ZUBOV'S STABILITY CRITERION

By Robert W. Bass

In his book, The Methods of A. Liapunov and Their Application (Moscow[1], 1957), Zubov states:

Proposition 1. *A n.a.s.c. for the stability of a closed*[2] *invariant set M is that no path from outside M possess an α-limit point in M.*

This proposition is correct when M is an *isolated* critical point of a planar flow ([2], p. 72).

But in general, Zubov's condition is merely necessary and not sufficient (as was observed both by Pinchas Mendelson and myself).

Here I shall follow the conventions of Lefschetz's survey *Liapunov and Stability in Dynamical Systems* [1], Chapter IV of which discusses the stability of invariant sets which possess a compact neighborhood. In particular, Lefschetz has substituted for Zubov's fallacious criterion the following condition.

Theorem 1. *A n.a.s.c. for the stability of M is that for each sufficiently small $\epsilon > 0$ there exist an $\eta = \eta(\epsilon) > 0$ such that*[3] *if $p \in H(\epsilon)$ then γ_p^- does not intersect $\mathcal{S}(\eta)$.*

Of course, the property indicated in Theorem 1 is not quite as sharp as Zubov's attempted characterization of stability. I would like to suggest as an alternative the following criterion, which is admittedly more complicated but does show precisely and to what extent Zubov's characterization fails. In particular, we shall see that Zubov's criterion is correct in the case of "structurally stable" planar flows.

This alternative criterion depends on a topological generalization of the concept of *saddle point* which is modeled upon Nemickii's definition of *saddle point at infinity* [2], and which has proved useful in connection with other studies of myself [3] and Mendelson [4].

Definition 1. We call the ω-limit set[4] $\Omega(p_0) \neq \emptyset$ of any point p_0 a *saddle set* when the following conditions hold: there exist a sequence of points $\{p_n\}$, a sequence of positive times $\{t_n\}$ and a point $q_0 \notin \Omega(p_0)$ such that $p_n \to p_0$, $t_n \to +\infty$, and $f(p_n, t_n) \to q_0$ as $n \to \infty$.

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[1] An English translation of this book is to be prepared for the U. S. Atomic Energy Commission and will become available as a public document.

[2] It is always assumed that the dynamical system is defined on a complete separable metric space R, of metric $d(p, q)$, and that *M possesses a neighborhood U whose closure $\bar{U}$ is compact.*

[3] Here $\mathcal{S}(\delta)$ denotes the set of points p whose distances $d(p, M)$ from M are less than $\delta > 0$; for sufficiently small δ, the closure $\overline{\mathcal{S}(\delta)}$ of this set is by hypothesis[2] compact, and $H(\delta)$ represents its (compact) boundary. The set $\gamma_p^- \equiv \{q \mid q = f(p, t), t \leqq 0\}$ denotes the negative semi-orbit or *path* through p.

[4] $\Omega(p_0)$ is necessarily a closed invariant set (G. D. Birkhoff; cf. [2]); if it is compact, then it is also necessarily connected.

DEFINITION 2. A *positively linked* sequence of saddle sets consists of sequences of points $\{p_k\}$, $\{q_k\}$, and associated saddle sets $\{\Omega(p_k) \equiv \Omega_k\}$, the situation for each k being precisely as for p_0, q_0 and $\Omega(p_0)$ in Definition 1, with the added proviso that

$$q_k = p_{k+1}, \qquad (k = 1, 2, 3, \cdots)$$

DEFINITION 3. A *strongly* positively linked sequence of saddle sets consists of the sequences $\{p_k\}$, $\{\Omega_k\}$ of Definition 2 plus further sequences of points $\{p_{1n}\}$ and times $\{T_{kn}\}$ $(k = 1, 2, 3, \cdots)$ with the following properties:

(i) $0 = T_{1n} < T_{2n} < \cdots < T_{kn} < T_{k+1n} < \cdots \quad (n = 1, 2, \cdots);$

(ii) $T_{kn} \to +\infty$ as $n \to \infty \quad (k = 1, 2, \cdots);$

(iii) $f(p_{1n}, T_{kn}) \to p_k$ as $n \to \infty \quad (k = 1, 2, \cdots).$

REMARK. As usual, Definitions 1–3 continue to hold if, throughout, we interchange the concepts alpha & omega, A & Ω, positive(ly) & negative(ly), $t > 0$ & $t < 0$, and $+\infty$ & $-\infty$.

