TURNING POINTS IN LINEAR ASYMPTOTIC THEORY

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1. Introduction

In the theory of linear ordinary differential equations important chapters are appropriately devoted to the dependence of such equations upon parameters. A matter of concern, then, is the asymptotic form of the equation's solutions relative to the parameter when the latter is numerically large (or small, in which case the reciprocal can be considered); strictly, the limiting forms of the solutions as the parameter becomes infinite. The general form of the equation with which this paper will deal is

$$\frac{d^n u}{dz^n} + \lambda P^{(n-1)}(z,\lambda)\,\frac{d^{n-1}u}{dz^{n-1}} + \cdots + \lambda^n P^{(0)}(z,\lambda)\,u = 0. \tag{1}$$

In terms of the operator

$$\mathfrak{L} \equiv \frac{d^n}{dz^n} + \lambda P^{(n-1)}(z,\lambda)\,\frac{d^{n-1}}{dz^{n-1}} + \cdots + \lambda^n P^{(0)}(z,\lambda), \tag{2}$$

this is $\mathfrak{L}(u) = 0$. The parameter is λ, and the coefficients $P^{(\nu)}(z, \lambda)$ are power series (possibly only polynomials or mere single terms) in $1/\lambda$, thus

$$P^{(\nu)}(z,\lambda) = \sum_{\mu=0}^{\infty} \frac{p_\mu^{(\nu)}(z)}{\lambda^\mu}, \qquad \nu = 0, 1, \cdots, (n-1). \tag{3}$$

A differential equation depends for its essential character upon its coefficients, and these—insofar as they are not constants—depend upon the domain of the variable z. A study of an equation (1) is therefore defined only when the z-domain has been specified. In different domains the equation may well have quite different solution forms. This matter will remain at the fore in the following discussion. As an over-all specification, however, the z-domain will in every instance be taken to be a closed bounded region of the complex plane in which the functions $p_\mu^{(\nu)}(z)$ of the series (3) are all analytic.

By a familiar change of the dependent variable, the coefficient $P^{(n-1)}(z, \lambda)$ of an equation (1) may be reduced to zero. That normalizes the equation, and makes the variable u specific. We shall generally suppose this normalization to have been made in order to preclude ambiguities in the formulas.

The method by which the asymptotic solution forms of an equation (1) are ordinarily determined depends in the first instance upon a derivation of *formal solutions*, namely of expressions which fulfill the differential equation term by term, but which may not be actual solutions because they happen to diverge. Truncations of these formal solutions are free of the stigma of divergence, but in their turn do not fulfill the given equation in more than an approximate sense. To achieve the original purpose it remains, therefore, to give rigorous proof that the formal solution truncations do represent true solutions asymptotically. That can generally be done, each result, however, often having validity only for suitably delimited sub-domains of λ and z.

In this paper only the formal part of this program is to be considered. The limitation of space is one, but not the primary, reason for this. For the formalism must be pursued inventively and must be adapted to different classes of differential equations (1) which are defined by it, whereas the lines of the concluding rigorous analysis have been pretty well laid down.

2. The classical algorithm

For purposes of subsequent comparison and discussion, there is point to setting forth briefly the algorithm for formally solving an equation (1) which is now of sufficient venerability to be appropriately designated as "classical." This takes its cue from the elementary case in which the coefficients $P^{(\nu)}$ are mere constants, and accordingly bases itself upon an exponential expression.

$$\zeta = e^{\lambda \int \theta(z) dz} A(z, \lambda), \tag{4}$$

in which $\theta(z)$ and $A(z, \lambda)$ are to be determined.* It is found that $\theta(z)$ must be a root of the *auxiliary equation*

$$\theta^n + p_0^{(n-2)}(z)\theta^{n-2} + \cdots + p_0^{(0)}(z) = 0. \tag{5}$$

To illustrate this as well as the subsequent procedure, we turn to the case $n = 2$, in which a minimum of complications need be confronted.

