NOTE ON A THEOREM OF SUGAWARA

By George W. Whitehead

1. Sugawara [8] has proved that, if X is an H-space, then there is a fibre map $p:E \to B$ whose fibre F is of the same weak homotopy type[1] as X and is contractible in E, while E and B have the same weak homotopy type, respectively, as the join $X * X$ and the suspension $S(X)$.[2] A similar theorem, using the notion of quasi-fibration, has been proved by Dold and Lashof [3].

The object of this note is to point out a simple proof of the semi-simplicial analogue of Sugawara's theorem, from which the latter then follows by standard techniques using Milnor's notion [5] of realization. Specifically, we shall prove:

Theorem 1. *If X is an H-complex, then there is a twisted cartesian product $(X, E(X), T(X))$ such that X is contractible in $T(X)$.*

Theorem 2. *If X is a regular H-complex, then the projection $\pi:T(X) \to E(X)$ is a semi-simplicial fibre map. The realizations $|T(X)|$ and $|E(X)|$ have the same weak homotopy type (same homotopy type if X is countable) as $|X| * |X|$ and $S(|X|)$ respectively.*

Corollary 1. *If X is a connected minimal H-complex, then the conclusions of Theorem 2 hold; in addition π is a minimal fibre map.*

Corollary 2. *If X is a 0-connected H-space all of whose homotopy groups are countable, then there is a fibre bundle $(E, S(X), p)$ whose fibre $\tilde{X}$ has the same weak homotopy type as X and is contractible in T.*

In general, the terminology used here will follow that of [7] (cf. also [1, 2], as well as the forthcoming book by D. M. Kan). The author is indebted to P. J. Hilton and J. C. Moore for many stimulating discussions.

2. By "complex" we shall mean a semi-simplicial complex X with base point e; e is a vertex of X, and we let $e_q = s_0^q e \in X_q$. Let $i_1, i_2:X \to X \times X$ be the semi-simplicial maps defined by

$$i_1(x) = (x, e_q), \qquad i_2(x) = (e_q, x) \qquad\qquad (x \in X_q).$$

An *H-complex* is a complex (X, e), together with a map $\mu:X \times X \to X$ such that $\mu i_1 = \mu i_2 : X \subset X$; equivalently, (X, e) is an H-complex if and only if each X_q has a multiplication with identity e_q such that all face and degeneracy operators are homomorphisms. The H-complex X will be called $\begin{Bmatrix} \textit{left} \\ \textit{right} \end{Bmatrix}$*-regular* if and only if, for each $a \in X_q$, the map $\begin{Bmatrix} x \to ax \\ x \to xa \end{Bmatrix}$ is a one-to-one map of X_q onto X_q.

[1] Two spaces X and Y have the same weak homotopy type if and only if there is a space Z and maps $f:Z \to X$, $g:Z \to Y$ such that f_* and g_* induce isomorphisms of the homotopy groups in all dimensions.

[2] This is a slight distortion of Sugawara's theorem, but it is easily deduced from it.

Furthermore, X is called *regular* if and only if X is both left- and right-regular. *Every left-regular H-complex is a Kan complex*; in fact Moore's proof [2, 7] of the corresponding fact for group complexes holds without essential change.

Let (X, e) be an H-complex. Imitating the W-construction [7] associated with a monoid complex, we define, for each $q > 0$, a set W_q by

$$W_q = X_q \times \cdots \times X_0 .$$

Furthermore, let $\partial_i : W_q \to W_{q-1}$, $s_i : W_q \to W_{q+1}$ $(i = 0, \cdots, q)$ be the maps defined by

$$\partial_i(x_q, \cdots, x_0) = (\partial_i x_q, \cdots, \partial_1 x_{q-1+1}, (\partial_0 x_{q-i}) \cdot x_{q-i-1}, x_{q-i-2}, \cdots, x_0)$$

$$s_i(x_q, \cdots, x_0) = (s_i x_q, \cdots, s_0 x_{q-i}, e_{q-i}, x_{q-i-1}, \cdots, x_0).$$