THEOREM 2. *A n.a.s.c. for the stability of a closed*[2] *invariant set M is that no point of M be either an α-limit point of a path outside M or a limit point of a strongly negatively linked sequence of saddle sets outside M.*

PROOF OF NECESSITY. Let M be stable, and let $q \in M$, $q \in A(r)$, $r \notin M$. Then there exists an unbounded monotone increasing sequence of times $\{t_n\}$ such that $\{q_n \equiv f(r, -t_n)\}$ satisfies $d(q_n, q) < 1/n$. Thus $f(q_n, t_n) = r$, whence $d(f(q_n, t_n), M) = d(r, M) > 0$, while $t_n \to +\infty$ as $n \to \infty$. Therefore M is not stable.

In the second case, let $q \in M$, where $d(p_k, q) < 1/2k$, $(k = 1, 2, \cdots)$ and $(\{p_k\}, \{A_k\})$ is a strongly negatively linked sequence of saddle sets $\{A_k\}$. By Definition 2, there are points $\{p_{1n}\}$ and positive times $\{T_{kn}\}$ such that $d(f(p_{1n}, -T_{kn}), p_k) < 1/2k$ for all $n \geqq n(k)$. Hence $d(f(p_{1n(k)}, -T_{kn(k)}), q) < 1/k$. Notice that $p_{1k} \to p_1$ as $k \to \infty$, whence for all $k \geqq k_0$ we have $d(p_{1n(k)}, M) \geqq \frac{1}{2}d_0 > 0$, where $d_0 \equiv d(p_1, M)$. Thus finally, setting $r_k \equiv f(p_{1n(k)}, -T_{kn(k)})$, we have: $d(r_k, M) < 1/k$, $d(f(r_k, T_{kn(k)}), M) \geqq \frac{1}{2}d_0 > 0$ for all $k \geqq k_0$, while $0 < T_{kn(k)} \to +\infty$ as $k \to \infty$. Hence M is not stable if either condition fails.

PROOF OF SUFFICIENCY. Lefschetz's criterion of Theorem 1 is clearly sufficient for the stability of M. In fact, if it holds then for each $q \in \mathcal{S}(\eta)$ we know that $d(f(p, t), M) < \epsilon$ for all $t \geqq 0$; i.e. $\eta = \eta(\epsilon)$ satisfies Liapunov's stability definition.

Now if Lefschetz's criterion should fail, then there exist sequences $\{p_{1n}\} \subset H(\epsilon)$ and $\{t_n\}$, $0 < t_n \to +\infty$ such that $d(f(p_{1n}, -t_n), M) \to 0$ as $n \to \infty$. Because M has a compact neighborhood, and because, if $\epsilon > 0$ be sufficiently small, $H(\epsilon)$ is compact, we can (by preliminary selections) assume without loss of generality that $p_{1n} \to p_1 \in H(\epsilon)$ and $f(p_{1n}, -t_n) \to q \in M$ as $n \to \infty$.

We shall now prove that

(*) $A(p_1) \neq \varnothing$.

For if $A(p_1)$ is empty, we can further assume that $f(p_{1n}, -T_n)$ is outside the compact set $\overline{S(\epsilon)}$ for some $T_n > 0$. (Indeed, if $\gamma_{p_{1n}}^- \subset \overline{S(\epsilon)}$ for all large n then $f(p_1, -T)$ cannot leave $\overline{S(\epsilon)}$ for any $T > 0$, because by continuity then $f(p_{1n}, -T) \notin \overline{S(\epsilon)}$ if n is large enough; but $\gamma_{p_1}^- \subset \overline{S(\epsilon)}$ would imply (*) by one of Birkhoff's fundamental theorems [2].) Thus when (*) fails we can by a preliminary selection assume that $f(p_{1n}, -T)$, $T > 0$, passes within a distance of $1/n$ from M *before* its first exit from $\overline{S(\epsilon)}$, while $f(p_1, -T)$ eventually leaves $\overline{S(\epsilon)}$ for all $T \geqq T_0 > 0$. However, if $f(p_1, -T)$, $T > 0$, does not come closer to M than $\delta > 0$ before its first exit from $\overline{S(\epsilon)}$, then for sufficiently large k, by continuity, $f(p_{1k}, -T)$, $T > 0$, does not come closer to M than $\delta/2$ before its own first exist, which is a contradiction. Therefore (*) is correct.

