Boletín de la Sociedad Matemática Mexicana Vol. 39, 1994

CONTROLLED SEMI-MARKOV MODELS WITH DISCOUNTED UNBOUNDED COSTS

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We consider semi-Markov control models with Borel state and control spaces, unbounded costs and not necessarily compact constraint sets. The basic control problem we are concerned with, is to minimize the expected total discounted cost. We show the existence of optimal stationary policies and we provide characterizations of the optimal cost function and optimal policies. Criteria for asymptotic optimality, convergence of value iteration, policy iteration and other approximation procedures are also discussed.

1. Introduction

This paper deals with semi-Markov control models (SMCMs) with Borel state and control spaces, allowing unbounded one-stage cost functions and the control constraint sets are not necessarily compact. The basic optimal control problem is to minimize the total expected discounted cost. Most of literature related to this problem is concentred on the countable state case, bounded costs or compact control set (see [1], [6], [8] and their references).

In this paper we extend the assumptions in [3] to the context of SMCMs. We show that under suitable conditions, V_α (the optimal cost function), is solution to the optimality equation and alse we show the existence of optimal stationary policies. Other questions we are concerned with are: How can we "approximate" V_α?, what are the conditions for a control policy to be optimal? and when a given policy is "close" to being optimal?.

In section 2 we introduce the SMCM, the performance criterion, the assumptions (regularity, continuity and compactness) and a measurable selection Theorem. In section 3 we show the main result in this paper (see Theorem (3.3) and provide conditions for a control policy to be optimal (see Theorem (3.5)). In section 4, we provide some answers to the questions: How can we "approximate" V_α? and in section 5, we obtain several optimality criteria and we briefly discuss the notion of asymptotic optimality.

NOTATION: A Borel space X is a Borel subset of a complete separable metric space and we denote by $B(X)$ its Borel σ-algebra and "measurable" always means Borel-measurable. Given a Borel space, we denote by $M(X)_+$ the family of measurable and no-negative functions on X, $L(X)_+$ denotes the subclass of lower semicontinuous functions in $M(X)_+$.

If X and Y are Borel spaces then a stochastic kernel $P(.|.)$ on X given Y is a function such that: $P(.|y)$ is a probability measure on X for each $y \in Y$ and $P(B|.)$ is a measurable function on Y for each $B \in B(X)$.

2. The semi-Markov control model

DEFINITION (2.1). A semi-Markov control model (SMCM) written (X, A, $\{A(x) : x \in X\}$, Q, F, D, d), consists of:

1. A nonempty Borel space X, called the state space.
2. A nonempty Borel space A, the control (or action) space.
3. A collection $\{A(x) : x \in X\}$ of nonempty Borel subsets of A. For each $x \in X$, $A(x)$ is the set of admissible controls (or actions) in the state x. Moreover we assume that the set $\mathcal{K} = \{(x, a) : x \in X, a \in A(x)\}$ is a Borel subset of $X \times A$ and contains the graph of a measurable map from X to A. We denote by $\mathcal{F}$, the class of measurable functions $f: X \to A$ such that $f(x) \in A(x)$ for all $x \in X$.
4. A stochastic kernel $Q(.|.)$ on X given $\mathcal{K}$, called the transition law.
5. A function $F(t|x, a, y)$ which is a distribution function, for each $(x, a, y) \in \mathcal{K} \times X$, and we assume to be jointly measurable in (x, a, y) for each $t \in \mathbb{R}$.
6. The functions D, $d \in M(\mathcal{K})_+$ are the so called cost functions.

The SMCM represents a stochatic system that evolves in the next way: At time $t = 0$ the system is in the state $x_0 \in X$ and a control $a_0 \in A(x_0)$ is applied, then the following things happen:

An immediate cost is incurred.

The system moves to a new state $x_1 \in X$ according to the probability measure $Q(.|x_0, a_0)$.

Conditional on the next state x_1 the time δ_1 until the transition occurs has the distribution function $t \to F(t|x_0, a_0, x_1)$.

A cost rate $d(x_0, a_0)$ is imposed until the transition occurs.

After the transition occurs, a control $a_1 \in A(x_1)$ is chosen and the process continues in this way indefinitely.

We will represent by x_n, a_n, δ_{n+1} $(n \geq 0)$, the state of the system after n^{th} transition, the action chosen in that state and the corresponding sojourn (or holding) time, respectively.

For each $t = 0, 1, \ldots$, define the space of admissible histories up to time t by $H_0 := X$ and

$$H_t := \mathcal{K}^t \times X = \mathcal{K} \times H_{t-1}, \qquad t = 1, 2, \ldots$$

An element $h_t \in H_t$ is a vector or history, of the form

$$h_t = (x_0, a_0, \ldots, x_{t-1}, a_{t-1}, x_t)$$

where $(x_n, a_n) \in \mathcal{K}$ for $n = 0, \ldots, t-1$ and $x_t \in X$.

DEFINITION (2.2). a) A control policy, is a sequence $\pi = \{\pi_t\}$ of stochastic kernels π_t on A given H_t, satisfying the constraint $\pi_t(A(x_t)|h_t) = 1$ for all $h_t \in H_t$, $t = 0, 1, \ldots$ We denote by Δ, the class of all policies.

b) A control policy $\pi = \{\pi_t\}$ is said to be stationary, if there exists a function $f \in \mathcal{F}$ such that, $\pi_t(.|h_t)$ is concentrated at $f(x_t)$, for all $h_t \in H_t$, $t = 0, 1, \ldots$ and we will identify π with f, and refer to $\mathcal{F}$ the set of statinary policies.

