Boletín de la Sociedad Matemática Mexicana Vol. 39, 1994

MULTIVARIABLE SPECTRAL THEORY OF ALGEBRAS OF ANALYTIC FUNCTIONS

BY GENARO ALMENDRA* AND ANTONI WAWRZYŃCZYK†

1. Introduction

The present paper is devoted to the study of cospectra of ideals J of algebras $\mathscr{A}(\Omega)$ of holomorphic functions with restricted growth defined in an open set $\Omega \subset \mathbb{C}^n$. We consider also the joint spectra of k-tuples of elements of the quotient algebra $\mathscr{A}(\Omega)/J$.

The principal innovation is the addition of "points at infinity" to the classical cospectrum $Z(J) = \{z \in \Omega \mid f(z) = 0,\ f \in J\}$. The obtained extended cospectrum is briefly speaking the set of common zeroes of the functions $f \in J$ extended continuously to the Stone-Čech compactification $\beta\Omega$ of Ω. The extended cospectrum is then a subset of $\beta\Omega$ although we consider later also the cospectra in other compactifications of Ω.

The appearence of $\beta\Omega$ in the spectral analysis of algebras of continuous functions on Ω is not surprising. In the case of the algebra $\mathscr{C}(\Omega)$ of all continuous functions on Ω the theorem of Gelfand-Kolmogoroff identifies the space $\beta\Omega$ with the set of the maximal ideals of the algebra. To the point $z \in \beta\Omega$ there correspond under this identification the ideal

$$M_z = \{f \in C(\Omega) |\ z \in \overline{Z(f)}\}.$$

In the above formula $Z(f)$ denotes the set of zeros of the function f and the closure is taken in the compact space $\beta\Omega$.

In case of the algebras of functions with restricted growth the relation between the points of $\beta\Omega$ and maximal ideals must be modified. Briefly speaking we treat the cospectrum of an ideal of the form $f\mathscr{A}(\Omega)$ as the obstacle to the invertibility of f. Even if the function $1/f$ makes sense in some subset of Ω the invertibility depends upon the behaviour of this function at infinity which is restricted by the growth condictions defining the algebra $\mathscr{A}(\Omega)$. A point $z \in \beta\Omega$ is cospectral ($z \in \zeta(J)$) if for every neighbourhood O of z there exists an element $f \in J$ such that $1/f$ does not behave in O as elements of $\mathscr{A}(\Omega)$ should do. In the same way we define the cospectrum $\zeta(J, C)$ of J in an arbitrary compactification C of Ω. First of all we succeed in proving that nontrivial ideal J have nonempty cospectrum for an arbitrary compactification C.

*Research supported by CONACyT
†Partially supported by SNI

It results that $\zeta(J) = \zeta(J, \beta\Omega)$ determines $\zeta(J, C)$ for an arbitrary compactification C. If $P_C: \beta\Omega \to C$ is the natural projection which leaves invariant the points of Ω then

$$\zeta(J, C) = P_C\zeta(J).$$

If we associate to a point $z \in \beta\Omega$ the ideal

$$\mathcal{M}_z = \{f \in \mathcal{A}(\Omega) |\ z \in \zeta(f\mathcal{A}(\Omega))\}$$

we can obtain all maximal ideals of $\mathcal{A}(\Omega)$ as $\mathcal{M}_z$ for appropiate $z \in \beta\Omega$.

The continuous multiplicative functionals of the algebra $\mathcal{A}(\Omega)$ correspond however to the points of Ω and are of the form $f \to f(z)$ with $z \in \Omega$.

Our interest in the algebras of the form $\mathcal{A}(\Omega)/J$ is motivated by the rôle played by this type of algebras in the spectral analysis of translation invariant function spaces on $\mathbb{R}^n$. In particular if $V \subset C^\infty(\mathbb{R}^n)$ is a linear translation invariant closed subspace the annihilator

$$V^\perp = \{T \in C^\infty(\mathbb{R}^n)' |\ T(f) = 0,\ f \in V\}$$

is isomorphic by means of the Fourier transform to an ideal J of the algebra

$$\mathcal{A}_p(\mathbb{C}^n) = \{f \in \mathcal{A}(\mathbb{C}^n) |\ |f(z)| \leq A \exp(B(\log(1 + \|z\|) + \|\mathrm{Im} z\|))\},$$

while the dual space V' is isomorphic to the quotient algebra $\mathcal{A}_p(\mathbb{C}^n)/J$.

This being the case an exponential function $\mathbf{e}_z: \mathbb{R}^n \ni x \to \exp((z|x))$ belongs to V if and only if $z \in Z(J)$. Using the traditional terminology: the spectral analysis holds in V if $Z(J) = \zeta(J) \cap \mathbb{C}^n$ is nonvoid. It is well known however that the latter set can be empty.

Now, it can be seen that the set $\zeta(J) \cap \mathbb{C}^n$ is equal to the joint spectrum of the n-tuple $\mathcal{Z} = ([z_1], \ldots, [z_n]) \in (\mathcal{A}(\mathbb{C}^n)/J)^n$. In section 3 we introduce the notion of the extended joint spectrum and we prove that the latter is equal just to $\zeta(J)$ in case of the n-tuple $\mathcal{Z}$. It means that both the study of the generalized cospectrum $\zeta(J)$ as well as of the extended joint spectrum can be treated as a subsequent step in the development of the spectral analysis and synthesis.

The mentioned extended joint spectrum of a k-tuple $\mathcal{F} = ([f_1], \ldots, [f_k]) \in (\mathcal{A}(\Omega)/J)^n$ is defined as a closed subset of an arbitrary compactification K of $\mathbb{C}^k$ and is denoted by $\sigma(\mathcal{F}, J, K)$. Our definition assures that in $\mathbb{C}^k \setminus \sigma(\mathcal{F}, J, K)$ there exists a generalized resolvent that is a k-tuple of functions $R_j(z, \mu)$ such that

$$\sum_{j=1}^{k} R_j(z, \mu)(z_j - \mu_j) - 1 \in J, \tag{1}$$

for $\mu \in \mathbb{C}^k \setminus \sigma(\mathcal{F}, J, K)$. Moreover, for every $\lambda \in K \setminus \sigma(\mathcal{F}, J, K)$ there exists a neighbourhood O of λ such that the set $\{R_j(\cdot, \mu) |\ \mu \in O \cap \mathbb{C}^k\}$ is bounded in $\mathcal{A}(\Omega)$.

It results that the joint spectrum in the Stone-Čech compactification $\beta\mathbb{C}^k$ determines joint spectra in other compactifications:

$$\sigma(\mathcal{F}, J, K) = P_K\sigma(\mathcal{F}, J, \beta\mathbb{C}^k).$$

In what follows we write just $\sigma(\mathcal{F}, J)$ in place of $\sigma(\mathcal{F}, J, \beta\mathbb{C}^k)$.

The joint spectrum is never empty for proper ideals J because we prove that

$$\tilde{\mathcal{F}}(\zeta(J)) \subset \sigma(\mathcal{F}, J)$$

and

$$P_K\tilde{\mathcal{F}}(\zeta(J)) \subset \sigma(\mathcal{F}, J, K).$$

In the formula above $\tilde{\mathcal{F}}$ denotes the unique continuous extension to $\beta\Omega$ of the mapping $\Omega \ni z \to (f_1, \ldots, f_k) \in \beta\mathbb{C}^k$.

In the case when K is equal to the real or complex projective space $\mathbf{P}^{2k}(\mathbb{R})$ or $\mathbf{P}^k(\mathbb{C})$ we have the equality

$$P_K\tilde{\mathcal{F}}(\zeta(J)) = \sigma(\mathcal{F}, J, K).$$

In both cases the spectrum $\sigma(\mathcal{F}, J, K)$ have the spectral mapping property what means the following:

For every polynomial mapping $\mathcal{P}: \mathbb{C}^k \to \mathbb{C}^m$ which extends to a continuous application $\hat{\mathcal{P}}: \mathbf{P}^k(\mathbb{C}) \to \mathbf{P}^m(\mathbb{C})$ we have

$$\hat{\mathcal{P}}\sigma(\mathcal{F}, J, \mathbf{P}^k\mathbb{C}) = \sigma(\mathcal{P} \circ \mathcal{F}, J, \mathbf{P}^m\mathbb{C}).$$

The analogous formula for the real projective space is also valid.