With the equation (1) specialized to

$$\frac{d^2u}{dz^2} + \lambda^2 P(z, \lambda)\, u = 0, \tag{1a}$$

it is found from (4) that

$$\mathcal{L}(\zeta) = \lambda^2 e^{\lambda \int \theta(z) dz} \left\{ (\theta^2 + P)\, A + \frac{1}{\lambda}\, (2\theta A' + \theta' A) + \frac{1}{\lambda^2} A'' \right\}, \tag{6}$$

the accents denoting differentiations with respect to z. The expression within the brace of this is representable as a formal power series in $1/\lambda$. The leading term drops from this series if $\theta(z)$ is chosen to be either root $\theta_j(z)$, $j = 1, 2$, of the auxiliary equation

$$\theta^2 + p_0(z) = 0. \tag{5a}$$

With such a choice of θ, the term in $1/\lambda$ drops from the brace in (6) if $a_0(z)$

* Throughout the paper an expression denoted by a capital letter, with λ and possibly other variables as arguments, shall be understood, in every case, to be formally a power series in $1/\lambda$ with coefficients that are analytic in the other variables. These coefficients will then be designated by the respective lower case letter with attached subscript, in the manner of the formulas (3).

is chosen so that

$$p_1(z)a_0 + 2\theta(z)a_0' + \theta'(z)a_0 = 0,$$

namely, if with $\theta_j(z)$ we associate

$$a_0^{(j)}(z) = \theta_j^{-\frac{1}{2}} \exp\left\{-\int \frac{p_1}{2\theta_j}\, dz\right\}. \tag{7}$$

The term in $(1/\lambda)^{\mu+1}$ drops from the brace, successively for $\mu = 1, 2, 3, \cdots$, if $a_\mu(z)$ is chosen so that

$$\sum_{\nu=0}^{\mu} p_{\mu+1-\nu}(z)a_\nu + 2\theta(z)a_\mu' + \theta'(z)a_\mu + a_{\mu-1}''(z) = 0,$$

namely, if with $\theta_j(z)$ we associate

$$a_\mu^{(j)}(z) = -a_0^{(j)}(z) \int \left\{\frac{a_{\mu-1}''(z) + \sum_{\nu=0}^{\mu-1} p_{\mu+1-\nu}\, a_\nu}{2a_0^{(j)}\theta_j}\right\} dz. \tag{8}$$

For the differential equation (1a) the expressions

$$\zeta^{(j)} = e^{\lambda \int \theta_j(z)dz} A^{(j)}(z, \lambda), \quad j = 1, 2, \tag{9}$$

so obtained are formal solutions in the sense described in §1.

3. Critique

The algorithm that has thus been set forth yields a complete set of formal solutions if the z-region in which the differential equation is being considered is one in which the roots of the axuxiliary equation (5) are all simple, namely one in which these auxiliary roots remain distinct. If root multiplicities are present —namely multiplicities that maintain identically over the z-region—the resulting set is incomplete. Modifications of the algorithm to adapt it to that contingency are, however, known [1].

The facts shape up differently when in the z-region two or more auxiliary roots which are distinct elsewhere fall into a coincidence at an isolated point. Such a point is called a *turning point*. The equation (1a), for instance, has a turning point at any zero of the function $p_0(z)$, since the roots of the equation (5a) coincide there at the value 0. That the algorithm of §2 fails in the presence of a turning point is clearly shown by the formulas (7) and (8), for they evidently do not define analytic functions where $\theta_j(z)$ can vanish.

Even the most casual observations give evidence that a turning point is apt to be critical in asymptotic theory. Because about such a point at least some of the solutions of the differential equation undergo radical changes in functional character. For example, the solutions of the equation

$$u'' + \lambda^2 p(z)u = 0,$$

with λ, z and $p(z)$ real, are respectively of oscillatory and exponential types where $p(z)$ is positive and negative, and make the transition between these types as the turning point is traversed. In the instances of less simple equations, and in the broader domain of the complex z, a turning point may be expected to mark very intricate metamorphoses of the solution forms.

There is mathematical challenge in this fact. And the challenge has immediacy because many modern theories of applied mathematics depend upon it. In quantum mechanics, in hydrodynamics, in microwave propagation, in diffraction, etc., turning points are encountered.

The problem posed by the presence of a turning point is far from simple. In a primitive way its solution has been sought—and by some is still being sought—by excising the turning point neighborhood and sub-dividing the residual region into parts in each of which a derivation by the algorithm of §2 is feasible. In the turning point neighborhood solutions in the form of power series in λ are obtainable. A patching together of the representations so found for the several subregions is then sought by the familiar process of identifications in overlapping domains.