REMARK (2.3). By a theorem of C. Ionescu Tulcea (see [2], Proposition C.3 in Apendix C), Given $x \in X$ and $\pi \in \Delta$, there exist a probability space (Ω, F, P_x^π) and three sequences of random variables x_n, a_n, δ_{n+1} $(n = 0, 1, \ldots)$ such that:

a) $P_x^\pi[x_0 = x] = 1$

b) $P_x^\pi[x_{n+1} \in B|h_n, a_n] = Q(b|x_n, a_n)$ for all $B \in \mathcal{B}(X)$, $h_n \in H_n$ and $a_n \in A(x_n)$, $n = 0, 1, \ldots$

c) $P_x^\pi[a_n \in C|h_n] = \pi_n(C|h_n)$ for all $C \in \mathcal{B}(A)$, $h_n \in H_n$, $n = 0, 1, \ldots$

d) $P_x^\pi[\delta_{n+1} \leq t|h_{n+1}] = F(t|x_n, a_n, x_{n+1})$ for all $t \in \mathbb{R}$, $h_{n+1} \in H_{n+1}$

e) The random variable δ_1, δ_2, $\ldots$ are conditionally independent given the process $(x_0, a_0, \ldots, x_n, a_n, \ldots)$. The expectation with respect P_x^π is denoted by E_x^π. In order to ensure that an infinite number of transitions does not occur in a finite interval, we need to impose a condition. To do this, we introduce the following notation:

$$H(t|x, a) := \int_X F(t|x, a, y)Q(dy|x, a)$$

This represents the distribution function of the holding time in the state $x \in X$, when the control $a \in A(x)$ is chosen,

ASUMPTION (2.4). *There exist $\epsilon > 0$, $\theta > 0$ such that*

$$H(\theta|x, a) \leq 1 - \epsilon$$

for all $(x, a) \in \mathcal{K}$.

Throughout the following, we will suppose that $\alpha > 0$ and define, for every $(x, a) \in \mathcal{K}$,

$$\Delta_\alpha(x, a) := \int_0^\infty \exp(-\alpha t)H(dt|x, a)$$

and

$$\tau_\alpha(x, a) := \frac{[1 - \Delta_\alpha(x, a)]}{\alpha}$$

LEMMA (2.5). *(see [12] proposition 2.4 b), c)). If assumption 2.4 holds, then*

1. $\Delta_\alpha < 1$, *where* $\Delta_\alpha := \sup_{\mathcal{K}} \Delta_\alpha(x, a)$
2. $P_x^\pi\left[\sum_{n=1}^\infty \delta_n = \infty\right] = 1\ \forall x \in X$, $\pi \in \Pi$

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We assume that costs are continuously discounted i.e., a cost C incurred at time t is equivalent to a cost $Ce^{-\alpha t}$ at time 0. The cost-per-stage function $C_\alpha(x, a)$ when the process is in the state x and action a is chosen is defined by

$$C_\alpha(x, a) := D(x, a) + d(x, a) \int_0^\infty \int_0^t e^{-\alpha s} ds H(dt|x, a)$$

where $D(x, a)$ represents an immediate cost and $d(x, a)$ the rate cost imposed until the transition ocurrs.

Observe that for all $(x, a) \in K$,

$$C_\alpha(x, a) = D(x, a) + \tau_\alpha(x, a) d(x, a), \qquad (x, a) \in \mathcal{K} \tag{2.1}$$

and let $T_0 \equiv 0$ and $T_n = T_{n-1} + \delta_n$, $n = 1, 2, \ldots$.

Definition (2.6) a) Given $x \in X$ and $\pi \in \Pi$, let

$$V(\pi, x) := E_x^\pi \sum_{n=0}^\infty \exp(-\alpha T_n) C_\alpha(x_n, a_n)$$

be the α-discounted expected total cost when using the policy π, given the initial state $x_0 = x$.

b. The *optimal α-discounted cost* when the initial state is $x_0 = x$ is defined by

$$V_\alpha(x) := \inf_\pi V(\pi, x).$$

c. A policy $\pi \in \Pi$ is said to be α-optimal if

$$V(\pi, x) = V_\alpha(x) \qquad \text{for all } x \in X.$$

REMARK (2.7). a. For all $x \in X$, $\pi \in \Pi$ observe that

$$V(\pi, x) = E_x^\pi [C_\alpha(x_0, a_0) + \sum_{n=1}^\infty \Delta_\alpha(x_0, a_0) \ldots \Delta_\alpha(x_{n-1}, a_{n-1}) C_\alpha(x_n, a_n)]$$

b. If we denote by $V_n(\pi, x)$ the α-discounted cost until the n^{th} transition occurs then

$$V_1(\pi, x) = E_x^\pi C_\alpha(x_0, a_0)$$

$$V_n(\pi, x) = E_x^\pi \sum_{k=0}^{n-1} \exp(-\alpha T_k) C_\alpha(x_k, a_k)$$

$$= E_x^\pi [C_\alpha(x_0, a_0) + \sum_{k=1}^{n-1} \Delta_\alpha(x_0, a_0) \ldots \Delta_\alpha(x_{k-1}, a_{k-1}) C_\alpha(x_k, a_k)], \; n > 1$$

c. If we write $\Lambda_n^\alpha := \Pi_{k=0}^{n-1}\Delta_\alpha(x_k, a_k)$ for $n = 1, 2, \ldots$ and $\Lambda_0^\alpha := 1$, then, by a. and b., we have

$$V(\pi, x) = E_x^\pi \sum_{n=0}^{\infty} \Lambda_n^\alpha C_\alpha(x_n, a_n)$$

and

$$V_n(\pi, x) = E_x^\pi \sum_{k=0}^{n-1} \Lambda_k^\alpha C_\alpha(x_k, a_k) \qquad n = 1, 2, \ldots$$

Semi-continuity and compactness conditions

In order to guarantee that $\mathcal{F}$ (the set introduced in Definition (2.1)) contains suitable "minimizers", we require to impose semi-continuity and inf-compactness conditions on the SMCM.

Definition (2.8). A real valued function on $\mathcal{K}$ is said to be *inf-compact* on $\mathcal{K}$ if the set $\{a \in A(x) : v(x, a) \leq r\}$ is compact for every $x \in X$ and $r \in \mathbb{R}$.

ASSUMPTION (2.9). *a. Both $D(x, a)$ and $d(x, a)$ belong to $L(\mathcal{K})_+$, and $D(x, a)$ is inf-compact on $\mathcal{K}$.*

b. The transition law Q is weakly continuous, i.e., the function

$$v'(x, a) := \int v(y) Q(dy|x, a)$$

is bounded and continuous in $(x, a) \in \mathcal{K}$ for each bounded and continuous function v on X.

c. $F(t|x, a, y)$ is continuous in $(x, a) \in \mathcal{K}$ for each $y \in X$ and $t \in \mathbb{R}$.

d. For each $v \in L(\mathcal{K})_+$, the function $v^*(x) := \inf_{a \in A(x)} v(x, a)$ is l.s.c. on X.