In the last section we prove that all multiplicative functionals on $\mathcal{A}_p(\Omega)$ are the evaluations of $f \in \mathcal{A}_p(\Omega)$ at a fixed $z \in \Omega$.

The present paper presents the results of an investigation not finished yet. The interest of the authors concentrates at this moment at those properties of ideals which can be expressed in terms of their cospectrum $\zeta(J)$. It can be seen for example that the slowly decreasing n-tuple $\mathcal{F}$ (see [1] for the definition and fundamental properties) whose cospectrum consists uniquely of "points at infinity" generates $\mathcal{A}_p(\Omega)$. It explains the fact that proper ideals generated by slowly decreasing tuples have always nontrivial the classical cospectrum. The slowly decreasing tuples were created as those which generate closed ideals. This property is also related with the appearence of the points at infinity in $\zeta(J_{\mathcal{F}})$.

It is easy to see that the ideal $f\mathcal{A}_p(\Omega)$ is not closed if and only if f is a topological divisor of zero in $\mathcal{A}_p(\Omega)$, exactly as in the case of commutative Banach algebras without divisors of zero. More generally, an ideal J consists of joint topological divisors of zero if and only if $J_{\mathcal{F}}$ is not closed for every k-tuple of elements of J.

If the spectral analysis is valid in $\mathscr{A}_p(\Omega)$ then every ideal J whose spectrum consists of points at infinity is dense in $\mathscr{A}_p(\Omega)$. It consists of joint topological divisors of zero, just like for example the maximal ideals corresponding to the points of the Shilov boundary in the Banach algebras case. If we are interested in the development of concepts like the Shilov boundary or of the peak points of the spectrum for $\mathscr{A}_p(\Omega)$ we must study profoundly the behaviour of the elements of $\mathscr{A}_p(\Omega)$ on $\beta\Omega$. In some particular cases described in [8] (specially for $\Omega = \mathbb{C}$) it is possible to formulate determinate results. In the case of the unit disc $D \subset \mathbb{C}$ the structure of ideals can be also described in terms of the classical and extended spectrum. These results involving the topology of the Stone-Čech compactification $\beta\Omega$ will be published in separate paper. The present one is devoted almost exclusivelly to the properties of the extended spectrum which follows by Hörmander's theorem 2.1.

The authors are very indebted to Professor K. Jarosz and to Professor W. Żelazko for interesting conversations and important suggestions.

2. Preliminaries

In what follows Ω will denote an open subset of $\mathbb{C}^n$ and p a function on Ω which is positive plurisubharmonic and satisfies the following conditions

1. $\log(1 + \|z\|)/p(z)$ is bounded on Ω.
2. there exist A, B, C, $D > 0$ such that for all $z \in \Omega$ the condition $\|z - w\| < \exp(-Cp(z) - D)$ implies $w \in \Omega$ and $p(w) \leq Ap(z) + B$.

In case of $\Omega = \mathbb{C}^n$ the above conditions are satisfied for $p(z) = |z|^r$, $r > 0$ as well as for the function $p(z) = \log(1 + \|z\|) + \|\text{Im } z\|$ which was already introduced in the previous section. If Ω is a domain of holomorphy we obtain a function satisfying 1 and 2 putting $p(z) = -\log(d(z, \partial\Omega))$, where $\partial\Omega$ is the boundary of Ω and d denotes the Euclidean distance in $\mathbb{C}^n$.

This type of pairs (Ω, p) was introduced by Hörmander in [6] with the purpose to study the finitely generated ideals of the algebra of functions which are holomorphic in Ω and have its growth determinated by p.

We denote by $\mathscr{C}(\Omega)$ the space of all continuous functions on Ω and by $\mathscr{A}(\Omega)$ the space of functions holomorphic in Ω. For $r > 0$ let $\mathscr{C}_p^r$ denote the set of continuous functions which satisfy

$$\|f\|_r = \sup_{\Omega} |f(z)| \exp(-rp(z)) < \infty$$

and $\mathscr{C}_p(\Omega) = \bigcup_{r>0} \mathscr{C}_p^r$.

We also introduce $\mathscr{A}_p^r = \mathscr{A}(\Omega) \cap \mathscr{C}_p^r(\Omega)$ and $\mathscr{A}_p(\Omega) = \bigcup_{r>0} \mathscr{A}_p^r$. In the spaces $\mathscr{A}_p(\Omega)$ and $\mathscr{C}_p(\Omega)$ we define the inductive limit topologies of the normed spaces $(\mathscr{A}_p^r, \|\cdot\|_r)$ and $(\mathscr{C}_p^r, \|\cdot\|_r)$ respectively.

The spaces $\mathscr{A}_p(\Omega)$ and $\mathscr{C}_p(\Omega)$ are topological algebras which constitute the principal subject of our research.

As proved in [6] the conditions 1 and 2 assure that

i. The space of all polynomials belongs to $\mathcal{A}_p(\Omega)$.
ii. If $f \in \mathcal{A}_p(\Omega)$ then $\frac{\partial}{\partial z_j} f \in \mathcal{A}_p(\Omega)$ for any $1 \leq j \leq n$.

The following theorem of Hörmander (and its extension given later) is the fundamental argument used to develope the notion of the cospectrum and joint spectrum in algebras $\mathcal{A}_p(\Omega)$.

THEOREM (2.1) *[6] Suppose that the function $p > 0$ on $\Omega \subset \mathbb{C}^n$ is plurisubharmonic and satisfies the conditions 1 and 2. Then the ideal generated by the functions $f_1, \ldots, f_k \in \mathcal{A}_p(\Omega)$ is the whole $\mathcal{A}_p(\Omega)$ if and only if there exist c_1, $c_2 > 0$ such that*

$$\sum_{j=1}^{k} |f_j| \geq c_1 \exp(-c_2 p). \tag{2}$$

The proof of the above theorem uses the celebrated results of Hörmander about the solution of the equation $\bar{\partial} h = g$ in spaces of differential forms with measurable coefficients which behave at infinity as elements of $\mathcal{C}_p(\Omega)$.

In case of the algebra $\mathcal{C}_p(\Omega)$ the above theorem is trivially valid and its proof is just the first step in the Hörmander's proof. Given the functions $f_1, \ldots, f_k \in \mathcal{C}_p(\Omega)$ which satisfy the condition(2) we can construct

$$h_j = \frac{\bar{f}_j}{\sum_{j=1}^{k} |f_j|^2}.$$

Thanks to the condition (2) and the relation $\sum_{j=1}^{k} |f_j| \leq k^{1/2} (\sum_{j=1}^{k} |f_j|^2)^{1/2}$ we see that $h_i \in \mathcal{C}_p(\Omega)$. The direct calculation gives $\sum_{j=1}^{k} h_j f_j = 1$, hence for arbitrary $\phi \in \mathcal{C}_p(\Omega)$ we have $\phi = \sum_{j=1}^{k} f_j (h_j \phi)$. The ideal generated by the functions $\{f_j\}$ is just $\mathcal{C}_p(\Omega)$. We shall need however a strengthened version of Theorem 2.1 providing several informations about the coefficients $h_j \in \mathcal{A}_p(\Omega)$ which permit to generate the function 1 from $\mathcal{F}$.