We verify that all the semi-simplicial identities are satisfied, except that

$$\partial_i \partial_{i+1}(x_q, \cdots, x_0)$$
$$= (\partial_i^2 x_q, \cdots, \partial_1^2 x_{q-i+1}, (\partial_0^2 x_{q-i}) \cdot ((\partial_0 x_{q-i-1}) \cdot x_{q-i-2}), x_{q-i-3}, \cdots, x_0),$$

$$\partial_i^2(x_q, \cdots, x_0)$$
$$= (\partial_i^2 x_q, \cdots, \partial_1^2 x_{q-i+1}, ((\partial_0^2 x_{q-i}) \cdot (\partial_0 x_{q-i-1})) \cdot x_{q-1-2}, x_{q-i-3}, \cdots, x_0).$$

Define $s : W_q \to W_{q+1}$ by

$$s(x_q, \cdots, x_0) = (e_{q+1}, x_q, \cdots, x_0)$$

Then

$$\partial_0 s = \text{identity}, \qquad \partial_{i+1} s = s \partial_i \quad \text{for} \quad i \geqq 0,$$

$$s_0 s = s^2, \qquad s_{i+1} s = s s_i \quad \text{for} \quad i \geqq 0.$$

Define $i : X_q \to W_q$ by

$$i(x) = (x, e_{q-1}, \cdots, e_0);$$

then

$$i \partial_j = \partial_j i, \qquad i s_j = s_j i \quad \text{for all } j.$$

Thus W fails to be a contractible semi-simplicial complex containing X because of the non-associativity of the multiplication in X.

Let $T_q = \{(x_q, \cdots, x_0) \in W_q \mid x_i = e_i \text{ for at most one } i \text{ with } 0 \leqq i < q\}$. Then $\partial_i T_q \subset T_{q-1}$, $s_i T_q \subset T_{q+1}$, and $\partial_i \partial_{i+1} \mid T_q = \partial_i^2 \mid T_q$. Moreover $si(X_q) \subset T_{q+1}$. Hence T is a semi-simplicial complex containing X, and X is contractible in T.

Let $E(X)$ be the suspension of X in the sense of Milnor [7], and define $\pi : T \to E(X)$ by

$$\pi(x_q, e_{q-1}, \cdots, e_{i+1}, x_i, e_{i-1}, \cdots, e_0) = s_0^{q-i-1} E x_i ;$$

then π is a semi-simplicial map; in fact, if $x_j = e_j$ for all j such that $i \neq j < q$, then

$$\partial_j \pi(x_q, \cdots, x_0) = \pi \partial_j (x_q, \cdots, x_0)$$
$$= \begin{cases} s_0^{q-i-1} E \partial_{j-q+i} x_i & (j \geqq q-i), \\ s_0^{q-i-2} E x_i & (i < q-1, j \leqq q-i-1), \\ b_{q-1} & (j = 0, i = q-1); \end{cases}$$

$$s_j \pi(x_q, \cdots, x_0) = \pi s_j (x_q, \cdots, x_0)$$
$$= \begin{cases} s_0^{q-i-1} E s_{j-q+i} x_i & (j \geqq q-i), \\ s_0^{q-i} E x_i & (j < q-i). \end{cases}$$

Define $f: X \times E(X) \to T$ by

$$f(x, s_0^i E y) = (x, e_{q-1}, \cdots, e_{q-i}, y, e_{q-i-2}, \cdots, e_0) \qquad (x \in X_q, y \in X_{q-i-1}).$$

