}_q(N)$ and $\sum_{i=1}^{n''+1} \operatorname{codim} H_i \geq n'' + 1$. Therefore $\{H_i\}_{i=1,\dots,(n''+1)}$ are not in general position in $T^{\mathcal{F}}_q(N)$. This means that there exist $\{w_i\}_{i=1,\dots(n''+1)} \subset T^{\mathcal{F}}_q(N)$ such that the equations $w_i = h_i + z$ have no solution for $h_i \in H_i$ and $z \in T^{\mathcal{F}}_q(N)$. But this is in contradiction with the hypothesis (+).

Now let $\tau \in \mathcal{C}^\infty_{\mathcal{F}}(F^*TN)$. From (2), applied with $S = \Sigma^{\mathcal{F}}_q(F)$, and the fact that the projections $J^k(\mathcal{C}^\infty_{\mathcal{F}}(TM)) \to J^k_S(\mathcal{C}^\infty_{\mathcal{F}}(TM))$, $J^k(\mathcal{C}^\infty_{\mathcal{F}}(TN)) \to J^k_S(\mathcal{C}^\infty_{\mathcal{F}}(TN))$ are onto, it follows that there exist $\xi_q \in \mathcal{C}^\infty_{\mathcal{F}}(TM)$, $\eta_q \in \mathcal{C}^\infty_{\mathcal{F}}(TN)$ such that $(tF)(\xi_q) + (\omega F)(\eta_q)|_{U_q} = \tau|_{U_q}$ for a suitable open neighbourhood U_q of $\Sigma^{\mathcal{F}}_q(F)$ in M. Since M is compact, it results that $F(Cl_M\Sigma^{\mathcal{F}}(F)\backslash U_q)$ is closed in N. Let V_q be its complement in N. Since $\Sigma^{\mathcal{F}}_q(F) \subset U_q$ it follows that $q \in V_q$. Let $\{\rho_\alpha\}_{\alpha\in I}$ be a $\mathcal{C}^\infty$-basic partition of unity which is subordinate to $\{V_q\}_{q\in N}$. The existence of such a partition subordinate to a locally finite covering by open (saturated) subsets of a foliated manifold is proved by R. Wolak in [8]. Let $\xi_1 = \sum_{\alpha\in I} (\rho_\alpha \circ F)\xi_{q(\alpha)}$, $\eta_1 = \sum_{\alpha\in I} \rho_\alpha \eta_{q(\alpha)}$, where for $\alpha \in I$ we denote by $q(\alpha)$ a point $q(\alpha) \in N$ such that $\operatorname{supp}\rho_\alpha \subset V_{q(\alpha)}$. Writing by $U = \bigcap_{\alpha\in I} (U_{q(\alpha)} \cup F^{-1}(N - \operatorname{supp}\rho_\alpha))$ we have that U is open and $(tF)(\rho_\alpha \circ F)\xi_{q(\alpha)} + (\omega F)(\rho_\alpha\eta_{q(\alpha)}) - (\rho_\alpha \circ F)\tau$ is identically 0 on $U_{q(\alpha)} \cup F^{-1}(N - \operatorname{supp}\rho_\alpha)$ so that we obtain:

$$[(tF)(\xi_1) + (\omega F)(\eta_1)]|_U = \tau|_U.$$

Since $Cl_M\Sigma^{\mathcal{F}}(F) \subseteq U$, it follows that $\tau - (tF)(\xi_1) - (\omega F)(\eta_1)$ vanishes on a neighbourhood of $Cl_M\Sigma^{\mathcal{F}}(F)$. Moreover, since $T_pF\colon T^{\mathcal{F}}_p(M) \to T^{\mathcal{F}}_{f(p)}(N)$ is onto for all $p \notin Cl_M\Sigma^{\mathcal{F}}(F)$ it follows that there exists $\xi_2 \in \mathcal{C}^\infty_{\mathcal{F}}(TM)$ such that $\tau - (tF(\xi_1) - (\omega F)(\eta_1) = (tF)(\xi_2)$. Hence $\xi = \xi_1 + \xi_2 \in \mathcal{C}^\infty_{\mathcal{F}}(TM), \eta = \eta_1 \in \mathcal{C}^\infty_{\mathcal{F}}(TN)$ and $\tau = (tF)(\xi + (\omega F)(\eta)$ and the Lemma is proved.