For given $\mathcal{F} = (f_1, \ldots, f_k)$ let us denote

$$\|\partial \mathcal{F}\|_r = \sum_{i=1}^{n} \sum_{j=1}^{k} \left\| \frac{\partial}{\partial z_i} f_j \right\|_r.$$

In [13] the following result is proved:

THEOREM (2.2) *For every $k \in \mathbf{N}$ and $s \geq 0$ there exist t, $r \geq 0$ and a polynomial $W_{k,s}$ with positive coefficients such that for every $g \in \mathcal{A}_p^s$ and every k-tuple $\mathcal{F} \in \mathcal{A}_p(\Omega)^k$ which satisfies the condition (2) and $\|\partial\mathcal{F}\|_r \leq \infty$ there exist $h_j \in \mathcal{A}_p^t$ obeying*

$$\sum_{j=1}^{k} h_j f_j = g$$

and

$$\|h_j\|_t \leq \frac{1}{c_1} W_{k,s}\left(\frac{\|\partial\mathcal{F}\|_r}{c_1}\right) \|g\|_s.$$

3. The cospectrum

In what follows we present a number of definitions and results parallely for the algebras $\mathcal{E}_p(\Omega)$ and $\mathcal{A}_p(\Omega)$. In order to simplify the exposition we understand by $\mathcal{A}$ anyone of these algebras. Let (C, j) be a compactification of Ω, that is to say C is a compact completely regular space and $j\colon \Omega \to C$ is a continuous injection with dense image.

Definition (3.1) For a given ideal $J \in \mathcal{A}$ and a compactification C of Ω we denote

$$\rho(J, C) = \Big\{z \in C | \exists f_j \in J, K > 0 \text{ and a neighbourhood } \mathcal{O} \text{ of } z \text{ in } C$$
$$\text{such that } \sum_{j=1}^{k} |f_j(w)| \geq \exp(-Kp(w)) \text{ for all } w \in \mathcal{O} \cap \Omega\Big\}.$$

The points belonging to $\rho(J, C)$ are called *regular points* for J in C. The complement of $\rho(J, C)$ is called *the extended cospectrum* of J in C and is denoted by $\zeta(J, C)$. We denote by $Z(J)$ the classical cospectrum of J that is the set of common zeros of the elements of J in Ω.

It is easily seen that $Z(J) = \zeta(J, C) \cap \Omega$ independently of the particular compactification of Ω.

THEOREM (3.1) *The set $\zeta(J, C)$ is empty if and only if $J = \mathcal{A}$.*

Proof. Suppose that $\rho(J, C) = C$. For every $z \in C$ there exist $\mathcal{O}(z)$, $K > 0$ and $f_j^z \in J$ such that

$$\sum_{j=1}^{k_z} |f_j^z(w)| \geq \exp(-Kp(w)),\ w \in \mathcal{O}(z) \cap \Omega.$$

By the compactness of C we can choose a finite subcovering $\{\mathcal{O}(z_i)\}_{i=1,\ldots,r}$ from the covering $\{\mathcal{O}(z)\}_{z\in C}$. Let $\{f_j^{z_i}\}, \ldots, \{f_j^{z_r}\}$ and $K_1, \ldots, K_r$ be the corresponding elements of J and corresponding positive constants. Taking $c = \max(K_1, \ldots, K_r)$ we obtain

$$\sum_{i=1}^{r} \sum_{j=1}^{k_z} |f_j^{z_i}(z)| \geq \exp(-cp(z)),\ z \in \Omega.$$

By Theorem 2.1 (or its version for $\mathscr{C}_p(\Omega)$) the functions $f_j^{z_1}, \ldots, f_j^{z_r}$ generate $\mathscr{A}$, that is $J = \mathscr{A}$. The converse is obvious. ■

In contrast to $\zeta(J, C)$ the classical cospectrum can be empty as shows the example found by Gurevich [5].

The family Comp Ω of all compactifications of Ω has its natural order defined as follows:

$$(C_1, j_1) \succ (C_2, j_2)$$

provided that there exists a continuous surjection $P: C_1 \to C_2$ such that $j_2 = P \circ j_1$. The maximal element of Comp Ω with respect to the order $\succ$ is the Stone-Čech compactification $\beta\Omega$ which is unique up to homeomorphism.

We denote: $\zeta(J) = \zeta(J, \beta\Omega)$ and $\rho(J) = \rho(J, \beta\Omega)$.

The Stone-Čech compactification can be defined equivalently up to a homeomorphism as the compact space containing Ω and such that any continuous map $\phi: \Omega \to K$ valued in a compact space K can be uniquely extended to a continuous map $\tilde{\phi}: \beta\Omega \to K$. This property of $\beta\Omega$ permits us to describe the cospectrum of J as the zero set of a family of functions associated to J.

THEOREM (3.4) *Let J be an ideal of $\mathscr{A}$. Then*

$$\zeta(J) = \{z \in \beta\Omega |\ (f \exp(cp))\tilde{}\,(z) = 0 \textit{ for all } c > 0,\ f \in J\}.$$

Proof. If J is a proper ideal and $z \in \zeta(J)$ then by the very definition of the cospectrum for arbitrary $\epsilon > 0, c > 0$ and every neighbourhood $\mathscr{O}$ of z there exists $w \in \mathscr{O}$ such that

$$|f(w)| \exp(cp(w)) \leq \epsilon.$$

By the continuity of the extension we obtain the assertion. The converse is obvious. ■

The description of the cospectrum obtained above implies in particular the following:

THEOREM (3.2) *Let I, J be proper ideals of $\mathscr{A}_p$ and let $\mathscr{J}$ be the ideal generated by I and J. Then*

$$\zeta(\mathscr{J}) = \zeta(I) \cap \zeta(J).$$

Proof. Assume that $z \in \zeta(I) \cap \zeta(J)$. For every $f \in \mathcal{J}$ there exist $g \in I$, $h \in J$ and $\varphi, \psi \in \mathcal{A}_p$ such that $f = \varphi g + \psi h$. There exist also constants $a, b, c, d > 0$ for which $|\varphi| \le a \exp(bp)$ and $|\psi| \le c \exp(dp)$. For an arbitrary $\gamma > 0$ we obtain

$$|f \exp(\gamma p)| \le a|g| \exp((b+\gamma)p) + c|h| \exp((d+\gamma)p).$$

Both terms of the last sum are continuous functions which by Theorem 3.2 extend to functions vanishing at z. Applying the same theorem we obtain $z \in \zeta(\mathcal{J})$. We have proved that

$$\zeta(I) \cap \zeta(J) \subset \zeta(\mathcal{J}).$$

Since $J \subset \mathcal{J}$ and $I \subset \mathcal{J}$ the opposite relation is also valid. ∎

In case of the algebra $\mathcal{C}_p(\Omega)$ the description of the cospectrum can be simplified because for $f \in J$ the function $f \exp(cp)$ belongs to J. We obtain:

$$\zeta(J) = \{z \in \beta\Omega |\ \tilde{f}(z) = 0,\ f \in J\}.$$

It follows immediately from this observation that the cospectrum of $J \subset \mathcal{A}_p(\Omega)$ is equal to the cospectrum of the ideal of $\mathcal{C}_p(\Omega)$ generated by J. In particular if we consider the cospectrum of an ideal $J_{\mathcal{F}}$ generated by several elements $f_1, \ldots, f_k \in \mathcal{A}_p(\Omega)$ it does not matter in which of two algebras we generate the ideal. By this reason we have decided to simplify the notation of $\zeta(J)$ the cospectrum avoiding to anote the algebra in question.

It is important to observe that the cospectrum $\zeta(J)$ of J determines completely the cospectra of J in other compactifications as shows the following

THEOREM (3.3) *Let C be an arbitrary compactification of Ω and let $P\colon \beta\Omega \to C$ be the projection which leaves invariant the points of Ω. Then for every ideal $J \subset \mathcal{A}$*

$$P(\zeta(J)) = \zeta(J, C).$$

Proof. We prove the relation $P(\zeta(J)) \subset \zeta(J, C)$ by showing that $P(z) \in \rho(J, C)$ implies $z \in \rho(J)$. In fact, if $\mathcal{O}(P(z))$, $f_j \in J$, $c > 0$ are such that

$$\sum_{j=1}^{k} |f_j(w)| \ge \exp(-cp(w))$$

for $w \in \mathcal{O} \cap \Omega$, then $\mathcal{U} = P^{-1}(\mathcal{O}(P(z)))$ is the neighbourhood of z such that the same inequality is valid for the points of $\mathcal{U} \cap \Omega$. It means that $P^{-1}(\rho(J, C)) \subset \rho(J)$, or equivalently $P(\zeta(J)) \subset \zeta(J, C)$.