REMARK (2.10). Assumption (2.9)d) holds if e.g. the multifunction $x \to A(x)$ is upper semicontinuous, and $A(x)$ is compact for every $x \in X$ ([9], Proposition 10.2), or if v is inf-compact and the multifunction $x \to A^*(x) := \{a \in A(x) : v^*(x) = v(x, a)\}$ is lower semicontinuous ([6], Lemma 3.2 (f)).

PROPOSITION (2.11). *If Assumption (2.9) holds, then*

a. $H(t|x, a)$ is continuous in $(x, a) \in \mathcal{K}$ for each $t \in \mathbb{R}$.

b. The functions $\tau_\alpha(x, a)$ and $\Delta_\alpha(x, a)$ are continuous in $(x, a) \in \mathcal{K}$.

c. $C_\alpha(x, a)$ is in $L(\mathcal{K})_+$ and it is inf-compact on $\mathcal{K}$. ■

Proof: a. follows from Proposition 14.2 in [9]. b) and c) can be obtained using the same arguments provided in Proposition 4.2 in [12]. ■

Measurable selection lemma

LEMMA (2.12). *a. If v is inf-compact, l.s.c. and bounded from below on $\mathcal{K}$, then the function v^* (defined in Assumption (2.9)d) is measurable, and there exists $f \in \mathcal{F}$ such that*

$$v^*(x) = v(x, f(x)) \qquad \forall x \in X.$$

b. If Assumption (2.9) holds and $\mu \in L(X)_+$, then the function

$$\mu^*(x) := \inf\{C_\alpha(x,a) + \Delta_\alpha(x,a) \int \mu(y) Q(dy|x,a)\}$$

belongs to $L(X)_+$, and there exists $f \in \mathcal{F}$ such that

$$\mu^*(x) = C_\alpha(x, f(x)) + \Delta_\alpha(x, f(x)) \int \mu(y) Q(dy|x, f(x)) \qquad \forall x \in X.$$

Proof: Part a. follows from Corollary 4.3 in [7]. To prove b), observe that, from Assumption (2.9)b. and Proposition (2.11)b.-c., the function

$$(x,a) \to C_\alpha(x,a) + \Delta_\alpha(x,a) \int \mu(y) Q(dy|x,a)$$

is nonnegative, l.s.c. and inf-compact on $\mathcal{K}$. Thus, using a. and Assumption (2.9)d., we obtain b. ∎

3. The optimality equation

In this section we give conditions under which $V_\alpha(x)$ is the (pointwise) minimal function in $L(X)_+$ that satisfies the so-called optimality equation (OE)

$$V_\alpha(x) = \inf_{a \in A(x)} \{C_\alpha(x,a) + \Delta_\alpha(x,a) \int V_\alpha(y) Q(dy|x,a)\}, \qquad x \in X, \tag{3.1}$$

and there exists $f^* \in \mathcal{F}$ such that $V(f^*, x) = V_\alpha(x)$ —see Theorem 3.3.

Definition (3.1). For $\mu \in M(X)_+$, define the function $T\mu$ on X by

$$T\mu(x) := \inf_{a \in A(x)} [C_\alpha(x,a) + \Delta_\alpha(x,a) \int \mu(y) Q(dy|x,a)].$$

REMARK (3.2). a. The OE (3.1) can be written as $V_\alpha(x) = TV_\alpha(x)$.

b. By Lemma (2.12)b, if Assumption (2.9) holds, then T maps $L(X)_+$ into itsef.

We also consider the sequence $\{v_n\}$ of value iteration functions defined recursively by:

$$v_0 \equiv 0$$

$$v_n(x) := Tv_{n-1}(x) = \inf_{a\in A(x)} \{C_\alpha(x,a) + \Delta_\alpha(x,a)\int v_{n-1}(y)Q(dy|x,a)\} \text{ for } n \geq 1.$$

By Remark (3.2)b, if Assumption (2.9) holds, then $v_n \in L(X)_+$, $\forall n \geq 0$. If $x \in X$ and $\pi \in \Pi$ we have:

$$v_n(x) \leq \int_A C_\alpha(x,a)\pi(da|x) + \int_A \Delta_\alpha(x,a)\int v_{n-1}(x_1)Q(dx_1|x,a)\pi(da|x)$$

and iteration of this inequality yields

$$v_n(x) \leq V_n(\pi, x) \tag{3.2}$$

where $V_n(\pi, x)$ is the α-discounted cost until the n^{th} transition (see Remark 2.7(b)).

THEOREM (3.3). *If Assumptions (2.4) and (2.9) hold and $V_\alpha(x) < \infty$ for every $x \in X$, then:*

a. $v_n \uparrow V_\alpha$
b. *V_α is the minimal (pointwise) function in $L(X)_+$ that satisfies the OE (3.1), i.e. $V_\alpha = TV_\alpha$.*
c. *There exists a stationary policy $f_\alpha \in \mathcal{F}$ such that $f_\alpha(x)$ minimizes the right-hand side of (3.1) for all $x \in X$, i.e.*

$$V_\alpha(x) = C_\alpha(x, f_\alpha(x)) + \Delta_\alpha\big(x, f_\alpha(x)\big)\int V_\alpha(y)Q\big(dy|x, f_\alpha(x)\big) \tag{3.3}$$

and f_α is α-optimal. Conversely, if $f_\alpha \in \mathcal{F}$ is an optimal stationary policy, then it satisfies (3.3).
d. *If π^* is a policy such that $V_\alpha(\pi^*, x) \in L(X)_+$ and it satisfies the OE and the condition*

$$\lim_{n\to\infty} E_x^\pi[\Lambda_n^\alpha V_\alpha(\pi^*, x_n)] = 0 \qquad \text{for all } x \in X, \pi \in \Pi, \tag{3.4}$$

then π^* is α-optimal, i.e. $V_\alpha(\pi^*, .) = V_\alpha(.)$. (See Remark 2.7(c) for the definition of Λ_n^α)

To prove Theorem (3.3) we need the next lemma.