Then $\partial_i f = f \partial_i$ for $i > 0$, $s_i f = f s_i$ for $i \geqq 0$, while

$$\partial_0 f(x, s_0^i E y) = \begin{cases} (\partial_0 x, e_{q-2}, \cdots, e_{q-i}, y, e_{q-i-2}, \cdots, e_0) & (i > 0) \\ ((\partial_0 x) y, e_{q-2}, \cdots, e_0) & (i = 0) \end{cases}$$

$$f \partial_0 (x, s_0^i E y) = \begin{cases} (\partial_0 x, e_{q-2}, \cdots, e_{q-i}, y, e_{q-i-2}, \cdots, e_0) & (i > 0) \\ (\partial_0 x, e_{q-2}, \cdots, e_0) & (i = 0). \end{cases}$$

Hence $\pi f \partial_0 (x, s_0^i E y) = \partial_0 (s_0^i E y)$, so that $(X, E(X), T)$ is a twisted cartesian product. This proves Theorem 1.

We remark that we may use the map f to identify T_q with $X_q \times E_q(X)$; then the formulas for the face and degeneracy operators become

$$\partial_0 (x, s_0^i E y) = \begin{cases} (\partial_0 x, \partial_0 s_0^i E y) & \text{if } i > 0, \\ ((\partial_0 x) y, b_q) & \text{if } i = 0; \end{cases}$$

$$\partial_j (x, u) = (\partial_j x, \partial_j u) \qquad \text{if } j > 0;$$

$$s_j (x, u) = (s_j x, s_j u) \qquad \text{if } j \geqq 0,$$

while π becomes the projection on the second factor.

We now prove Theorem 2 by showing that π is a semi-simplicial fiber map. Let $z = s_0^l E y \in E_{q+1}(X)$, so that $y \in X_{q-l}$, and let $x_i \in X_q$ ($i \neq k$, $0 \leqq i \leqq q+1$) such that, if $t_i = (x_i, \partial_i z)$, then $\partial_i t_j = \partial_{j-1} t_i$ for all i, j such that $i < j$, $i \neq k \neq j$. We must prove the existence of $t \in T_{q+1}$ such that $\partial_i t = t_i$ for $i \neq k$, $\pi(t) = z$; t must have the form (x, z) for some $x \in X_{q+1}$.

Case I. ($l = 0, k > 0$): The above conditions on the $\partial_i t_j$ become

$$(\partial_0 x_i)(\partial_{i-1} y) = \partial_{i-1} x_0 \qquad (0 < i \neq k),$$
$$\partial_i x_j = \partial_{j-1} x_i \quad (0 < i < j, i \neq k \neq j).$$

Let x_0' be the element of X_q such that $x_0'y = x_0$. Then if $0 < i \neq k$,

$$(\partial_{i-1}x_0')(\partial_{i-1}y) = \partial_{i-1}(x_0'y) = \partial_{i-1}x_0 = (\partial_0 x_i)(\partial_{i-1}y)$$

and therefore

$$\partial_{i-1}x_0' = \partial_0 x_i .$$

Hence $x_0', x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{q+1}$ satisfy the Kan consistency conditions, and therefore there is an $x \in X_{q+1}$ such that

$$\partial_0 x = x_0', \qquad \partial_i x = x_i \qquad (0 < i \neq k).$$

Then

$$\partial_0(x, z) = ((\partial_0 x)y, b_q) = (x_0'y, b_q) = (x_0, b_q) = t_0$$

$$\partial_i(x, z) = (\partial_i x, \partial_i z) = (x_i, \partial_i z) = t_i \qquad (0 < i \neq k)$$

and therefore $t = (x, z)$ is the desired element.

CASE II. ($l = 1, k > 0$): Our conditions become

$$(\partial_0 x_0)y = (\partial_0 x_1)y,$$

$$\partial_0 x_i = \partial_{i-1}x_0 \qquad (1 < i \neq k),$$

$$\partial_i x_j = \partial_{j-1}x_i \qquad (0 < i < j, i \neq k \neq j).$$

Hence $\partial_0 x_0 = \partial_0 x_1$ and therefore the x_i satisfy the Kan consistency conditions. Therefore we can choose $x \in X_{q+1}$ such that $\partial_i x = x_i$ for $i \neq k$, and it again follows that $\partial_i(x, z) = t_i$ for $i \neq k$.