In order to prove the opposite inclusion, we assume that $z \in \zeta(J, C)$ but $P^{-1}(z) \subset \rho(J)$. This will lead to a contradiction.

Suppose that for every $x \in P^{-1}(z)$ there exist $f_j \in J$ and $\epsilon, c > 0$ such that $(\sum_{j=1}^{k} |f_j(y)| \exp(cp(y)))\check{} > \epsilon$ in some neighbourhood of x. Since $P^{-1}(z)$ is compact we can find an N-tuple $f_1, \ldots, f_N \in J$ and $\epsilon > 0, c > 0$ such that

$$(\sum_{i=1}^{N} |f_i| \exp(cp))\check{}(x) > \epsilon$$

for all $x \in P^{-1}(z)$.

On the other hand if $z \in \zeta(J, C)$ we know that in each neighbourhood $\mathcal{U}$ of z in C there exists $z_{\mathcal{U}} \in \mathcal{U} \cap \Omega$ such that

$$\sum_{i=1}^{N} |f_i(z_{\mathcal{U}})| \exp(cp(z_{\mathcal{U}})) \leq \epsilon/2.$$

By the compactness of $\beta\Omega$ the net $(z_{\mathcal{U}})$ has an accumulation point x which must belong to $P^{-1}(z)$ by the definition of the net. It implies that $(\sum_{i=1}^{N} |f_i| \exp(cp))\check{}(x) \leq \epsilon/2$. This is the contradiction. ∎

The description of the maximal ideals plays the fundamental rôle in the spectral theory of topological algebras. If J is a maximal ideal of $\mathcal{A}$ (not neccessarily closed) and $z \in \zeta(J)$ then by the very definition of the cospectrum $J \subset J_z = \{f \in \mathcal{A}_p | (f \exp(Cp))\check{}(z) = 0 \text{ for all } C > 0\}$.

The set J_z is a proper ideal of $\mathcal{A}$ hence by the maximality of J both ideals coincide. It follows that all maximal ideals of $\mathcal{A}$ are of the form J_z for $z \in \beta\Omega$. The question arise if $\zeta(J_z) = \{z\}$ and if the ideal J_z is maximal for each point $z \in \beta\Omega$. It is true for the algebra $\mathcal{C}_p(\Omega)$ because continuous bounded functions separate points of $\beta\Omega$. This being the case we obtain the identification between the space $\beta\Omega$ and the space $\mathcal{M}(\mathcal{C}_p)$ of all maximal ideals of $\mathcal{C}_p(\Omega)$. For the algebra $\mathcal{A}_p(\Omega)$ this is not the case. One construct distinct points z, $w \in \beta\Omega \setminus \Omega$ which can not be separated by elements of $\mathcal{A}_p(\Omega)$, hence $J_z = J_w$.The construction was suggested by the reviewer.et us take two sequences (z_j) and (w_j) whose elements form discrete disjoint sets and such that $\|z_j - w_j\| \exp(cp(z_j))$ tends to zero for all $c > 0$. It follows that $|f(z_j) - f(w_j)| \exp(cp(z_j)) \to 0$ for all $c > 0$. The closures of the sets $\{z_j\}$ and $\{w_j\}$ in $\beta\Omega$ are disjoint. Let Φ be an ultrafilter of subsets of $\{z_j\}$ which defines an element $z \in \beta\Omega \setminus \Omega$ which belongs to the closure of $\{z_j\}$. Substituting in each subset from Φ the element z_j by w_j we obtain an ultrafilter which defines $w \in \beta\Omega$ belonging to the closure of $\{w_j\}$. Hence $z \neq w$. On the other hand the elements of $\mathcal{A}_p(\Omega)$ do not separate z from w.

It is an open problem if for general $\mathcal{A}_p(\Omega)$ every point $z \in \beta\Omega$ gives us J_z which is maximal. At least in some particular cases it seems to be true.

The construction was suggested by the reviewer.L

4. The quotient algebra $\mathscr{A}/J$

Let us consider an ideal $J \subset \mathscr{A}$ and the quotient algebra $\mathscr{A}/J$. The invertibility of an element $[f] \in \mathscr{A}/J$ can be expressed in terms of the corresponding cospectra.

THEOREM (4.1) *Let $f \in \mathscr{A}$. The following conditions are equivalent:*

a. The class $[f]$ is invertible in $\mathscr{A}/J$.
b. $\zeta(J_f) \cap \zeta(J) = \emptyset$.
c. $\exists c > 0$ such that $(f \exp(cp))\tilde{}$ does not vanish on $\zeta(J)$.

Proof. The class $[f]$ is invertible if there exist $g \in \mathscr{A}$ and $h \in J$ such that $fg - 1 = h$. Equivalently, the ideal generated by f and J is equal to $\mathscr{A}$. Its cospectrum is empty and by Theorem 3.3 it is equal to $\zeta(J_f) \cap \zeta(J)$. This proves $a) \Rightarrow b)$.

Now, suppose that $c)$ is not valid, that is $\forall c > 0$ the function $(f \exp(cp))\tilde{}$ vanishes somewhere in $\zeta(J)$. Denote by A_c the (compact) set of zeroes in $\zeta(J)$ of the function $(f \exp(cp))\tilde{}$. Obviously $c < d$ implies $A_d \subset A_c$. By the compactness of $\zeta(J)$ the set $\bigcap_{c>0} A_c \subset \zeta(J_f) \cap \zeta(J)$ is nonvoid. We proved that $b) \Rightarrow c)$.

Finally assume that the condition $c)$ is satisfied and consider the ideal $\mathscr{J}$ generated by f and J, whose cospectrum is $\zeta(J_f) \cap \zeta(J)$ according to Theorem 3.3. If $\mathscr{J}$ is nontrivial and $z \in \zeta(\mathscr{J})$ then in particular for every $c > 0$ the extension of the function $f \exp(cp)$ vanishes at z. This contradicts c) and the invertibility of $[f]$ is proved. ∎

Let us compare the situation with the classical model of the Banach algebra of $\mathscr{C}(X)$ of continuous functions on a compact set X. If J is a closed ideal in $\mathscr{C}(X)$ and $\zeta(J)$ is the set of common zeroes of the elements of J then the invertibility of $[f]$ in $\mathscr{C}(X)/J$ is equivalent to the condition that f does not vanish on $\zeta(J)$ or equivalently that $\zeta(J_f) \cap \zeta(J) = \emptyset$. In this case the spectrum of $[f]$ in the quotient algebra coincides with the set of values of f on $\zeta(J)$. The set $\zeta(J)$ plays also the rôle of the spectrum of the quotient algebra if the latter concept is defined as the set of multiplicative functionals on $\mathscr{C}(X)$.

Now we pass to define the extended spectrum of elements of the algebra $\mathscr{A}$. To avoid repetitions of arguments we define at once the extended joint spectrum of a k-tuple of elements in an arbitrary compactification K of $\mathbb{C}^k$.

Definition (4.1) Denote by $\mathscr{F}$ a k-tuple $(f_1, \dots, f_k)$ of elements of $\mathscr{A}$. Let $\lambda \in K$. We say that λ is regular for $\mathscr{F}$ relative to J if there exist $c > 0$, $d > 0$ and a neighbourhood $\mathscr{O}(\lambda)$ of λ in K such that for all $\mu \in \mathscr{O}(\lambda) \cap \mathbb{C}^k$ and $z \in \zeta(J)$

$$(4) \qquad (\sum_{j=1}^{k} |f_j - \mu_j| \exp(cp))\tilde{}(z) \geq d.$$

The set of regular points for $\mathcal{F}$ in K is denoted by $\varrho(\mathcal{F}, J, K)$. It is open in K. Its complement is called the joint spectrum of $\mathcal{F}$ in K with respect to J and is denoted $\sigma(\mathcal{F}, J, K)$. We denote by $\sigma(\mathcal{F}, J)$ (resp. $\varrho(\mathcal{F}, J)$) the spectrum (resp. the set of regular points) of $\mathcal{F}$ in the Stone-Čech compactification $\beta\mathbb{C}^k$.