We shall use now the notation from [4]. Let $J^k(M, N) = \{j^k_p f, f \in \mathcal{C}^\infty_{\mathcal{F}}(U, N)$ where U is some open neighbourhood of $p\}$, let $L^k(m)$ be the Lie group consisting of k-jets of origin preserving diffeomorphisms $(\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$, $L^k(m, l) = \{j^k_0\psi \in L^k(m), \psi\colon (\mathbb{R}^{m-l} \times \mathbb{R}^l, 0) \to (\mathbb{R}^{m-l} \times \mathbb{R}^l, 0),\ \psi(x, y) =$

$(\psi_1(x,y),\psi_2(x,y)),\forall(x,y)\in\mathbb{R}^{m-l}\times\mathbb{R}^l\}$. By Prop. 2.12 from [4] it is known that the mapping source-target $(\alpha,\beta)\colon J^k_{\mathcal{F}}(M,N)\to M\times N$ defines a structure of a subbundle of the bundle $(\alpha,\beta)\colon J^k(M,N)\to M\times N$ with structure group $L^k(m,m-l)\times L^k(n,n-l')(J^k(\mathbb{R}^m,\mathbb{R}^n)\cap J^k_{\mathcal{F}}(\mathbb{R}^m,\mathbb{R}^n)\simeq\mathbb{R}^m\times\mathbb{R}^n\times J^k(m,n-l')\times J^k(l,l'))$.

The natural action of $L^k(m,m-l)\times L^k(n,n-l')$ on $J^k(m,n-l')\times J^k(l,l')$ is given by

$$((j^k\Phi))(0),(j^k\Psi)(0))\cdot(j^kF)(0)=j^k_o(\Psi\circ F\circ\Phi^{-1}).$$

Let $\mathcal{U}^k_{\mathcal{F}}$ be a good coordinate system of $J^k(M,N)$ with respect to $\mathcal{F}_N$ in the sense of [4], and let $\mathcal{U}^k_{\mathcal{F},p}$ be the associated good coordinate system of $J^k_{\mathcal{F}}(M,N)$ with respect to $\mathcal{F}_M\times\mathcal{F}_N$,

$$\mathcal{U}^k_{\mathcal{F},p}=\{J^k(U_i,V_i)\cap J^k_{\mathcal{F}}(M,N),J^k(U_i,V_i)\in\mathcal{U}^k_{\mathcal{F}}\}.$$

This system induces canonically the foliated structure $\mathcal{F}^k(M,N)$ on $\mathcal{J}^k_{\mathcal{F}}(M,N)$ by the local submersions:

$$p_i\colon J^k(U_i,V_I)\cap J^k_{\mathcal{F}}(M,N)\to\mathbb{R}^{l'}$$
$$p_i=pr_2\circ\Psi_i\circ\Pi\circ\Phi_{U_i,V_i}$$

where

$$\Phi_{U_i,V_i}\colon J^k(U_i,V_i)\cap J^k_{\mathcal{F}}(M,N)\to U_i\times V_i\times J^k(m,n-l')\times J^k(l,l')$$

is the coordinate system on $J^k_{\mathcal{F}}(M,N)$ induced by the foliated coordinate systems (U_i,Φ_i) and (V_i,Ψ_i) on $(M,\mathcal{F}_M)$ and $(N,\mathcal{F}_N)$ respectively and $\Pi\colon U_i\times V_i\times J^k(m,n-l')\times J^k(l,l')\to V_i$, $pr_2\colon\mathbb{R}^n\to\mathbb{R}^{l'}$ are the canonical projections.