LEMMA (3.4). *Under the Hypothesis of Theorem (3.3),*

a. *If $v \in L(X)_+$ satisfies $v \geq Tv$, then $v \geq V_\alpha$.*
b. *If v is a measurable function on X, such that Tv is well defined and is such that $v \leq Tv$ and*

$$\lim_{n\to\infty} E_x^\pi[\Lambda_n^\alpha v(x_n)] = 0 \qquad \forall x, \pi,$$

then $v \leq V_\alpha$.

Proof a) If $v \geq Tv$, then, by Lemma (2.12)b, there exists $f \in \mathcal{F}$ such that:

$$v(x) \geq C_\alpha\big(x, f(x)\big) + \Delta_\alpha\big(x, f(x)\big) \int v(y) Q\big(dy|x, f(x)\big).$$

Iterating this inequality we obtain

$$v(x) \geq E_x^f\left[C_\alpha\big(x_0, f(x_0)\big) + \sum_{k=1}^{n-1} \Delta_\alpha\big(x_0, f(x_0)\big)\cdots\Delta_\alpha(x_{k-1}, f(x_{k-1}))C_\alpha\big(x_k, f(x_k)\big)\right]$$
$$+ E_x^f[\Delta_\alpha\big(x_0, f(x_0)\big)\cdots\Delta_\alpha(x_{n-1}, f(x_{n-1}))v(x_n)]$$

and, since $v \geq 0$, we have

$$v(x) \geq V_n(f, x) \qquad \forall n \in \mathbb{N}.$$

Letting $n \to \infty$ we obtain

$$v(x) \geq V(f, x) \geq V_\alpha(x).$$

b. If $x \in X$, $\pi \in \Pi$ and $n = 0, 1, 2 \ldots$, then by Remark (2.3)b,

$$\begin{aligned} E_x^\pi[\Lambda_{n+1}^\alpha v(x_{n+1})|h_n, a_n] &= \Lambda_{n+1}^\alpha \int v(y) Q(dy|x_n, a_n) \\ &= \Lambda_n^\alpha[C_\alpha(x_n, a_n) + \Delta_\alpha(x_n, a_n) \int v(y) Q(dy|x_n, a_n) - C_\alpha(x_n, a_n)] \\ &\geq \Lambda_n^\alpha[v(x_n) - C_\alpha(x_n, a_n)] \qquad \text{(by the hyphotesis } Tv \geq v). \end{aligned}$$

Hence

$$\Lambda_n^\alpha C_\alpha(x_n, a_n) \geq -E_x^\pi[\Lambda_{n+1}^\alpha v(x_{n+1}) - \Lambda_n^\alpha v(x_n)|h_n, a_n]. \tag{3.5}$$

Taking expectations $E_x^\pi(.)$ in (3.5) and summing over $n = 0, 1, \ldots, t-1$, we obtain

$$V_t(\pi, x) \geq v(x) - E_x^\pi[\Lambda_t^\alpha v(x_t)]$$

Letting $t \to \infty$ in the latter inequality, we obtain $V(\pi, x) \geq v(x)$. Thus, since π was arbitrary, $V_\alpha(x) \geq v(x)$. ■

Proof of Theorem (3.3) a.–b. Clearly, the operator $T: L(X)_+ \to L(X)_+$ is monotone, i.e. $v \geq u$ implies $Tv \geq Tu$. This implies that, $\{v_n\}$ is a nondecreasing sequence in $L(X)_+$ and, therefore, there exists $\mu \in L(X)_+$ such that $v_n \uparrow \mu$. By the Monotone Convergence Theorem we obtain

$$C_\alpha(x, a) + \Delta_\alpha(x, a) \int v_{n-1}(y) Q(dy|x, a) \uparrow C_\alpha(x, a) + \Delta_\alpha(x, a) \int \mu(y) Q(dy|x, a)$$

and by Lemma 2.7(c) in [5] it follows that $\lim_{n\to\infty} Tv_{n-1}(x) = T\mu(x)$. Hence, $\mu = T\mu$, i.e. μ $(\in L(X)_+)$ satisfies the OE.

Now we prove that $\mu = V_\alpha$. By Lemma (3.4)a it follows that $\mu \geq V_\alpha$ and by (3.2), we obtain

$$v_n(x) \leq V_n(\pi, x) \leq V(\pi, x) \qquad \forall n,\ \pi,\ x.$$

Letting $n \to \infty$ we get

$$\mu(x) \leq V(\pi, x) \qquad \forall \pi \in \Pi,\ x \in X.$$

and hence

$$\mu(x) \leq V_\alpha(x).$$

We have thus shown that $\mu = V_\alpha$.

To prove b, we note that, if $\mu' \in L(X)_+$ satisfies $\mu' = T\mu'$, then Lemma (3.4)a yields $\mu' \geq V_\alpha$.

c. By Lemma (2.12), there exists $f_\alpha \in \mathcal{F}$ that satisfies (3.3). Iterating (3.3) we obtain:

$$\begin{aligned} V_\alpha(x) &= E_x^{f_\alpha}[C_\alpha(x_0, a_0) + \sum_{n=0}^{N} \Lambda_n^\alpha C_\alpha(x_n, a_n)] + E_x^{f_\alpha}[\Lambda_{N+1}^\alpha V_\alpha(x_{n+1})] \\ &\geq V_N(f_\alpha, x) \qquad \forall N = 1, 2, \ldots, \end{aligned}$$

and letting $N \to \infty$, $V_\alpha(x) \geq V(f_\alpha, x)$. Since the reverse inequality trivially holds (see Definition (2.6)b) it follows that $V_\alpha(x) = V(f_\alpha, x)$. The converse follows from the fact that, for any stationary policy $f \in \mathcal{F}$, $V(f, .)$ satisfies (by the Markov property (Remark (2.3)b) and Remark (2.7)a)

$$V(f, x) = C_\alpha(x, f(x)) + \Delta_\alpha(x, f(x)) \int V(f, y) Q(dy|x, f(x))$$

d. Apply Lemma (3.4)b to obtain $V_\alpha(x) \geq V(\pi^*, x)$. ■

Sufficient conditions for (3.4).