Let us observe that the sets $\varrho(\mathcal{F}, J, K)$ and $\sigma(\mathcal{F}, J, K)$ depend in fact only on classes $[\mathcal{F}] = ([f_1], \ldots, [f_k])$ hence we deal with an extension of the concept of the joint spectrum of a k-tuple of elements of $\mathcal{A}/J$. First of all we shall prove that for all finite points μ's in some neighbourhood of each regular element $\lambda \in K$ there exists a generalized resolvent $R(\mu)$ and the set of all $R(\mu)_{\mu\in\mathcal{U}}$ is bounded.

THEOREM (4.2) *Let $\lambda \in \varrho(\mathcal{F}, J, K)$ and let $\mathcal{O} \subset K$ be a neighbourhood of λ in which the inequality* (4) *is valid for some c, $d > 0$. Denote $\mathcal{U} = \mathcal{O} \cap \mathbb{C}^k$. There exist functions $R_j(\cdot, \mu) \in \mathcal{A}$, $\mu \in \mathcal{U}$ such that*

$$\sum_{j=1}^{k} (f_j - \mu_j) R_j(\cdot, \mu) - 1 \in J \tag{5}$$

for all $\mu \in \mathcal{U}$. There exist $r > 0$, $b > 0$ such that $\|R_j(\cdot, \mu)\|_r \leq b$ for all $\mu \in \mathcal{U}$.

Proof. If $z \notin \zeta(J)$ then there exists $g_z \in J$ and $c_z > 0$ such that

$$|g_z(w)| > \exp(-c_z p(w))$$

for all w in some neighbourhood of z in Ω.

By the supposition that λ is regular for $\mathcal{F}$ we obtain that

$$\sum_{j=1}^{k} |f_j(w) - \mu_j| \exp(cp(w)) \geq d$$

for appropriate $c > 0, d > 0$ and for all $w \in \Omega$ in some neighbourhood of $\zeta(J)$.

By the compactness of $\beta\Omega$ we can choose a finite number of functions $g_1, \ldots, g_m \in J$ such that for some $c, d > 0$ and for all $(w, \mu) \in \Omega \times \mathcal{U}$

$$\sum_{j=1}^{k} |f_j(w) - \mu_j| + \sum_{i=1}^{m} |g_i(w)| \geq \exp(-cp(w)). \tag{6}$$

By Theorem 2.1 (or its version for $\mathcal{C}_p(\Omega)$) the functions

$$\{f_j - \mu_j, g_i\}_{1\leq j\leq k,\, 1\leq i\leq m}$$

generate the space $\mathcal{A}$. In particular there exist functions $R_j \in \mathcal{A}$ such that

$$\sum_{j=1}^{k} R_j(w, \mu)(f_j(w) - \mu_j) + \sum_{j=k+1}^{k+m} R_j(w, \mu) g_{j-k}(w) = 1. \tag{7}$$

Since the second sum is an element of J we obtain the formula (5). The second part of the assertion follows by Theorem 2.2 in case of the algebra $\mathscr{A}_p(\Omega)$ and directly by the definition of the resolvents defined in section 2 when the algebra $\mathscr{C}_p(\Omega)$ is considered. ■

By the usual application of a partition of unity we can construct easily a global resolvent of the class $C^\infty(\Omega)$. Nevertheless the nonuniqueness of the resolvent makes difficult the study of its global properties.

The equation (7) permits us to formulate the condition of regularity of a point in a slightly stronger form.

COROLLARY (4.1) *Let K be a compactification of $\mathbb{C}^k$. A point $\lambda \in K$ is regular for $\mathscr{F}$ relative to J if and only if there exist a neighbourhood $\mathbb{O}(\lambda)$, $d > 0$, $c > 0$ and a neighbourhood $\mathscr{V}$ of $\zeta(J)$ such that* (4) *is valid for all $z \in \mathscr{V}$ and for all $\mu \in \mathbb{O}(\lambda) \cap \mathbb{C}^k$.*

Proof. Assume that λ satisfies all conditions determined in Definition (4.1). Let $c > 0$, $a > 0$ be such that the functions $R_j(w,\mu)$ in (7) satisfy $|R_j(\cdot,\mu)| \le a\exp(cp)$ for all finite $\mu \in \mathbb{O}(\lambda)$. Next define

$$\mathscr{V} = \{z \in \beta\Omega | (a\exp(cp)\sum_{j=k+1}^{k+m} |g_{j-k}|)^{\widehat{}}(z) < \frac{1}{2}\}.$$

The set $\mathscr{V}$ is a neighbourhood of $\zeta(J)$. Thanks to (7) we obtain for every finite $\mu \in \mathbb{O}(\lambda)$ and for $z \in \mathscr{V}$:

$$1 \le a(\sum_{j=1}^{k} |f_j - \mu_j| \exp(cp))^{\widehat{}}(z) + \frac{1}{2}.$$

Choosing $d < \frac{1}{2a}$ we obtain the desired result. ■

Exactly as in the case of the cospectrum the knowledge of the spectrum $\sigma(\mathscr{F}, J)$ permits us to obtain $\sigma(\mathscr{F}, J, K)$ for an arbitrary compactification K by simple projection. Let us denote by $P_K\colon \beta\mathbb{C}^k \to K$ the natural projection which leaves invariant the points of $\mathbb{C}^k$.

PROPOSITION (4.1) $P_K(\sigma(\mathscr{F}, J)) = \sigma(\mathscr{F}, J, K)$.

Proof. Suppose that $\lambda \in K$ is regular for $\mathscr{F}$ relative to J. Let $\mathbb{O}(\lambda)$ be a neighbourhood of λ such that (4) is satisfied. Then $P_K^{-1}(\mathbb{O}(\lambda))$ is the neighbourhood of an $P_K^{-1}(\lambda)$ for which (4) is also satisfied. It means that all elements of $P_K^{-1}(\lambda)$ are regular and consequently $P_K(\sigma(\mathscr{F}, J)) \subset \sigma(\mathscr{F}, J, K)$.

Now assume that $\lambda \in \sigma(\mathcal{F}, J, K)$. For all $d, c > 0$ and for each neighbourhood O of λ there exist $\mu^O \in O \cap \mathbb{C}^k$ and $z^O \in \sigma(J)$ such that

$$(\sum_{j=1}^{k} |f_j - \mu_j^O| \exp(cp))\check{}(z^O) < d.$$

By the compactness of $\beta\mathbb{C}^K$ the generalized sequence $\{\mu^O\}$ has an accumulation point, say $\mu \in \beta\mathbb{C}^k$. It follows by the construction that $P_K\mu = \lambda$.

The point μ is singular because at least for a subsequence μ^U which converges to μ we have

$$(\sum_{j=1}^{k} |f_j - \mu_j^U| \exp(cp))\check{}(z^U) < d. \qquad \blacksquare$$

Any k-tuple $\mathcal{F} = (f_1, \dots, f_k)$ of elements of $\mathcal{A}$ can be treated as a continuous mapping

$$\Omega \ni z \to (f_1(z), \dots, f_k(z)) \in \mathbb{C}^k \subset \beta\mathbb{C}^k.$$

There exists a unique continuous extension of this map on the domain $\beta\Omega$ which is denoted in the sequel by $\tilde{\mathcal{F}}$. The following result determines the relation between $\tilde{\mathcal{F}}(\zeta(J))$ and $\sigma(\mathcal{F}, J)$.