DEFINITION 4. We call $\mathcal{O}\subset J^k(M,N)$ a *pseudo-orbit* in the weak sense with respect to $\mathcal{U}^k_{\mathcal{F},p}$ if $\mathcal{O}=\bigcup\limits_{i=1}\mathcal{O}_i(c^i,z^i)$ for some $c_i\in\mathbb{R}^{l'}$, $z^i\in J^k(m,n-l')\times J^k(l,l')$ where $\mathcal{O}_i(c_i,z^i)=U_i\times\Psi_i^{-1}(c^i\times\mathbb{R}^{n-l'})\times[L^k(m,m-l)\times L^k(n,n-l')]\cdot z^i$, (V_i,ψ_i) beeing a foliated coordinate system on $(N,\mathcal{F}_N)$.

REMARK. Here $[L^k(m,m-l)\times L^k(n,n-l')]\cdot z^i$ denotes the orbit of z^i by the above action in $J^k(m,n-l')\times J^k(l,l')$. If in the previous definition instead of this orbit we take the orbit $(L^k_{m-l}(m)\times L^k_{n-l'}(n))(z^i)$ with $L^k_{m-l}(m)=\{j^k_o\Psi\in L^k(n,m-l),\Psi_2(y)=0\}$, we obtain the concept of pseudo-orbit in $J^k_{\mathcal{F}}(M,N)$ given by Izumiya in [4] (Def.3.1).

We obtain the following result:

LEMMA 9. *Any pseudo-orbit $\mathcal{O}$ in the weak sense with respect to $\mathcal{U}^k_{\mathcal{F},p}$ is a local submanifolds correction in $\mathcal{F}^k(M,N)$ with respect to $\mathcal{U}^k_{\mathcal{F}},p$ (in Izumiya's sense). Moreover, if $\mathcal{O}'$ is a pseudo-orbit in the weak sense with respect to $\mathcal{U}^k_{\mathcal{F},p}$ and $z \in \mathcal{O} \cap \mathcal{O}'$ then we have $T_z\mathcal{O} = T_z\mathcal{O}'$.*

Proof. It is clear that each $\mathcal{O}_i(c^i, z^i)$ is a submanifold of $J^k(U_i, V_i) \cap J^k_{\mathcal{F}}(M,N)$. Let $\mathcal{O}_i(c^i,z^i) \cap \mathcal{O}_j(c^j,z^j)$ be non-empty. Since any $L^k(m, n-l) \times L^k(n, n-l')$-orbit in $J^k(m, n-l') \times J^k(l, l')$ is mapped on a $L^k(m, m-l) \times L^k(n, n-l')$-orbit by the action of $L^k(m, m-l) \times L^k(n, n-l')$ and, by definition, $\mathcal{O}_i(c^i,z^i) \subset J^k(U_i,V_i) \cap J^k_{\mathcal{F}}(M,N) \cap L$ for some $L \in \mathcal{F}^k(M,N)$, there results that the second condition from the definition of local submanifolds correction (Definition 2.7 from [4]) is satisfied. The last part of the Lemma is obvious.

By this result, the notion of which J^kF as a section of $J^k_{\mathcal{F}}(M,N)$ is transverse to a pseudo-orbit (in the weak sense) with respect to a good coordinate system is not dependent on the choice of good coordinate systems. Thus we say that j^kF is transverse to a pseudo-orbit $\mathcal{O}$ in the weak sense if j^kF is transverse to $\mathcal{O}$ with respect to a good coordinate system.