There are now only two possibilities:

$$\text{(i) } q \in A(p_1);$$

$$\text{(ii) } A(p_1) \cap M = \varnothing;$$

indeed, if $q' \in A(p_1) \cap M$ we can easily replace q by q' without essential loss of generality.

Now

(**) *suppose that* (i) *is false for* ALL *pairs of points* $q \in M$, $p_1 \in H(\epsilon)$, *and for* ALL $0 < \epsilon < \epsilon_0$.

Then we have the following situation: $p_{1n} \to p_1$, $f(p_{1n}, -t_n) \to q \in M$ for $t_n \to +\infty$ as $n \to \infty$, while $d(A(p_1), M) \equiv \delta > 0$. Clearly, for all sufficiently large n, there are positive times $\{T_{2n}\}$, $0 < T_{2n} < t_n$, $T_{2n} \to +\infty$, such that $f(p_{1n}, -T_{2n}) \in H(\delta/2)$. Hence by a preliminary selection we may further assume that as $n \to \infty$, $f(p_{1n}, -T_{2n}) \to p_2 \in H(\delta/2)$, where $p_2 \notin A(p_1)$. Now by (*), $A(p_2) \neq \varnothing$, and by (**), $A(p_2) \cap M = \varnothing$. Hence we can repeat the preceding construction to find $p_3 \notin A(p_2)$. Similarly, we can construct $\{p_k\}$ satisfying Definition 3. That is, if (**) holds and if Lefschetz's sufficient criterion fails, then

(***) *M contains a limit point of a strongly negatively linked sequence of saddle points.*

Hence the truth of (**) and the failure of (***) are sufficient for Lefschetz's (sufficient) criterion. This concludes the proof of Theorem 2.

ALTERNATIVE PROOF (Added May 1, 1959). According to a result of Ura [7], an invariant set M is stable if and only if M contains all of its positive *indirect prolongation points*. But according to an argument of Seibert [8], either M contains all such points, or (***) occurs for a sequence of saddle points outside M, or M contains an α-limit point of a point outside M. Hence Ura's criterion and our Theorem 2 are equivalent.

Seibert also suggests the desirability of extending our Definitions 2 and 3 to cover uncountable collections of saddle sets.

REMARK. It is not difficult to construct a stable invariant set M, some point of which is a limit point of a sequence of saddle sets. In fact, let M be a critical point corresponding to a plane "center", i.e. let a neighborhood of M in E^2 consist solely of closed (periodic) paths. Now in the well known manner continuously deform the flow so as to leave the paths unchanged except that exactly two (distinct) critical points (p_{1k}, p_{2k}) are produced on each member C_k of a disjoint sequence $\{C_k\}$ of closed curves whose common limit point is M, say $d(C_k, M) < \frac{1}{2}d(C_{k-1}, M)$, $(k = 1, 2, \cdots)$. Clearly each $p_{ik}(i = 1, 2)$ is a saddle point; moreover, M remains a stable invariant point. *This example* (suggested by Peter Seibert) *shows that it is solely the "strongly linked" feature of the sequence of saddle sets which is sufficient for the instability of M in Theorem* 2.

CONCLUSION. It is apparent that Zubov's criterion fails only to the extent that the dynamical system in R may contain convergent sequences of saddle sets. In a planar flow which is defined by two sufficiently smooth simultaneous first order differential equations in E^2, we know that if the system is structurally stable ([5], [6]) then the only saddle sets are elementary saddle points in the usual sense (for cf. the Poincaré-Bendixson Theorem and the hypothesis that there is a compact region, e.g. a disk, in E^2 from which no path exits). Thus, as a corollary of Theorem 2, we have

THEOREM 3. *In a structurally stable planar dynamical system Proposition* 1 *is correct.*

It seems likely that the analog of Theorem 3 may hold for structurally stable systems of higher dimension [6b, c].

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