THEOREM (4.3)

$$\tilde{\mathcal{F}}(\zeta(J)) \subset \sigma(\mathcal{F}, J). \tag{8}$$

Proof. Let $\lambda = \tilde{\mathcal{F}}(w)$ where $w \in \zeta(J)$. Suppose that λ is regular for $\mathcal{F}$ with respect to J that is

$$\sum_{j=1}^{k} |f_j - \mu_j| \exp(cp)\check{}(z) \geq d \tag{9}$$

for all $z \in \zeta(J)$, $\mu \in \mathcal{O}(\lambda) \cap \mathbb{C}^k$.

By Corollary 4.1 the above inequality remains valid for $\mu \in \mathcal{O}(\lambda) \cap \mathbb{C}^k$ and $z \in \mathcal{V}$ where $\mathcal{V}$ is a neighbourhood of $\zeta(J)$. Let us choose an arbitrary finite point z from $\mathcal{V} \cap \tilde{\mathcal{F}}^{-1}(\mathcal{O}(\lambda))$. Take $\mu_0 = \mathcal{F}(z)$. Obviously $\mu_0 \in \mathcal{O}(\lambda)$ but the left hand side of (9) vanishes at z. This is a contradiction. $\blacksquare$

By applying to both sides of (8) the projection P_K for an arbitrary compactification K of $\mathbb{C}^k$ we obtain immediately

COROLARY (4.2)

$$P_K\tilde{\mathcal{F}}(\zeta(J)) \subset \sigma(\mathcal{F}, J, K). \tag{10}$$

5. Towards the spectral mapping theorem

The question if the inclusion (8) is in fact an equality remains open. In the general case we can only prove that $\sigma(\mathcal{F}, J) \setminus \tilde{\mathcal{F}}(\zeta(J)) \subset \beta\mathbb{C}^k \setminus \mathbb{C}^k$ that is to say inside $\mathbb{C}^k$ both sets coincide.

THEOREM (5.1)

$$\sigma(\mathcal{F}, J) \cap \mathbb{C}^k = \tilde{\mathcal{F}}(\zeta(J)) \cap \mathbb{C}^k.$$

Proof. In virtue of Theorem 4.3 it remains to prove that $\sigma(\mathcal{F}, J) \cap \mathbb{C}^k \subset \tilde{\mathcal{F}}(\zeta(J))$. If $\lambda \in \sigma(\mathcal{F}, J) \cap \mathbb{C}^k$ then for arbitrary $c > 0$ and natural n we can find $\mu^{(n)} \in B(\lambda, \frac{1}{n})$ and $z_n \in \zeta(J)$ such that

$$\left(\sum_{j=1}^{k} |f_j - \mu_j^{(n)}| \exp(cp) \right)^{\sim} (z_n) \leq \frac{1}{n}.$$

The factor $\exp(cp)$ is no less than 1 in the whole compactified domain which implies

$$\sum_{j=1}^{k} |\tilde{f}_j(z_n) - \mu_j^{(n)}| \leq \frac{1}{n}.$$

The sequences $\{\tilde{\mathcal{F}}(z_n)\}$ and $\{\mu^{(n)}\}$ are equivalent hence both of them tend to λ.
Since the cospectrum $\zeta(J)$ is compact the sequence (z_n) has an accumulation point in some $z \in \zeta(J)$. It means that $\tilde{\mathcal{F}}(z) = \lambda$. ■

The above result means that if there exists $\lambda \in \sigma(\mathcal{F}, J) \setminus \tilde{\mathcal{F}}(\zeta(J))$ then for the finite elements of some neighbourhood of λ the generalized resolvents do exist but they form an unbounded subset in $\mathcal{A}$.

In the very special case of $\Omega = \mathbb{C}^n$ and $\mathcal{F} = \mathrm{id}_{\mathbb{C}^n}$ we have equality in (8).

Let us denote by $\mathcal{Z}$ the n tuple for which the i-th function is just $f_i(z) = z_i$. The corresponding mapping $\tilde{\mathcal{Z}}$ is the identity in $\beta\mathbb{C}^n$.

THEOREM (5.2) *Let $\Omega = \mathbb{C}^n$. Then*

$$\sigma(\mathcal{Z}, J) = \zeta(J). \tag{11}$$

Proof. The relation $\zeta(J) \subset \sigma(\mathcal{Z}, J)$ follows by Theorem (4.3). It remains to prove that $\sigma(\mathcal{Z}, J) \subset \zeta(J)$. Suppose that $\lambda \in \sigma(\mathcal{Z}, J)$. According to Corollary (4.1) for arbitrary c, $\epsilon > 0$ and for every neighbourhood O of λ and V of $\sigma(J)$ there exists $z \in V \cap \mathbb{C}^n$ and $\mu \in O \cap \mathbb{C}^n$ such that

$$\sum_{j=1}^{n} |z_j - \mu_j| \exp(cp(z)) < \epsilon. \tag{12}$$

Our aim is to prove that for every $f \in J$ and $d > 0$, the extension $(f\exp(dp))\check{}$ vanishes at λ. In virtue of Corollary (2.5) in [1] there exist functions $Q_j \in \mathcal{A}(\mathbb{C}^{2n})$ such that

$$f(z) - f(\mu) = \sum_{j=1}^{n} Q_j(z,\mu)(z_j - \mu_j)$$

and for some constants C, D depending only upon f the inequality

$$|Q_j(z,\mu)| \leq C\exp(D(p(z) + p(\mu))$$

is valid on $\mathbb{C}^{2n}$. Recall that the function p has the property that for appropriate $A, B > 0$ the relation $\|z - \mu\| \leq 1$ implies that $p(\mu) \leq Ap(z) + B$. Choose the neighbourhood V in such a way that $|f\exp(d(Ap + B))|\check{} < \epsilon$ in V. We obtain

$$\begin{aligned} |f(z) - f(\mu)|\exp(dp(\mu)) &\leq \exp(dp(\mu))\sum_{j=1}^{n}|Q_j(z,\mu)||z_j - \mu_j| \\ &\leq C\exp\big((D + d)p(\mu) + Dp(z)\big)\sum_{j=1}^{n}|z_j - \mu_j| \\ &\leq C\exp\big((D + d)B\big)\exp\big(((D + d)A + D)p(z)\big)\sum_{j=1}^{n}|z_j - \mu_j|. \end{aligned}$$

Since the constants A, B, C, D depend only on the functions f and p we can suppose that the constant c was chosen from the beginning as $((D + d)A + D)$ and in place of ϵ in (12) the value $\epsilon/C\exp((D + d)B)$ has been used. In this way we obtain

$$|f(z) - f(\mu)|\exp(dp(\mu)) \leq \epsilon.$$

Taking into account that $|f(z)|\exp(dp(\mu)) \leq |f(z)|\exp(d(Ap(z) + B)) \leq \epsilon$ we obtain $|f(\mu)\exp(dp(\mu))| \leq 2\epsilon$ for several $\mu \in O \cap \mathbb{C}^n$. Since the neighbourhood O is arbitrary we conclude: $(f\exp(dp))\check{}(\lambda) = 0$. ∎

Now applying Theorem 3.4 and Proposition 4.1 we obtain

COROLARY (5.1) *For an arbitrary compactification K of $\mathbb{C}^n$*

$$\sigma(\mathcal{Z}, J, K) = \zeta(J, K).$$

Coming back to the question of the equality in place of the inequality in (10) let us observe that for the compactification of Alexandroff K_a of $\mathbb{C}^k$ we have in fact

$$P_{K_a}\tilde{\mathcal{F}}(\zeta(J)) = \sigma(\mathcal{F}, J, K_a).$$

This equation asserts simply that Theorem 5.1 is valid and both sides can be unbounded only at the same time.

Let us observe also the following

PROPOSITION (5.1) *Assume that for certain compactification K we have the equality*

$$P_K\tilde{\mathcal{F}}(\zeta(J)) = \sigma(\mathcal{F}, J, K).$$

and that $K \succ C$. Then

$$P_C\tilde{\mathcal{F}}(\zeta(J)) = \sigma(\mathcal{F}, J, C).$$

Proof. It suffices to apply the projection $P_{KC}: K \to C$ to both sides of the first equation. ∎

We can also obtain the equality in (10) by imposing several topological condition on the compactification K of $\mathbb{C}^k$.