Throughout the following the Assumptions (2.4) and (2.9) are supposed to hold and $V_\alpha(x) < \infty \; \forall x \in X$.

We denote by C_0, C_1, C_2 and C_3 the next conditions:

C_0: $C_\alpha(x, a)$ is bounded on $\mathcal{K}$.

C_1: There exists a number $m > 0$ and a nonnegative measurable function $\omega(.)$ on X such that for all $(x, a) \in \mathcal{K}$,

1. $C_\alpha(x, a) \leq m\omega(x)$.
2. $\int \omega(y)Q(dy|x, a) \leq \omega(x)$.

C_2 : $C(x) := \sum_{t=0}^{\infty} C_t(x) < \infty \; \forall x \in X$, where

$$C_t(x) := \sup_{a \in A(x)} \Delta_\alpha(x, a) \int C_{t-1}(y)Q(dy|x, a)$$

for $t = 1, 2, \ldots$ and $C_0(x) := \sup_{a \in A(x)} C_\alpha(x, a)$

C_3: $\lim_{n \to \infty} E_x^\pi[\Lambda_n^\alpha V(\pi', x_n)] = 0 \; \forall \pi, \pi' \in \Pi, \, x \in X$.

THEOREM (3.5). *a. C_i implies C_{i+1} $(i = 0, 1, 2)$ and C_3 implies (3.4). Hence:*
b. If C_i holds for any $i = 0, 1, 2, 3$, then the policy π^* is optimal if and only if $V(\pi^*, .)$ is in $L(X)_+$ and satisfies the OE.

Proof: a. $C_0 \Rightarrow C_1$. Let $m > 0$ be any upper bound for $C_\alpha(x, a)$ and $\omega(.) \equiv 1$.

$C_1 \Rightarrow C_2$. By an induction argument, we can show that

$$C_t(x) \leq \Delta_\alpha^t m\omega(x) \qquad \forall x \in X, \;\; t = 0, 1, 2 \ldots$$

where $\Delta_\alpha = \sup_{\mathcal{K}} \Delta_\alpha(x, a)$ (< 1, see Lemma (2.5)1). Thus

$$C(x) \leq \frac{m\omega(x)}{1 - \Delta_\alpha} < \infty.$$

$C_2 \Rightarrow C_3$. Let $\pi \in \Pi$, $x \in X$ be arbitrary. We will first show that

$$V(\pi, x) \leq C(x). \tag{3.6}$$

Observe that, from Remark (2.3)b,

$$\begin{aligned} E_x^\pi[\Lambda_n^\alpha C_0(x_n)|h_{n-1}, a_{n-1}] &= \Lambda_{n-1}^\alpha \Delta_\alpha(x_{n-1}, a_{n-1}) \int C_0(y)Q(dy|x_{n-1}, a_{n-1}) \\ &\leq \Lambda_{n-1}^\alpha C_1(x_{n-1}). \end{aligned}$$

Taking expectations we obtain,

$$E_x^\pi[\Lambda_n^\alpha C_0(x_n)] \leq E_x^\pi[\Lambda_{n-1}^\alpha C_1(x_{n-1})]$$

and using the same argument,

$$\begin{aligned} E_x^\pi[\Lambda_n^\alpha C_0(x_n)] &\le E_x^\pi[\Lambda_{n-1}^\alpha C_1(x_{n-1})] \le \dots \\ &\le E_x^\pi[\Lambda_1^\alpha C_{n-1}(x_1)] \le E_x^\pi[C_n(x_0)] = C_n(x), \qquad n \ge 1. \end{aligned} \tag{3.7}$$

From (3.7) and the fact that $C_\alpha(x_n, a_n) \le C_0(x_n)$,

$$E_x^\pi[\Lambda_n^\alpha C_\alpha(x_n, a_n)] \le E_x^\pi[\Lambda_n^\alpha C_0(x_n)] \le C_n(x),$$

which implies (3.6). Now, if π, π' are two arbitrary policies,

$$E_x^\pi[\Lambda_n^\alpha V(\pi', x_n)] \le E_x^\pi[\Lambda_n^\alpha C(x_n)]. \tag{3.8}$$

Moreover,

$$\begin{aligned} E_x^\pi[\Lambda_n^\alpha C(x_n)|h_{n-1}, a_{n-1}] &= \Lambda_n^\alpha \int \sum_{t=0}^\infty C_t(y) Q(dy|x_{n-1}, a_{n-1}) \\ &= \Lambda_{n-1}^\alpha \sum_{t=0}^\infty \Delta_\alpha(x_{n-1}, a_{n-1}) \int C_t(y) Q(dy|x_{n-1}, a_{n-1}) \\ &\le \Lambda_{n-1}^\alpha \sum_{t=0}^\infty C_{t+1}(x_{n-1}). \end{aligned}$$

Hence, taking expectation E_x^π,

$$\begin{aligned} E_x^\pi[\Lambda_n^\alpha C(x_n)] &\le E_x^\pi[\Lambda_{n-1}^\alpha \sum_{t=0}^\infty C_{t+1}(x_{n-1})] \le \dots \\ &\le E_x^\pi[\sum_{t=0}^\infty C_{t+n}(x_0)] = \sum_{t=0}^\infty C_{t+n}(x) \end{aligned}$$

This in turn yields

$$E_x^\pi[\Lambda_n^\alpha V(\pi', x_n)] \le \sum_{t=n}^\infty C_t(x) \to 0$$

when $n \to \infty$ since $C(x)$ is finite.

b. Follows from a) and Theorem (3.3)b, d. ■

4. Approximations

In this section, we consider other types of approximations to the optimal cost function V_α, different to the one used in the Theorem (3.3)a.

a. Infinite horizon problems with bounded costs.

Let $D^n(x,a)$, $d^n(x,a)$ $(n = 0,1,2\ldots)$ be nonnegative, bounded, l.s.c., and inf-compact functions on $\mathcal{K}$, such that $D^n \uparrow D$, $d^n \uparrow d$.

