A NOTE ON H-SPACES

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1. Introduction

Let (X, x_0) be an H-space with multiplication $\phi_1 : (X \times X, (x_0, x_0)) \to (X, x_0)$ such that x_0 is a homotopy-unit. Let ϕ_2 be another H-structure on (X, x_0). It is known that the following three H-structures on $(\Omega X, x_0)$ are homotopy-commutative and homotopy-equivalent to one another:

$$\Omega\phi_1, \Omega\phi_2, \psi : \Omega X \times \Omega X \to \Omega X$$

are defined by

$$\Omega\phi_i(f, g)(t) = \phi_i(f(t), g(t))$$

and ψ is multiplication of loops. In this note we give an example where $\phi_i : X_i \times X_i \to X_i$, for $i = 1$ and 2, are homotopy-commutative H-spaces and ΩX_1 is of the same homotopy type as ΩX_2, but $\Omega\phi_1$ is not homotopy-equivalent to $\Omega\phi_2$.

2. The Example

Let $K(\pi, n)$ be an Eilenberg-MacLane space of type

$$(\pi, n); \quad \text{i.e.} \quad \pi_i(K(\pi, n)) = 0 \quad \text{if} \quad i \neq n, \quad \text{and} \quad \pi_n(K(\pi, n)) = \pi.$$

Each $K(\pi, n)$ is a homotopy-commutative H-space with a unique multiplication. Our spaces X_i will have two non-vanishing homotopy groups. Let

$$\iota_n \in H^n(K(Z, n); Z)$$

denote the fundamental class. Let $X_1 = K(Z, 3) \times K(Z, 5)$ and $\phi_1 : X_1 \times X_1 \to X_1$ be the product multiplication. Let X_2 be the fibre space over $K(Z, 3)$ with fibre $K(Z, 5)$ and with k-invariant

$$\delta^* \operatorname{Sq}^2(\iota_3) = (\iota_3)^2 \in H^6(K(Z, 3); Z),$$

where δ^* is the Bockstein operation associated to the exact coefficient sequence

$$0 \to Z \to Z \xrightarrow{\eta} Z_2 \to 0$$

If $\theta \in H^q(K(\pi, n); G)$, let ${}^1\theta \in H^{q-1}(K(\pi, n-1); G)$ denote the suspension of θ. Then $\delta^* \operatorname{Sq}^2(\iota_3) = {}^1({}^1(\delta^* \operatorname{Sq}^2(\iota_5)))$, where $\delta^* \operatorname{Sq}^2(\iota_5) \in H^8(K(Z, 5); Z)$. Hence X_2 has a homotopy-commutative multiplication because $X_2 = \Omega^2 Y$, where Y has k-invariant $\delta^* \operatorname{Sq}^2(\iota_5)$. Call this multiplication ϕ_2. Furthermore, $\Omega X_1 = K(Z, 2) \times K(Z, 4)$, and ΩX_2 is of the same homotopy type as ΩX_1 because ${}^1(\delta^* \operatorname{Sq}^2(\iota_3)) = 0$.

Let $\Delta_i = (\Omega\phi_i)^* : H^*(\Omega X_i) \to H^*(\Omega X_i \times \Omega X_i)$. In order to show that $\Omega\phi_1$ and $\Omega\phi_2$ are not homotopy equivalent, we shall show that $\Delta_1 \neq \Delta_2$.

3. Δ_2

Let us first study $H^*(X_2)$. Let $p: X_2 \to K(Z, 3)$ and $i: K(Z, 5) \to X_2$ be the projection onto the base and injection of the fibre respectively. By considering the cohomology spectral sequence of this fibre space, we see that $d_5(\iota_5) = \delta^* \operatorname{Sq}^2(\iota_3)$. Hence $d_5(2\iota_5) = 0$, and there exists an element $u \in H^5(X_2)$ such that $i^*(u) = 2\iota_5$. Furthermore, $H^5(X_2)$ is the infinite cyclic group generated by u. Let η denote reduction mod 2. Then $\eta(u) = p^*(\operatorname{Sq}^2(\iota_3))$, as $\eta(u)$ is the only non-zero element in $H^5(X_2; Z_2)$, and $p^*(\operatorname{Sq}^2(\iota_3)) \neq 0$.

Consider now the fibre space

$$\Omega K(Z, 5) = K(Z, 4) \xrightarrow{{}^1i} \Omega X_2 \xrightarrow{{}^1p} \Omega K(Z, 3) = K(Z, 2)$$

Since ${}^1u \in H^4(\Omega X_2)$ is a suspension, $\Delta_2({}^1u) = {}^1u \otimes 1 + 1 \otimes {}^1u$. Also, $({}^1i)^*({}^1u) = {}^1(i^*(u)) = 2({}^1\iota_5) = 2\iota_4$. Since

$$0 \to H^4(K(Z, 2)) \xrightarrow{({}^1p)^*} H^4(\Omega X_2) \xrightarrow{({}^1i)^*} H^4(K(Z, 4)) \to 0$$

is exact, and $H^4(K(Z, 2)) = Z$ and $H^4(K(Z, 4)) = Z$, we have that either ${}^1u$ is divisible by 2 or ${}^1u + ({}^1p)^*(\iota_2^2)$ is divisible by 2. However $\eta({}^1u) = {}^1(\eta(u)) = ({}^1p)^*(\operatorname{Sq}^2(\iota_2)) \neq 0$, thus ${}^1u$ is not divisible by 2. Define $v = \frac{1}{2}({}^1u + ({}^1p)^*(\iota_2^2)) \in H^4(\Omega X_2)$. For notational sake, let $y = ({}^1p)^*(\iota_2)$; i.e. $v = \frac{1}{2}({}^1u + y^2)$. Clearly, $\Delta_2(y) = y \otimes 1 + 1 \otimes y$, and thus $\Delta_2(y^2) = (y \otimes 1 + 1 \otimes y)^2 = y^2 \otimes 1 + 2y \otimes y + 1 \otimes y^2$. Thus, computing with rational coefficients, we have

$$\begin{aligned}\Delta_2(v) &= \tfrac{1}{2}({}^1u) \otimes 1 + 1 \otimes \tfrac{1}{2}({}^1u) + \tfrac{1}{2}(y^2) \otimes 1 + \tfrac{1}{2}(2y \otimes y) + 1 \otimes \tfrac{1}{2}(y^2) \\ &= v \otimes 1 + y \otimes y + 1 \otimes v.\end{aligned}$$

Thus $\eta(v) \in H^4(\Omega X_2; Z_2)$ is not primitive.

Consider $\Omega X_1 = K(Z, 2) \times K(Z, 4)$. $\eta(\iota_2^2)$ and $\eta(\iota_4)$ are the two generators of $H^4(\Omega X_1; Z_2)$. Note that $\Delta_1(\eta(\iota_4)) = \eta(\iota_4) \otimes 1 + 1 \otimes \eta(\iota_4)$, and $\Delta_1(\eta(\iota_2^2)) = [\eta(\iota_2) \otimes 1 + 1 \otimes \eta(\iota_2)]^2 = \eta(\iota_2)^2 \otimes 1 + 1 \otimes \eta(\iota_2)^2$. Thus every element of $H^4(\Omega X_1; Z_2)$ is primitive and $\Delta_1 \neq \Delta_2$.

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