THEOREM (5.3) *Let K be a compactification of $\mathbb{C}^k$ such that for all $\lambda, \nu \in K$ there exist neighbourhoods O of λ and U of ν such that*

$$\inf_{\mu \in O \cap \mathbb{C}^k, \omega \in U \cap \mathbb{C}^k} \|\mu - \omega\| > 0.$$

Then

$$P_K\tilde{\mathcal{F}}(\zeta(J)) = \sigma(\mathcal{F}, J, K).$$

Proof. In virtue of Corollary 4.2 it remains to prove that $\sigma(\mathcal{F}, J, K) \subset P_K\tilde{\mathcal{F}}(\zeta(J))$. We know that for finite points the inclusion is valid, hence suppose that $\lambda \in \sigma(\mathcal{F}, J, K) \setminus \mathbb{C}^k$ and that $\lambda \notin P_K\tilde{\mathcal{F}}(\zeta(J))$.

According to our supposition for every $w \in P_K\tilde{\mathcal{F}}(\zeta(J))$ there exist neighbourhoods O_w of λ and U_w of w as well as a constant ϵ_w such that $\|\mu - \omega\| > \epsilon_w$ for all $\mu \in O_w \cap \mathbb{C}^k$ and $\omega \in U_w \cap \mathbb{C}^k$. Since the set $A = P_K\tilde{\mathcal{F}}(\zeta(J))$ is compact we can choose a finite subcovering from the covering $\{U_w\}_{w \in A}$ obtaining in this way neighbourhoods

$$U = \bigcup_{1 \le i \le m} U_{w_i} \text{ and } O = \bigcap_{1 \le i \le m} O_{w_i}$$

such that $\|\mu - \omega\| > \epsilon = \min_{1 \le i \le m}\{\epsilon_{w_i}\}$ for all finite $\mu \in O$ and $\omega \in U$.

However by the singularity of λ there exists $\mu' \in O \cap \mathbb{C}^k$ and $z \in (P_K\tilde{\mathcal{F}})^{-1}(U)$ such that

$$(\sum_{j=1}^{k} |f_j - \mu'_j|)\check{}(z) < \epsilon.$$

By continuity the same is valid for some $z' \in (P_K\tilde{\mathcal{F}})^{-1}(U) \cap \mathbb{C}^n$. This is a contradiction, hence $\lambda \in P_K\tilde{\mathcal{F}}(\zeta(J))$. ∎

In particular it is easy to see that the projective spaces $\mathbf{P}^{2k}(\mathbb{R})$, $\mathbf{P}^k(\mathbb{C})$ which are compactifications of $\mathbb{C}^k$ satisfy the assumptions of the above theorem. We obtain

COROLARY (5.2) *For an arbitrary* $\mathcal{F} \in \mathcal{A}^k$ *and for* $K = \mathbf{P}^{2k}(\mathbb{R})$ *or* $K = \mathbf{P}^k(\mathbb{C})$ *we have*

$$P_K \tilde{\mathcal{F}}(\zeta(J)) = \sigma(\mathcal{F}, J, K).$$

Let us now observe that the spectrum $\sigma(\mathcal{F}, J)$ has other properties which are characteristic for various types of joint spectra considered in the theory of commutative and noncommutative Banach algebras.

The space $\beta\mathbb{C}^k$ can be considered in a natural way as a closed subset of the k-th Cartesian product $\beta\mathbb{C} \times \ldots \times \beta\mathbb{C}$. As the natural injection we consider the continuous extension of the mapping: $\mathbb{C}^k \hookrightarrow \prod_{i=1}^k \beta\mathbb{C}$. Using this convention we can assert the following:

PROPOSITION (5.2)

$$\sigma(\mathcal{F}, J) \subset \prod_{i=1}^{k} \sigma(\{f_i\}, J).$$

Proof. We assert only that if a point $\lambda \in \beta\mathbb{C}^k$ projected in $\prod_{i=1}^k \sigma(\{f_i\}, J)$ on the m-th variable gives us λ_m which is regular for f_m relative to J, then λ is regular for $\mathcal{F}$. This is obvious by the definition of $\varrho(\mathcal{F}, J)$. ■

Let $\mathcal{P}^{(m)} = (P_1, \ldots, P_m)$ be a tuple of polynomials of k complex variables. Denote by $\tilde{\mathcal{P}}^{(m)}$ the continuous extension to $\beta\mathbb{C}^k$ of the mapping $(z_1, \ldots, z_k) \to (P_1(z), \ldots, P_m(z)) \in \beta\mathbb{C}^m$.

THEOREM (5.4) *For every k-tuple* $\mathcal{F}$ *and for every* $\mathcal{P}^{(m)}$

$$\tilde{\mathcal{P}}^{(m)}(\sigma(\mathcal{F}, J)) \subset \sigma(\mathcal{P}^{(m)} \circ \mathcal{F}, J).$$

Proof. For simplicity we write just $\mathcal{P}$ in place of $\mathcal{P}^{(m)}$. Given finite points $z \in \Omega$ and $\mu \in \mathbb{C}^k$ let us write

$$P_i(\mathcal{F}) - P_i(\mu) = \sum_{|\alpha|=1}^{r} Q_{i\alpha}(\mu)(f_1 - \mu_1)^{\alpha_1} \ldots (f_k - \mu_k)^{\alpha_k},$$

where $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a multiindex, $Q_{i\alpha}$ are polynomials of k variables and r is the maximum of the orders of the polynomials P_i. Assume that $\lambda \in \beta\mathbb{C}^k$ belongs to the spectrum of $\mathcal{F}$ relative to J. Given c, ϵ we can find in each neighbourhood of λ and a finite point μ such that in every neighbouhood of $\zeta(J)$ there exists z which satisfies

$$\sum_{j=1}^{k} |f_j(z) - \mu_j| \exp(cp(z)) \leq \min\{\epsilon/M, 1/2\},$$

where $M = \sum_{i=1}^m \sum_{|\alpha|=1}^r |Q_{i\alpha}(\mu)|$. We obtain

$$\begin{aligned}\sum_{i=1}^m &|P_i(\mathcal{F}(z)) - P_i(\mu)| \exp(cp(z))\\ &\leq (\sum_{i=1}^m \sum_{|\alpha|=1}^r |Q_{i\alpha}(\mu)||f_1(z)-\mu_1|^{\alpha_1}\dots|f_k(z)-\mu_k|^{\alpha_k})\exp(cp(z))\\ &\leq M\sum_{i=1}^k \exp(cp(z))|f_i(z)-\mu_i| \leq \epsilon.\end{aligned}$$

This proves that $\tilde{\mathcal{P}}(\lambda) \in \sigma(\mathcal{P}\circ\mathcal{F}, J)$. ■

Using the terminology of the spectral theory of topological algebras the above result is the one-way spectral mapping theorem for $\sigma(\mathcal{F}, J)$.

If we consider the projective space as the compactification of $\mathbb{C}^k$ we can obtain a version of the complete spectral mapping theorem.