The remaining cases ($l = k = 0; l = 1, k = 0; l > 1$) are trivial, since, for all simplexes involved, T behaves like an ordinary (non-twisted) cartesian product.

We complete the proof of Theorem 2 by examining the realization $| T |$ of T [5]. Let C_q be the subset of W_q consisting of all sequences of the form $(e_q, \cdots, e_{i+1}, x, e_{i-1}, \cdots, e_0)$; then C is a subcomplex of T containing X; clearly $s(C) \subset C$, so that C is contractible. In fact, it is easily verified that $| C |$ is homeomorphic with the (reduced) cone[3] $C(| X |)$ over $| X |$ under a map which sends the point

$$<| x, t |, s> \in C(| X |) \quad (x \in X_q, t \in \Delta^q, s \in I)$$

into the point

$$|(e_{q+1}, x, e_{q-1}, \cdots, e_0), t'| \in | C |,$$

where t has barycentric coordinates $(t_0, \cdots, t_q)$, and $t' \in \Delta^{q+1}$ has barycentric coordinates $(1 - s, st_0, \cdots, st_q)$.

Define $g: X \times C \to T$ by

$$g(x; x_q, \cdots, x_0) = (xx_q, x_{q-1}, \cdots, x_0);$$

[3] The reduced cone $C(Y)$ over a space (Y, e) is the quotient space $Y \times I/(e \times I \cup Y \times 0)$; the image of the point (y, t) in $C(Y)$ is $<y, t>$.

g is a semi-simplicial map of $X \times C$ onto T and $g^{-1}(C) = X \vee C \cup X \times X$, while g maps simplexes of $X \times C$ not belonging to $X \vee C \cup X \times X$ in a one-to-one way onto the simplexes of T not belonging to C. Hence g induces an isomorphism

$$\bar{g}: X \times C/X \vee C \cup X \times X \approx T/C.$$

Let $X \# C$ be the reduced join $X \times C/X \vee C$; clearly the image of $X \times X$ in $X \# C$ is $X \# X$, and we have

$$X \times C/X \vee C \cup X \times X \approx X \# C/X \# X.$$

But $X \# C/X \# X$ is easily seen to be isomorphic with $X \# (C/X)$. On the other hand, the restriction p' to C induces an isomorphism of C/X onto $E(X)$. Thus

$$T/C \approx X \# E(X).$$

Now $|C|$ is contractible, and $|T/C| = |T|/|C|$; hence $|T|$ has the same homotopy type as $|T/C|$. On the other hand, the spaces $|X \# E(X)|$ and $|X| \# |E(X)|$ have the same weak homotopy type. Finally, $|E(X)| = S(|X|)$ and therefore $|X| \# |E(X)| = |X| \# S(|X|) = S(|X| \# |X|)$, and the latter space has the same weak homotopy type as $|X| * |X|$. Hence $|T|$ has the same weak homotopy type as $|X| * |X|$. This completes the proof of Theorem 2.

Corollary 1, except for the minimality of π, follows from Lemma 2, below. The easy proof that π is minimal if X is minimal is left to the reader.

To prove Corollary 2, let M be a minimal subcomplex of the total singular complex $S(X)$ of X such that the base-point e of X is a vertex of M. Since X is an H-space, $S(X)$ is an H-complex; since M is a deformation retract of $S(X)$, M is an H-complex. Let $\tilde{X} = |M|$; then $\pi: T(M) \to E(M)$ is a minimal fibre map; by Proposition 2.2 of [1], $(T(M), E(M), \pi)$ is a semi-simplicial fibre bundle. According to an unpublished theorem of M. G. Barratt, $E(M)$ has a simplicial subdivision K; since the homotopy groups of X are countable, M and K are countable. By Corollary 5.6 of [1], the induced bundle over the star of every vertex of K is a product bundle. Passing to the realizations, we deduce that the restriction of $|\pi|$ to the star of each vertex of $|K|$ is the projection of a product bundle. Hence $(|T(M)|, |K|, |\pi|)$ is a fibre bundle. (Note: the countability hypothesis on X was needed to ensure that the realization of the product is the product of the realizations).