Now, instead of (2.1), consider the cost-per-stage function

$$C_\alpha^n(x,a) := D^n(x,a) + \tau_\alpha(x,a)d^n(x,a)$$

and for $x \in X$ and $\pi \in \Pi$ define the corresponding cost functions

$$U_n(\pi,x) := E_x^\pi \sum_{k=0}^{\infty} \exp(-\alpha T_k) C_\alpha^n(x_k,a_k) \tag{4.1}$$

$$U_n(x) := \inf_\pi U_n(\pi,x) \tag{4.2}$$

For all $v \in L(X)_+$ we define the function $T_n v$ by

$$T_n v(x) = \inf_{a\in A(x)} \{C_\alpha^n(x,a) + \Delta_\alpha(x,a)\int v(y)Q(dy|x,a)\} \tag{4.3}$$

REMARK (4.1). If C_α is replaced by C_α^n, then (Theorem (3.3)b) the optimal cost function $U_n(x)$ is the unique bounded function in $L(X)_+$ which satisfies

$$U_n = T_n U_n \qquad \forall n. \tag{4.4}$$

The uniqueness follows from Lemma (3.4).

PROPOSITION (4.2). *The sequence $\{U_n\}$ is monotone increasing and converges to V_α.*

Proof: Since $C_\alpha^n \uparrow C_\alpha$, we have that $\{U_n\}$ is an increasing sequence in $L(X)_+$ and therefore there exists a function $\mu \in L(X)_+$ such that $U_n \uparrow \mu$. Moreover, by Lemma 2.7(c) in [5] and letting $n \to \infty$ in 4.4 we obtain that $\mu = T\mu$ and hence $\mu \geq V_\alpha$ (see Theorem 3.3 (b)). On the other hand $U_n \leq V_\alpha$ $\forall n$ (see 4.2) so that $\mu \leq V_\alpha$. Thus, $\mu = V_\alpha$, i.e. $U_n \uparrow V_\alpha$. ∎

b. Recursive bounded costs

Let T_n be as in (4.3) and let $\{\mu_n\}$ be the sequence defined recursively by

$$\mu_0 \equiv 0$$
$$\mu_n := T_n \mu_{n-1}, \qquad n \geq 1$$

i.e.

$$\mu_n(x) = \inf_{a\in A(x)} \{C_\alpha^n(x,a) + \Delta_\alpha(x,a)\int \mu_{n-1}(y)Q(dy|x,a)\}.$$

PROPOSITION (4.3). *The sequence* $\{\mu_n\}$ *is monotone increasing and converges to* V_α.

Proof. By analogous arguments to those used in the proof of Proposition (4.2), we get $v \geq V_\alpha$, where $v := \lim \mu_n$. On the other hand, $\mu_n(x) \leq V_n(\pi, x)$ for each $\pi \in \Pi$, $x \in X$ (see (3.2)). Hence, $\mu_n(x) \leq V(\pi, x)$ and therefore $v(x) \leq V_\alpha(x)$. Thus, $\mu_n \uparrow V_\alpha$. ■

Policy iteration

To begin, observe that if $f \in \mathcal{F}$ is a stationary policy and $x \in X$, then

$$(4.5) \qquad V(f,x) = C_\alpha\big(x, f(x)\big) + \Delta_\alpha\big(x, f(x)\big) \int V(f,y) Q\big(dy|x, f(x)\big)$$

(see the proof of Theorem (3.3)c)

Let $f_0 \in \mathcal{F}$ be a stationary policy with finite-valued discounted cost $V(f_0, .) := \omega_0(.) \in L(X)_+$. Then, by (4.5),

$$(4.6) \quad \omega_0(x) = C_\alpha\big(x, f_0(x)\big) + \Delta_\alpha\big(x, f_0(x)\big) \int \omega_0(y) Q\big(dy|x, f_0(x)\big) \qquad \forall x \in X.$$

If T is the operator in Definition (3.1), then by Lemma (2.12), there exists $f_1 \in \mathcal{F}$ such that:

$$(4.7) \qquad C_\alpha\big(x, f_1(x)\big) + \Delta_\alpha\big(x, f_1(x)\big) \int \omega_0(y) Q\big(dy|x, f_1(x)\big) = T\omega_0(x).$$

Write $\omega_1(.) = V(f_1, .)$. In general, given $f_n \in \mathcal{F}$, suppose that $\omega_n(.) := V(f_n, .) \in L(X)_+$, and let $f_{n+1} \in \mathcal{F}$ such that

$$(4.8) \quad C_\alpha\big(x, f_{n+1}(x)\big) + \Delta_\alpha\big(x, f_{n+1}(x)\big) \int \omega_n(y) Q\big(dy|x, f_{n+1}(x)\big) = T\omega_n(x)$$
$$= \min_{a \in A(x)} [C_\alpha(x,a) + \Delta_\alpha(x,a) \int \omega_n(y) Q(dy|x,a).$$

PROPOSITION (4.4). *There exists* $\omega \in M(X)_+$ *such that* $\omega_n \downarrow \omega$, *and* $T\omega = \omega$. *If, moreover,* ω *satisfies*

$$(4.9) \qquad \lim_{n \to \infty} E_x^\pi [\Lambda_n^\alpha \omega(x_n)] = 0 \qquad \forall x \in X, \ \pi \in \Pi,$$

then $\omega = V_\alpha$.