Assume that $\hat{\mathcal{P}}: \mathbf{P}^k(\mathbb{C}) \to \mathbf{P}^m(\mathbb{C})$ is a continuous application which maps $\mathbf{P}^k(\mathbb{C}) \setminus \mathbb{C}^k$ into $\mathbf{P}^m(\mathbb{C}) \setminus \mathbb{C}^m$ and such that its restriction to $\mathbb{C}^k$ is of the form $\mathcal{P} = (p_1, \dots, p_m)$, where p_i are polynomials of k complex variables. As a unique continuous extension of a mapping of polynomial type $\hat{\mathcal{P}}$ satisfies the relation

$$\hat{\mathcal{P}} \circ P_{\mathbf{P}^k(\mathbb{C})} = P_{\mathbf{P}^m(\mathbb{C})} \circ \tilde{\mathcal{P}}. \tag{13}$$

THEOREM (5.5) *Let $\mathcal{F} \in \mathcal{A}^k$. Then*

$$\hat{\mathcal{P}}(\sigma(\mathcal{F}, J, \mathbf{P}^k(\mathbb{C}))) = \sigma(\mathcal{P}\circ\mathcal{F}, J, \mathbf{P}^m(\mathbb{C})).$$

Proof. We calculate applying Corollary 5.2 and (13):

$$\begin{aligned}\hat{\mathcal{P}}(\sigma(\mathcal{F}, J, \mathbf{P}^k(\mathbb{C}))) &= \hat{\mathcal{P}}(P_{\mathbf{P}^k(\mathbb{C})}\tilde{\mathcal{F}}(\zeta(J)))\\ &= P_{\mathbf{P}^m(\mathbb{C})}(\tilde{\mathcal{P}}\circ\tilde{\mathcal{F}}(\zeta(J))) = \sigma(\mathcal{P}\circ\mathcal{F}, J, \mathbf{P}^m(\mathbb{C})).\end{aligned}$$ ■

If we use the real projective space $\mathbf{P}^{2i}(\mathbb{R})$ as the compactification of $\mathbb{C}^i$ we obtain in the same way the formula

$$\hat{\mathcal{P}}(\sigma(\mathcal{F}, J, \mathbf{P}^{2k}(\mathbb{R}))) = \sigma(\mathcal{P}\circ\mathcal{F}, J, \mathbf{P}^{2m}(\mathbb{R})), \tag{14}$$

valid for every

$$\hat{\mathcal{P}}: \mathbf{P}^{2k}(\mathbb{R}) \to \mathbf{P}^{2m}(\mathbb{R})$$

which is of complex polynomial type on $\mathbb{C}^k$ (canonically imbedded in $\mathbf{P}^{2k}(\mathbb{R})$) and which sends the points at infinity into points at infinity.

6. Multiplicative functionals on $\mathscr{A}_p(\Omega)$

The functional of the evaluation at a given point $z \in \Omega$

$$\chi_z \; : \; \mathscr{A}_p(\Omega) \ni f \to f(z) \in \mathbb{C}$$

is a continuous multiplicative functional on $\mathscr{A}_p(\Omega)$. It is natural to ask if the functionals of the form χ_z for $z \in \Omega$ are the unique continuous multiplicative functionals on $\mathscr{A}_p(\Omega)$. The answer is positive and in [4] we can find this result. It is deduced however from the same fact which is the subject of our Theorem 2.3 and it seems that its proof in [4] is not complete. In this section we prove a stronger result and the proof is simpler. In particular it does not make use of the functional calculus of Waelbroeck.

It results that all multiplicative functionals on $\mathscr{A}_p(\Omega)$ are given by the evaluation at a fixed point $z \in \Omega$. In particular it means that all multiplicative functionals are continuous and the maximal ideals $\mathscr{M}_z$ for $z \notin \Omega$ are of codimension > 1.

THEOREM (6.1) *Every nonzero multiplicative functional on $\mathscr{A}_p(\Omega)$ is of the form χ_z for some $z \in \Omega$.*

Proof. Let $\chi \colon \mathscr{A}_p(\Omega) \to \mathbb{C}$ be a nontrivial multiplicative functional. Its kernel $J_\chi = \{f \in \mathscr{A}_p(\Omega) | \chi(f) = 0\}$ is an ideal of codimension 1. Its extended cospectrum is nontrivial and for $\mu \in \zeta(J_\chi)$ we have for every $f \in J_\chi$:

$$\lim_{\Omega \ni w \to \mu} f(w) = 0$$

by Theorem 3.2.

Since $f - \chi(f) \in J_\chi$ for an arbitrary $f \in \mathscr{A}_p(\Omega)$ we obtain

$$\lim_{\Omega \ni w \to \mu} f(w) = \chi(f) \tag{15}$$

for all $f \in \mathscr{A}_p(\Omega)$. In particular taking $f_j(w) = w_j$ we observe that $\chi(f_j) = \lim_{\Omega \ni w \to \mu} w_j$. It implies in particular that $z = (\chi(f_1), \dots, \chi(f_n))$ belongs to $\overline{\Omega}$ (closure in $\mathbb{C}^n$!). If $z \in \Omega$ then the functional χ is equal to χ_z by (15) and we are done.

Suppose that $z \in \overline{\Omega} \setminus \Omega$. We need the following result which in the Waelbroeck's terminology says that Ω is a spectral set for $\mathcal{Z}$:

PROPOSITION (6.1) [4], [13] *There exists $t > 0$ such that for every $\lambda \in \mathbb{C}^n \setminus \Omega$ there exist $r_j(\lambda) \in \mathscr{A}_p^t$ which satisfy*

$$\sum_{j=1}^{n} (w_j - \lambda_j) r_j(\lambda) = 1 \tag{16}$$

for all $w \in \Omega$. The set of all functions $\{r_j(\lambda)\}$ is bounded in $\mathscr{A}^t_p$.

Putting $\lambda_j = \chi(f_j)$ in (16) and applying χ to both sides we obtain a contradiction. ■

The following corollaries are obvious.

COROLARY (6.1) *All multiplicative functionals on $\mathscr{A}_p(\Omega)$ are continuous. All maximal ideals of $\mathscr{A}_p(\Omega)$ of codimension 1 are of the form $\mathcal{M}_z$ where $z \in \Omega$.*

COROLARY (6.2) *For every $\mu \in \beta\Omega \setminus \Omega$ there exists $f \in \mathscr{A}_p(\Omega)$ such that* $\lim_{\Omega \ni w \to \mu} |f(w)| = \infty$.

GENARO ALMENDRA
UNIVERSIDAD AUTÓNOMA CHAPINGO
AREA DE MATEMÁTICAS

ANTONI WAWRZYŃCZYK
UNIVERSIDAD AUTÓNOMA METROPOLITANA-IZTAPALAPA
DEPARTAMENTO DE
MATEMÁTICAS

REFERENCES

[1] C.A.BERENSTEIN AND B.A.TAYLOR, *Interpolation problems in $\mathbb{C}^n$ with applications to harmonic analysis,* J. Anal. Math., **38**, (1980), 188–254.
[2] L.CARLESON, *Interpolation by bounded analytic functions and the corona problem,* Ann. of Math. 2, **76**, (1962), 547–559.
[3] L. EHRENPREIS, Fourier Analysis in Several Complex variables, Wiley-Interscience, New York, 1970.
[4] J-P. FERRIER, Spectral Theory and Complex Analysis, North-Holland, Mathematics Studies 4, 1973.
[5] D.I.GUREVICH, *Countrexamples to the problem of L.Schwartz, Funktsional.* Anal. i Prilozhen., 2, **9**, (1975), 29–35.
[6] L. HÖRMANDER, *Generators of some rings of analytic functions,* Ann. of Math., 2, **76**, (1962), 943–949.
[7] L. HÖRMANDER, An Introduction to Complex Analysis in Several Variables, Springer Verlag, 1990.
[8] J. J. KELLEHER, B. A. TAYLOR, *Closed ideals in locally convex algebras of analytic functions,* J. Reine Angew. Math. **255**, (1972), 190–209.
[9] R. C. WALKER, The Stone-Čech Compactification, Springer Verlag, 1976.
[10] L. WAELBROECK, Lectures in spectral theory, Dept. of Math., Yale University, 1963.
[11] A. WAWRZYŃCZYK, *How to make nontrivial the spectrum of a translation invariant space of smooth functions,* Informe de Investigación, UAM-I, 1991.
[12] A. WAWRZYŃCZYK, *El espectro extendido y el análisis espectral de espacios invariantes,* Aport. Mat., Comunicaciones **11**, (1992), 41–55.
[13] A. WAWRZYŃCZYK, *A note on generating ideals of holomorphic functions,* Informe de Investigación, UAM-I, 1993, submitted for publication.
[14] W. ZELAZKO, Banach Algebras, Elsevier Publ. Comp. and PWN, 1973.