3. Some remarks on minimal complexes

We shall need some facts about obstruction theory in Kan complexes; for details the reader is referred to the forthcoming book by D. Kan.

Let X be a Kan complex; for simplicity we assume that X has only one vertex. Two simplexes $x, y \in X_n$ are called *compatible* if and only if $\partial_i x = \partial_i y$ for all i. With each ordered pair x, y of compatible n-simplexes there is associated a *separation element* $d^n(x, y) \in \pi_n(X)$. The separation element has the following properties:

1) $d^n(x, x) = 0$;
2) $d^n(x, y) + d^n(y, z) = d^n(x, z)$;
3) given $x \in X_n$, $\alpha \in \pi_n(X)$, there exists $y \in X_n$ such that $d^n(x, y) = \alpha$;
4) if $x, y \in X_{n+1}$ are such that $\partial_i\partial_j x = \partial_i\partial_j y$ for all i, j and if X is n-simple, then $\sum_{i=0}^{n+1} (-1)^i d^n(\partial_i x, \partial_i y) = 0$;
5) if X is minimal, $d^n(x, y) = 0$, then $x = y$.

LEMMA 1. *Let X be a Kan H-complex with only one vertex. Let $a, x_1, x_2 \in X_n$ and suppose x_1 and x_2 are compatible. Then $d^n(ax_1, ax_2) = d^n(x_1, x_2)$.*

PROOF. We first show that, if $0 \leqq i \leqq n - 1$, then $d^n((s_i\partial_i a)x_1, (s_i\partial_i a)x_2) = d^n(ax_1, ax_2)$. Let $u_k = (s_{i+1}a)(s_i x_k)$ $(k = 1, 2)$. Hence

$$\partial_j u_k = \begin{cases} (s_i\partial_j a)(s_{i-1}\partial_j x_k) & (j < i), \\ (s_i\partial_i a)x_k & (j = i), \\ ax_k & (j = i + 1), \\ a(s_i\partial_{i+1}x_k) & (j = i + 2), \\ (s_{i+1}\partial_{j-1}a)(s_i\partial_{j-1}x_k) & (j > i + 2). \end{cases}$$

It follows that $\partial_j u_1 = \partial_j u_2$ unless $j = i$ or $j = i + 1$, while

$\partial_i u_1$ and $\partial_i u_2$ are compatible,

$\partial_{i+1}u_1$ and $\partial_{i+1}u_2$ are compatible.

Hence u_1, u_2 satisfy the conditions of 4) and therefore[4]

$$\begin{aligned} 0 &= \sum_{j=0}^{n+1} (-1)^j d^n(\partial_j u_1, \partial_j u_2) \\ &= (-1)^i\{d^n((s_i\partial_i a)x_1, (s_i\partial_i a)x_2) - d^n(ax_1, ax_2)\}. \end{aligned}$$

Applying the above result for $i = n - 1, n - 2, \cdots, 0$, we see that

$$d^n(ax_1, ax_2) = d^n(a'x_1, a'x_2),$$

where

$$\begin{aligned} a' &= s_0\partial_0 s_1\partial_1 \cdots s_{n-1}\partial_{n-1}a \\ &= s_0^n\partial_0\partial_1\cdots\partial_{n-1}a \in s_0^n X_0 ; \end{aligned}$$

Since X has only one vertex, $a' = e_n$, and our conclusion follows.

[4] If X is an H-complex with multiplication μ then $|\mu|$ induces a function μ' on $|X| \times |X|$ to $|X|$ whose restriction to every compact set is continuous and such that $\mu' i_1 = \mu' i_2 : |X| \subset |X|$. In such a space all Whitehead products $[\alpha, \beta]$ vanish, and, in particular, $|X|$ is n-simple for every n. Hence if X is also a Kan complex, X is n-simple for every n.