Proof. We will first show that $\{\omega_n\}$ is a decreasing sequence. From (4.6) and (4.7),

$$\omega_0(x) \geq C_\alpha\big(x, f_1(x)\big) + \Delta_\alpha\big(x, f_1(x)\big) \int \omega_0(y) Q\big(dy|x_1, f_1(x)\big).$$

Iterating the inequality we obtain:

$$\omega_0(x) \geq V(f_1, x) = \omega_1(x) \qquad \text{(see the proof of Lemma (3.4)a).}$$

By a similar argument we obtain

$$\omega_n \geq T\omega_n \geq \omega_{n+1}, \qquad \text{for } n = 1, 2, \ldots \tag{4.10}$$

Hence, there exists a function $\omega \in M(X)_+$ such that

$$\omega_n \downarrow \omega$$

Since $\omega_n(.) \geq V_\alpha(.)$ $\forall n$, we have

$$\omega \geq V_\alpha \tag{4.11}$$

Now, as is well known (see e.g. Lemma 3.3 in [6]), if $h_n: \mathcal{K} \to \mathbb{R}$ satisfies $h_n \downarrow h$, then

$$\lim_{n\to\infty} \inf_{a\in A(x)} h_n(x,a) = \inf_{a\in A(x)} h(x,a).$$

Thus, applying this result to (4.10), we get

$$\omega \geq T\omega \geq \omega$$

and therefore,

$$\omega = T\omega$$

If ω satisfies (4.9), then Lemma (3.4)b yields $\omega \leq V_\alpha$ and, by 4.11, we obtain $\omega = V_\alpha$. ∎

5. Other optimality criteria and asymptotic optimality

Recall that Assumptions (2.4) and (2.9) are supposed to hold and $V_\alpha(x) < \infty, \forall x \in X$. Consider the function $\Phi: \mathcal{K} \to \mathfrak{R}$ defined by

$$\Phi(x,a) := C_\alpha(x,a) + \Delta_\alpha(x,a) \int V_\alpha(y) Q(dy|x,a) - V_\alpha(x)$$

and observe that Φ is a nonnegative l.s.c. function and the OE (3.1) can be written as

$$\min_{a\in A(x)} \Phi(x,a) = 0.$$

Moreover, $\forall \pi \in \Pi$ and $x \in X$,

$$\Phi(x_n, a_n) = E_x^\pi[C_\alpha(x_n,a_n) + \Delta_\alpha(x_n,a_n)V_\alpha(x_{n+1}) - V_\alpha(x_n)|h_n, a_n]. \tag{5.1}$$

This follows from Remark (2.3)b and the properties of conditional expectations.

Let $\{M_n\}$ be the sequence defined by $M_0 := V_\alpha(x_0)$

$$M_n = \sum_{t=0}^{n-1} \Lambda_t^\alpha C_\alpha(x_t, a_t) + \Lambda_n^\alpha V_\alpha(x_n) \qquad \text{for } n = 1, 2, \ldots \tag{5.2}$$

and let $\{U_n\}$ be the sequence given by

$$U_n := \sum_{t=n}^{\infty} \Lambda_{n,t}^\alpha C_\alpha(x_t, a_t) \qquad n = 0, 1, 2, \ldots \tag{5.3}$$

where $\Lambda_{n,t}^\alpha = \Pi_{k=n}^{t-1} \Delta_\alpha(x_k, a_k)$ for $t > n$ and $\Lambda_{n,n}^\alpha = 1$.

From Remark (2.7)c. and (5.3) we have

$$V(\pi, x) = V_n(\pi, x) + E_x^\pi[\Lambda_n^\alpha U_n]. \tag{5.4}$$

Using (5.2), (5.3) and (5.4) we can also write $V(\pi, x)$ as

$$V(\pi, x) = E_x^\pi(M_n) + E_x^\pi[\Lambda_n^\alpha(U_n - V_\alpha(x_n))] \qquad \text{for } n = 0, 1, 2, \ldots \tag{5.5}$$

THEOREM (5.1). *Let π be a policy such that $V(\pi, x) < \infty\ \forall x \in X$. Then the following statements are equivalent:*

a. π is an optimal policy
b. $E_x^\pi(\Lambda_n^\alpha U_n) = E_x^\pi(\Lambda_n^\alpha V_\alpha(x_n))\ \forall x$, $n = 0, 1, 2, \ldots$
c. $E_x^\pi \Phi(x_n, a_n) = 0\ \forall n, x$
d. $\{M_n\}$ is a P_x^π-martingale $\forall x$.

To prove this theorem, we need the following result:

LEMMA (5.2). *Let π be a policy such that $V(\pi, x) < \infty\ \forall x \in X$. Then:*
a. $\sum_{t=n}^\infty E_x^\pi[\Lambda_t^\alpha \Phi(x_t, a_t)] = E_x^\pi[\Lambda_n^\alpha(U_n - V_\alpha(x_n))]$ for $n \geq 1$
a'. $\sum_{t=n}^\infty E_x^\pi[\Lambda_{n,t}^\alpha \Phi(x_t, a_t)] = E_x^\pi(U_n - V_\alpha(x_n))$
b. $E_x^\pi[\Lambda_n^\alpha U_n] \geq E_x^\pi[\Lambda_n^\alpha V_\alpha(x_n)]$ for $n = 0, 1, 2, \ldots$

Proof: First we prove that $E_x^\pi[\Lambda_m^\alpha V_\alpha(x_m)] \to 0$ as $m \to \infty$. From (5.4), $E_x^\pi[\Lambda_n^\alpha U_n] \to 0$ and by the properties of conditional probability we have

$$E_x^\pi[\Lambda_m^\alpha U_m] = E_x^\pi[E_x^\pi[\Lambda_m^\alpha U_m | h_m, a_m]] = E_x^\pi[\Lambda_m^\alpha E_x^\pi[\sum_{t=m}^{\infty} \Lambda_{m,t} C_\alpha(x_t, a_t) | h_m, a_m]]$$

$$= E_x^\pi[\Lambda_m^\alpha E_{x_m}^{\pi(m)}[\sum_{t=m}^{\infty} \Lambda_{m,t} C_\alpha(x_t, a_t)]] \geq E_x^\pi[\Lambda_m^\alpha V_\alpha(x_m)] \geq 0$$

where $\pi(m)$ is the "m-shifted policy" (see [2] pp. 5-6).