We obtain the following characterization for this notion:

LEMMA 10. *Let $F \in \mathcal{C}^\infty_{\mathcal{F}}(M,N)$, let $\mathcal{O} \subset J^k_{\mathcal{F}}(M,N)$ be a pseudo-orbit in the weak sense and let $p \in M$ such that $(j^kF)(p) \in \mathcal{O}$. Then j^k_F is transverse to $\mathcal{O}$ at p iff $(tF)((\mathcal{C}^\infty_{\mathcal{F}}(TM))_p) + (\omega F)((\mathcal{C}^\infty_{\mathcal{F}}(TN)_{F(p)}) + \mathcal{M}^{k+1}_{p.\mathcal{F}}(M)(\mathcal{C}^\infty_{\mathcal{F}}(F^*TN))_p = (\mathcal{C}^\infty_{\mathcal{F}}(F^*TN))_p$*

Proof. The proof is the same as in the usual case (cf. [5,V], §2). Namely, there is the following natural identification of $\mathbb{R}$-vector spaces,

$$T_z(J^k_p(M,N)) = (\mathcal{C}^\infty(F^*TN))_p / \mathcal{M}^{k+1}_p(\mathcal{C}^\infty(F^*TN))_p$$

where $z = j^k_pF \in J^k_p(M,N)$ (cf. [5,V], §2).

In terms of this identification we have:

$$T_z(J^k_{\mathcal{F}}(M,N)) = (\mathcal{C}^\infty_{\mathcal{F}}(F^*TN))_p / \mathcal{M}^{k+1}_{p.\mathcal{F}}(M)(\mathcal{C}^\infty_{\mathcal{F}}(F^*TN))_p$$

Now, by using the same arguments of Mather's paper ([5,V], §2) we obtain the following formula for the tangent space at z of the fiber $\mathcal{O}_p$ over p of local pseudo-orbit in the weak sense $\mathcal{O}(z \in \mathcal{O})$:

$$T_z(\mathcal{O}_p) = \big[(tF)(\mathcal{M}_{p,\mathcal{F}}(M)(\mathcal{C}^\infty_{\mathcal{F}}(TM))_p) + (\omega F)(\mathcal{C}^\infty_{\mathcal{F}}(TN)_{f(p)}) + \mathcal{M}^{k+1}_{p,\mathcal{F}}(M)(\mathcal{C}^\infty_{\mathcal{F}}(F^*TN))_p\big] \big/ \big[\mathcal{M}^{k+1}_{p,\mathcal{F}}(M)(\mathcal{C}^\infty_{\mathcal{F}}(F^*TN))_p\big]$$

We now give the following version of the transversality theorem in the tangential sense.

LEMMA 11. *Let $(M,\mathcal{F}_M)$ and $(N,\mathcal{F}_N)$ be smooth foliated manifolds, let $\mathcal{U}^k_{\mathcal{F},p}$ be a good coordinate system of $J^k_{\mathcal{F}}(M,N)$ with respect to $\mathcal{F}_M \times \mathcal{F}_N$ and let $A = \bigcup_{i\in I} A_i$ be a local submanifold correction in $J^k_{\mathcal{F}}(M,N)$ with respect to $\mathcal{U}^k_{\mathcal{F},p}$. Then the set $T_{A,\mathcal{F}} = \{F \in \overline{\mathcal{C}^\infty_{\mathcal{F}}(M,N)},\ j^kF \,\overline{\pitchfork}_{\mathcal{F},p} A\}$ is dense in $\overline{\mathcal{C}^\infty_{\mathcal{F}}(M,N)}$ where, as in [4], $j^kf \,\overline{\pitchfork}_{\mathcal{F},p} A$ if j^kf is transverse to A_i relative to $(\mathcal{F}^k(M,N), J^k(U_i,V_i)\cap J^k_{\mathcal{F}}(M,N))$, for all $i\in I$ (for the last notion see Def. 2.1 of [4]).*

Proof. We shall use the following result the proof of which follows, in the main, that of Lemma 2.2 from [4].

Transversality lemma in the tangential sense

Let B be a smooth manifold, let $(M,\mathcal{F}_M)$, $(N,\mathcal{F}_N)$ be foliated manifolds, let U be an open set of M, A a submanifold of U, and $j\colon B \to \mathcal{C}^\infty(M,N)$ a mapping. We denote by $\Phi\colon B\times M \to N$ the mapping given by $\Phi(b,p) = j(b)(p)$. If Φ is a smooth mapping and if $\Phi \,\overline{\pitchfork}_{(\mathcal{F}_N,U)}\, A$, then the set $\{b\in B,\ j(b)\,\overline{\pitchfork}_{(\mathcal{F}_N,U)}\, A\}$ is dense in B.