LEMMA 2. *Every connected minimal H-complex is regular.*

PROOF. We shall prove that X is left-regular; the proof that X is right-regular is similar. Let $a \in X_n$; we prove, by induction on n, that left multiplication by a is a one-to-one mapping to X_n onto itself. This is trivial if $n = 0$, for then $a = e_0$. Assume the result in dimensions less than n.

Let $x, y \in X_n$ such that $ax = ay$. Then $(\partial_i a)(\partial_i x) = \partial_i(ax) = \partial_i(ay) = (\partial_i a)(\partial_i y)$; by the induction hypothesis $\partial_i x = \partial_i y$ for all i, and therefore x, y are compatible. But $ax = ay$ and therefore $d^n(ax, ay) = 0$; by Lemma 1, $d^n(x, y) = 0$; by property 5), above, $x = y$.

Let $y \in X_n$. By the induction hypothesis, there exist $x_i \in X_{n-1}$ $(1 \leqq i \leqq n)$ such that $(\partial_i a)x_i = \partial_i y$. Then if $i < j$, we have

$$(\partial_i\partial_j a)(\partial_i x_j) = \partial_i((\partial_j a)x_j) = \partial_i\partial_j y,$$

$$(\partial_i\partial_j a)(\partial_{j-1} x_i) = (\partial_{j-1}\partial_i a)(\partial_{j-1} x_i) = \partial_{j-1}((\partial_i a)x_i) = \partial_{j-1}\partial_i y = \partial_i\partial_j y,$$

and hence

$$(\partial_i\partial_j a)(\partial_i x_j) = (\partial_i\partial_j a)(\partial_{j-1} x_i);$$

by the induction hypothesis, $\partial_i x_j = \partial_{j-1} x_i$. By the Kan condition there exists $x' \in X_n$ such that $\partial_i x' = x_i$ for $1 \leqq i \leqq n$. Then, if $i > 0$,

$$\partial_i(ax') = (\partial_i a)(\partial_i x') = (\partial_i a)x_i = \partial_i y;$$

since X is minimal, $\partial_0(ax') = \partial_0 y$; thus ax' and y are compatible, and $\alpha = d^n(ax', y)$ is defined. Choose $z \in X_n$ such that $d^n(z, e_n) = -\alpha$; then $d^n(x'z, x') = -\alpha$. Let $x = x'z$; then

$$\begin{aligned} d^n(ax, y) &= d^n(ax, ax') + d^n(ax', y) \\ &= d^n(x, x') + \alpha = 0; \end{aligned}$$

since X is minimal, $ax = y$. This completes the proof of Lemma 2.

Appendix

Let $\mathcal{W}_0'$ be the class of spaces with base points which have the same homotopy type as a countable CW-complex [6]; all spaces considered here will be members of $\mathcal{W}_0'$. It follows from the results of [6] that all the constructions made below will not take us outside the class $\mathcal{W}_0'$.

THEOREM. *The following conditions on a connected space X are equivalent:*

(1) *X is an H-space;*

(2) *there are spaces Y and Z such that $X \times \Omega Y$ has the same homotopy type as ΩZ;*

(3) *there is a fibration $X' \xrightarrow{i} Y \xrightarrow{p} Z$, where X' has the same homotopy type as X and i is null-homotopic;*

(4) *X is dominated by ΩSX.*

PROOF. We first recall the known fact: *every space dominated by an H-space is an H-space.* In fact, let X be an H-space and let $f:X \to Y$, $g:Y \to X$ be maps such that $fg \simeq 1$. Let $i_X:X \vee X \subset X \times X$,

$$\begin{array}{ccccc} X \vee X & \xrightarrow{i_X} & X \times X & \xrightarrow{\mu} & X \\ f \vee f \downarrow \uparrow g \vee g & & f \times f \downarrow \uparrow g \times g & & f \downarrow \uparrow g \\ Y \vee Y & \xrightarrow{i_Y} & Y \times Y & & Y \end{array}$$