Now, by (5.1), for $n \geq 1$ we have:

$$\begin{aligned}
&\sum_{t=n}^{\infty} E_x^\pi[\Lambda_t^\alpha \Phi(x_t, a_t)] \\
&\quad = \sum_{t=n}^{\infty} E_x^\pi\{\Lambda_t^\alpha E_x^\pi[C_\alpha(x_t, a_t) + \Delta_\alpha(x_t, a_t)V_\alpha(x_{t+1}) - V_\alpha(x_t)|h_t, a_t]\} \\
&\quad = \sum_{t=n}^{\infty} E_x^\pi \Lambda_n^\alpha \Lambda_{n,t}^\alpha C_\alpha(x_t, a_t) + \sum_{t=n}^{\infty} E_x^\pi[\Lambda_{t+1}^\alpha V_\alpha(x_{t+1}) - \Lambda_t^\alpha V_\alpha(x_t)] \\
&\quad = E_x^\pi[\Lambda_n^\alpha U_n] - E_x^\pi[\Lambda_n^\alpha V_\alpha(x_n)].
\end{aligned}$$

This proves a. The proof of a'. is analogous, and b. follows from a. since $\Phi(.,.) \geq 0$ and $\Delta_\alpha(.,.) > 0$. ■

We also need the following result

LEMMA (5.3). *For any $\pi \in \Pi$ and $x \in X$, $\{M_n\}$ is a P_x^π-submartingale, i.e.*

$$E_x^\pi(M_{n+1}|h_n) \geq M_n \qquad P_x^\pi - \text{a.s. } \forall n.$$

Therefore,

$$E_x^\pi M_{n+1} \geq E_x^\pi M_n \geq \cdots \geq E_x^\pi M_0 = V_\alpha(x) \qquad \forall n. \tag{5.6}$$

Proof: From (5.2), for $n \geq 1$,

$$M_{n+1} = M_n + \Lambda_n^\alpha[C_\alpha(x_n, a_n) + \Delta_\alpha(x_n, a_n)V_\alpha(x_{n+1}) - V_\alpha(x_n)].$$

Therefore, by properties of conditional expectations,

$$E_x^\pi(M_{n+1}|h_n) = M_n + E_x^\pi[\Lambda_n^\alpha \Phi(x_n, a_n)|h_n] \geq M_n. \tag{5.7}$$ ■

*Proof of Theorem (5.1).*a. $\Rightarrow$ b. If π is an optimal policy, then $V(\pi, x) = V_\alpha(x)$ and from (5.5) and Lemma 5.3,

$$\begin{aligned}
V_\alpha(x) &= E_x^\pi M_n + E_x^\pi[\Lambda_n^\alpha(U_n - V_\alpha(x_n))] \\
&\geq V_\alpha(x) + E_x^\pi[\Lambda_n^\alpha(U_n - V_\alpha(x_n))]
\end{aligned}$$

Thus b. follows from Lemma (5.2).c.

b. $\Rightarrow$ a. Take $n = 0$.

b. $\Rightarrow$ c. From Lemma (5.2)a, $E_x^\pi(\Lambda_n^\alpha \Phi(x_n, a_n)) = 0$ and since $\Delta_\alpha(.,.) > 0$,

$$E_x^\pi \Phi(x_n, a_n) = 0 \qquad \forall n$$

c. $\Rightarrow$ b. Follows from Lemma (5.2).a'.

The equivalence of c. and d. follows from (5.7), $\Delta_\alpha(.,.) > 0$, $\Phi \geq 0$ and the properties of conditional expectations. ■

To finish this section, we briefly discuss the notion of asymptotic optimality which was introduced in the analysis of adaptive control problems (see [2,3,4,10]) and which allows us to say when a given policy is close to being optimal.

For $x \in X$ and $\pi \in \Pi$, we write $V^n(\pi, x) = E_x^\pi U_n$.

Definition (5.4). A policy $\pi \in \Pi$ is said to be asymptotically discount optimal (ADO) if for each $x \in X$,

$$V^n(\pi, x) - E_x^\pi V_\alpha(x_n) \to 0 \qquad \text{as } n \to \infty$$

THEOREM (5.5). *Let $\pi \in \Pi$ be such that $V(\pi, x) < \infty$ for each $x \in X$. Then the following statements are equivalent*

a. π is ADO
b. $\lim_{n\to\infty}\{\sum_{t=n}^{\infty} E_x^\pi \Lambda_{n,t}\Phi(x_t, a_t)\} = 0$
c. For every $x \in X$, $E_x^\pi \Phi(x_n, a_n) \to 0$ as $n \to 0$.

Proof. The equivalence of a) and b) follows from Lemma (5.2)a'., and the equivalence of b. and c. is trivial. ■

Observe that by Theorem (5.1).c. and Theorem (5.5).c., if a policy is optimal then it is ADO. Also observe that if $C_\alpha(x, a)$ is bounded, then Theorem (5.5).c. is equivalent to: For each $x \in X$, $\Phi(x_n, a_n) \to 0$ in P_x^π-probability as $n \to \infty$.

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Erratum

"Average optimality in semi-Markov control models on Borel Spaces: unbounded cost and controls" Bol. Soc. Mat. Mexicana, Vol. 38, 1993, 47–60, by O. Vega-Amaya.

(1) The Lemma (3.1), page 50, should read as follows:

LEMMA (3.1). Let $\{c_n : u = 0, 1, \ldots\}$ be a sequence of nonnegative numbers and $\{b_n : n = 0, 1, \ldots\}$ a sequence of positive numbers such that

$$\limsup_n n^{-1} b_n \leq 1 \tag{3.8}$$

Then

$$\limsup_{\beta \uparrow 1} (1-\beta) \sum_{n=0}^{\infty} \beta^n c_n \leq \limsup_n b_n^{-1} S_n, \tag{3.9}$$

where $S_n = \sum_{k=0}^{n-1} c_k$, $n = 1, 2, \ldots$, and $S_n = 0$.

Proof.
$$\begin{aligned}\limsup_n n^{-1} S_n &\leq \left(\limsup_n b_n^{-1} S_n\right)\left(\limsup_n n^{-1} b_n\right)\\ &\leq \limsup_n b_n^{-1} S_n.\end{aligned}$$

Now, the inequality in (3.9) is an immediate consequence of the (Tauberian) Theorem A.2 in [1].

(II) The constant M in Remark (3.2)(d), page 51, and Assumption (5.1) (a), page 53, should be equal to 1.

(III) The line 13, page 51, should read as follows: if the condition in (3.8) holds.