For each $i\in I$, let $\{A_i^j\}_{j\in\mathbf{N}}$ be an open covering of A_i such that for every $(i,j)\in I\times\mathbf{N}$ we have:

a) the closure of A_i^j in $J^k(U_i,V_i)$ is contained in A_i;

b) $\overline{A_i^j}$ is compact.

We consider the sets:

$T_{A_i^j} = \{f \in \mathcal{C}^\infty(M,N), f\,\overline{\pitchfork}\, A \text{ on } \overline{A_i^j}\}$,

$T_{A_i^j,\mathcal{F}} = T_{A_i^j} \cap \mathcal{C}^\infty_{\mathcal{F}}(M,N)$,

$T_{A_i^j,\overline{\mathcal{F}}} = T_{A_i^j} \cap \overline{\mathcal{C}^\infty_{\mathcal{F}}(M,N)}$.

Since $T_{A,\mathcal{F}} = \bigcap_{i\in I} T_{A_i^j,\overline{\mathcal{F}}}$ and $\overline{\mathcal{C}^\infty_{\mathcal{F}}(M,N)}$ is a Baire space, it is enough to show that for every $(i,j)\in I\times\mathbf{N}$, $T_{A_i^j}$ is open and $T_{A_i^j,\mathcal{F}}$ is dense in $\mathcal{C}^\infty_{\mathcal{F}}(M,N)$. The first result is proved by S.Izumiya in [4]. For the second, let $(U_i,\Phi_i),(V_i,\Psi_i)$, $\Psi_i(V_i) = V_1^i \times V_2^i \subset \mathbb{R}^{n-l'}\times\mathbb{R}^{l'}$ be foliated coordinate systems so that $(\alpha,\beta)(\overline{A_i^j}) \subset U_i\times V_j$ and $\rho\in\mathcal{C}^\infty_{\mathcal{F}}(\mathbb{R}^m,[0,1])$, $\rho_1'\in\mathcal{C}^\infty(\mathbb{R}^{l'},[0,1])$, $\rho_2'\in\mathcal{C}^\infty(\mathbb{R}^{n-l'},[0,1])$ such that:

$$\rho = \begin{cases} 1 & \text{on a neighbourhood of } \Phi_i\circ\alpha(\overline{A_i^j}) \\ 0 & \text{off } \Phi_i(U_i) \end{cases}$$

$$\rho_1' = \begin{cases} 1 & \text{on a neighbourhood of } pr_2\circ\Psi_i\circ\beta(\overline{A_i^j}) \\ 0 & \text{off } V_1^i \end{cases}$$

$$\rho_2' = \begin{cases} 1 & \text{on a neighbourhood of } pr_1\circ\Psi_i\circ\beta(\overline{A_i^j}) \\ 0 & \text{off } V_2^i. \end{cases}$$

The existence of a function ρ is assured by Lemma 3 of [8]. Let B_2 (resp. B_1) be the space of basic polynomial mappings, $\mathbb{R}^m \to \mathbb{R}^{l'}$ (resp. of polynomial mappings of $\mathbb{R}^m \to \mathbb{R}^{n-l'}$) of degree at most k.