$\phi_X:X \vee X \to X$ the natural map ($\phi(x_0, x) = \phi(x, x_0) = x$), and let $\mu:X \times X \to X$ be the multiplication in X, so that $\mu i_X \simeq \phi_X$. Define $\mu':Y \times Y \to Y$ by $\mu' = f\mu(g \times g)$; then $\mu' i_Y = f\mu(g \times g)i_Y = f\mu i_X(g \vee g) \simeq f\phi_X(g \vee g) = fg\phi_Y \simeq \phi_Y$, and therefore Y is an H-space.

It follows that (2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1). That (1) $\Rightarrow$ (3) is Sugawara's theorem. That (1) $\Rightarrow$ (4) is due to James [4]. It remains to prove that (3) $\Rightarrow$ (2).

We may safely assume that $X = X'$. Let $h:X \times I \to Y$ be a null-homotopy of i; then $ph:X \times I \to Z$ defines a map $h':X \to \Omega Z$. The map $p:Y \to Z$ induces a map $\Omega p:\Omega Y \to \Omega Z$. Let $\mu:\Omega Z \times \Omega Z \to \Omega Z$ be the usual multiplication of loops. Define $f:X \times \Omega Y \to \Omega Z$ by

$$f = \mu(h' \times \Omega p).$$

We claim that f is a homotopy equivalence.

We may assume that Y and Z are 0-connected. Let $\tilde{Y}$, $\tilde{Z}$ be the universal covering spaces[5] of Y, Z. Then p induces a map $\tilde{p}:\tilde{Y} \to \tilde{Z}$ such that the square in the diagram

$$\begin{array}{ccccc} & & \tilde{Y} & \xrightarrow{\tilde{p}} & \tilde{Z} \\ & \overset{\tilde{\imath}}{\nearrow} & \downarrow \pi_Y & & \downarrow \pi_Z \\ X & \xrightarrow[i]{} & Y & \xrightarrow[p]{} & Z \end{array}$$

is commutative. Since i is null-homotopic, there is a map $\tilde{\imath}:\tilde{X} \to \tilde{Y}$ such that $\pi_Y \tilde{\imath} = i$, and $\tilde{\imath}$ is homotopic to a map of X into the (discrete) fibre of π_Y; since X is connected, $\tilde{\imath}$ is null-homotopic. It is easily verified that $X \xrightarrow{\tilde{\imath}} \tilde{Y} \xrightarrow{\tilde{p}} \tilde{Z}$ is a fibration. Hence we may assume that Y and Z are 1-connected; it follows that ΩY and ΩZ are 0-connected.

Since all the spaces involved belong to $\mathcal{W}_0'$ it suffices to prove that f_* maps the homotopy groups of $X \times \Omega Y$ isomorphically onto those of ΩZ. This follows immediately from the known direct sum decomposition

$$\pi_{i+1}(Z) \approx \pi_{i+1}(Y) \oplus \pi_i(X)$$

[5] Let $X \in \mathcal{W}_0'$ and let W be a countable CW-complex, $f:X \to W$ a homotopy equivalence. Let $p:\tilde{W} \to W$ be the universal covering; it follows easily that the fibration $p':\tilde{X} \to X$ induced by f is the universal covering of X.

and commutativity of the diagram

$$\begin{array}{ccccc}
\pi_i(X) & \longrightarrow & \pi_{i+1}(Z) & \xleftarrow{\;p_*\;} & \pi_{i+1}(Y) \\
 & {\scriptstyle h'_*}\searrow & \big\downarrow \rho & & \big\downarrow \rho \\
 & & \pi_i(\Omega Z) & \xleftarrow[(\Omega p)_*]{} & \pi_i(\Omega Y).
\end{array}$$

The details are left to the reader.

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