Now we shall use the transversality lemma to show that $T_{A_i^j,\mathcal{F}}$ is dense in $\mathscr{C}^\infty_{\mathcal{F}}(M,N)$, for every $(i,j) \in I \times \mathbb{N}$. Let $F \in \mathscr{C}^\infty_{\mathcal{F}}(M,N)$ and for $(b_1,b_2) \in B_1 \times B_2$, let $G_{(b_1,b_2)}: M \to N$ given by

$$G_{(b_1,b_2)}(p) = \begin{cases} F(p) & \text{if } p \in U_i \text{ or } F(p) \notin V_i \\ \Psi_i^{-1}(\rho(\Phi_i(p))\rho_1'(F_1(p))b_1(\Phi_i(p))+ & \\ F_1(p), \rho(\Phi_i(p)) \times \rho_2'(F_2(p))b_2(\Phi_i(p)) + F_2(p)) & \text{otherwise.} \end{cases}$$

where $(\Psi_i \circ F)(p) = (F_1(p), F_2(p)), \forall p \in F^{-1}(V_i)$.

The choice of ρ, ρ_1', ρ_2' guarantees that $G_{(b_1,b_2)}$ is a smooth foliation preserving mappings from M to N and $G_{(0,0)} = F$. Let

$$j: B_1 \times B_2 \to \mathscr{C}^\infty(M, J^k_{\mathcal{F}}(M,N)), j(b_1,b_2) = j^k G_{(b_1,b_2)}$$
$$\Phi: B_1 \times B_2 \times M \to J^k_{\mathcal{F}}(M,N),\ \Phi(b_1,b_2,p) = j(b_1,b_2)(p).$$

By the same techniques as in the proof of the ordinary jet transversality theorem we may prove that there are neighbourhoods B_1', B_2' of the origin in B_1, B_2 respectively and a neighbourhood $A_i'^j$ of $\overline{A_i^j}$ in A_i such that, denoting by $\widetilde{\Phi} = \Phi|_{M \times B_1' \times B_2'}$ we have $\widetilde{\Phi}\, \overline{\pitchfork}_{\mathcal{F}^k(M,N), J^k_{\mathcal{F}}(M,N)} A_i$ on $A_i'^j$. We apply the transversality lemma and we obtain that $\{(b_1,b_2) \in B_1' \times B_2', j^k G_{(b_1,b_2)}\, \overline{\pitchfork}_{\mathcal{F}^k(M,N) J^k_{\mathcal{F}}(M,N)} A_i$ on $A_i'^j\}$ is dense in $B_1' \times B_2'$. Then we can find a sequence $\{b_n\} \subset B_1 \times B_2$ such that $\lim_{n\to\infty} b_n = (0,0) \in B_1' \times B_2'$ and $j^k G_{(b_n)}\, \overline{\pitchfork}_{\mathcal{F}^k(M,N), J^k_{\mathcal{F}}(M,N)} A_i$ on $A_i'^j$. But $G_{(0,0)} = F$ and, by construction, $G_b = F$ off U_i. Thus it follows that $\lim_{n\to\infty} G_{b_n} = F$ in $\mathscr{C}^\infty_{\mathcal{F}}(M,N)$ and $T_{A_i'^j,\mathcal{F}}$ is dense in $\mathscr{C}^\infty_{\mathcal{F}}(M,N)$. This completes the proof.

By using the techniques from [5] and [4] it is easy to generalize the above result to a multijet version. With the usual multijet notations we obtain the following:

LEMMA 12. *Let $(M,\mathcal{F}_M)$ and $(N,\mathcal{F}_N)$ be smooth foliated manifolds, let S be a finite subset of M, let ${}_S\mathcal{U}^k_{\mathcal{F},p}$ be a good coordinate system of ${}_S J^l_{\mathcal{F}}(M,N)$ with respect to $\mathcal{F}_M \times \mathcal{F}_N$ and let A be a local submanifold correction in ${}_S J^l_{\mathcal{F}}(M,N)$ with respect to ${}_S\mathcal{U}^k_{\mathcal{F},p}$. Then $T^S_{A,\mathcal{F}} = \{F \in \overline{\mathscr{C}^\infty_{\mathcal{F}}(M,N)}, {}_S j^k F\, \overline{\pitchfork}_{\mathcal{F},p} A\}$ is dense in $\overline{\mathscr{C}^\infty_{\mathcal{F}}(M,N)}$.*

Analogously we obtain the multijet versions for Lemmas 9 and 10.

LEMMA 13. *Let $(M,\mathcal{F}_M)$ and $(N,\mathcal{F}_N)$ be smooth foliated manifolds, M compact, $F \in \mathscr{C}^\infty_{\mathcal{F}}(M,N), k \in \mathbb{N}, k \geq n', s \geq n'' + 1^k$. If F is a $\mathscr{C}^\infty$-$\mathcal{F}$-stable* *mapping, then for every subset S of M having S or fewer points, and such that $F(S)$ is a single point, we have:*

$$(tF)((\mathscr{C}^{\infty}_{\mathscr{F}}(TM))_S) + (\omega F)((\mathscr{C}^{\infty}_{\mathscr{F}}(TN))_{F(S)}) + \mathscr{M}^{k+1}_{p,\mathscr{F}}(M)(\mathscr{C}^{\infty}_{\mathscr{F}}(F^*TN))_S = (\mathscr{C}^{\infty}_{\mathscr{F}}(F^*TN))_S \tag{13}$$

Proof. By Lemma 9 (the multijet version), $\mathbb{O}$ is a local submanifold correction in $\mathscr{F}^k(M,N)$ with respect to a good coordinate system ${}_S\mathscr{U}^k_{\mathscr{F},p}$ and, by Lemma 12, we have that $T^S_{\mathbb{O},\mathscr{F}} = \{G \in \overline{\mathscr{C}^{\infty}_{\mathscr{F}}(M,N)}, {}_S j^k G \,\bar{\pitchfork}_{\mathscr{F},p}\, \mathbb{O}\}$ is dense in $\overline{\mathscr{C}^{\infty}_{\mathscr{F}}(M,N)}$. Since F is a $\mathscr{C}^{\infty}$-$\mathscr{F}$-stable mapping there is a neighbourhood $V_F \subset \mathscr{C}^{\infty}_{\mathscr{F}}(M,N)$ such that for every $G \in V_F$ there are diffeomorphisms $H \in \mathrm{Diff}^{\infty}_{\mathscr{F}}(M), K \in \mathrm{Diff}^{\infty}_{\mathscr{F}}(N)$ such that $G = K \circ F \circ H$. Thus V_F is a neighbourhood of F in $\overline{\mathscr{C}^{\infty}_{\mathscr{F}}(M,N)}$ and $V_F \cap T^S_{\mathbb{O},\mathscr{F}} \neq \emptyset$. Let $G \in V_F \cap T^S_{\mathbb{O},\mathscr{F}}$, so ${}_S j^k G \,\bar{\pitchfork}_{\mathscr{F},p}\, \mathbb{O}$. By Lemma 9 (the multijet version) it follows that $K^{-1} \circ \mathbb{O} \circ H^{-1} = \bigcup_i K^{-1} \circ \mathbb{O}_i \circ H^{-1}$ is a pseudo-orbit in the weak sense with respect to the good coordinate system $K^{-1} \circ_S \mathscr{U}^k_{\mathscr{F},p} \circ H^{-1}$. Since ${}_S j^k G \,\bar{\pitchfork}_{\mathscr{F},p}\, \mathbb{O}$, we have ${}_S j^k F \,\bar{\pitchfork}_{\mathscr{F}} (K^{-1} \circ \mathbb{O} \circ H^{-1}$. By using Lemma 10 (multijet version) it follows that for any element $x = (x_1, \dots, x_s) \in M^{(s)}$ with ${}_S j^k F(x) \in K^{-1} \circ \mathbb{O} \circ H^{-1}$ we have (3).

Since by Lemma 8, the last condition is equivalent to the fact that F is a $\mathscr{C}^{\infty}$-$\mathscr{F}$-infinitesimally stable mapping, we obtain the proof of the implication:

F is a $\mathscr{C}^{\infty}$-$\mathscr{F}$-stable mapping $\Longrightarrow$ F is a $\mathscr{C}^{\infty}$-$\mathscr{F}$-infinitesimally stable mapping.

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