Insofar as it may, this method must seek its assets in the fact that the patches are obtainable by classical means, and that it can be resorted to when no subtler way is discerned. It must concede as liabilities its laboriousness—when the patching can be accomplished at all—and its mathematical inelegance. It throws little or no light upon the nature of the configurations which characterize the problem. In the following we set forth a method based on a fresh algorithm. This calls for no dismemberment of the z-region, and gives the characteristic configurations of the problem their determinative roles from the start.

4. The turning point algorithm

An equation (1), in a specific z-region having been given, consider in association with it another differential equation

$$(10) \qquad \frac{d^n w}{dx^n} + \lambda^{n-2} Q^{(n-2)}(x, \lambda) \frac{d^{n-2} w}{dx^{n-2}} + \cdots + \lambda^n Q^{(0)}(x, \lambda) w = 0.$$

We defer the precise specification of this equation, contenting ourselves at this point with the particulars that the functions $Q^{(\nu)}(x, \lambda)$ shall be power series in $1/\lambda$ with coefficients $q_\mu^{(\nu)}(x)$ that are analytic in x, and that x itself be some analytic function of z.

With coefficients $A^{(\nu)}(z, \lambda)$ that at this point are undetermined, except that they shall be analytic in z, set

$$(11) \qquad \zeta = A^{(0)}(z, \lambda) w + \frac{1}{\lambda} A^{(1)}(z, \lambda) \frac{dw}{dx} + \cdots + \frac{1}{\lambda^{n-1}} A^{(n-1)}(z, \lambda) \frac{d^{n-1} w}{dx^{n-1}}.$$

This relation is differentiable, and after the differentiation the derivative $d^n w/dx^n$ can be eliminated by the use of the equation (10). By a repetition of this process the successive derivatives of ζ can be expressed in terms of w and its first $(n - 1)$ derivatives, thus

$$\frac{d^j\zeta}{dz^j} = \lambda^j\left[A^{(0,j)}(z,\lambda)w + \frac{1}{\lambda}A^{(1,j)}(z,\lambda)\frac{dw}{dx} + \cdots \right. \tag{12}$$
$$\left. \cdots + \frac{1}{\lambda^{n-1}}A^{(n-1,j)}(z,\lambda)\frac{d^{n-1}w}{dx^{n-1}}\right], \qquad j = 1, 2, \cdots, n,$$

the coefficients $A^{(\nu,j)}(z, \lambda)$ being specific in terms of those of the equation (10) and the formula (11). These yield an evaluation

$$\mathfrak{L}(\zeta) = \lambda^n\left[G^{(0)}(z,\lambda)w + \frac{1}{\lambda}G^{(1)}(z,\lambda)\frac{dw}{dx} + \cdots \right. \tag{13}$$
$$\left. \cdots + \frac{1}{\lambda^{n-1}}G^{(n-1)}(z,\lambda)\frac{d^{n-1}w}{dx^{n-1}}\right],$$

in which each coefficient $G^{(\nu)}(z, \lambda)$ is formally a power series in $1/\lambda$, and is analytic in z.

The objective is now to drop the terms in $(1/\lambda)^\mu$ for successive μ, from the series $G^{(\nu)}(z, \lambda)$, $\nu = 0, 1, 2, \cdots, (n - 1)$, by making appropriate choices of the equation (10), of the function $x(z)$, and of the hitherto unspecified coefficients $a_\mu^{(\nu)}(z)$. The possibility of doing that in the realm of functions that are analytic over the given z-region will depend upon the differential equation (10), as will become clear. With the choices indicated (when such are possible), the formula (11) associates with each solution w of the equation (10) an expression ζ which formally fulfills the equation (1). It will be clear that when the forms of the functions w are known the formula (11) in fact yields formal solutions of the given differential equation.

In review we see, now, that the implementation of this algorithm depends upon the possibility of choosing for the role (10) a differential equation whose solution forms are known, or are in some way determinable, and which, over and above that, conforms to the stipulations of analyticity that were imposed upon the determinations that have to be made.

5. The case $n = 2$

For simplicity in illustrating the actual operation of the algorithm, the appeal to the differential equation (1a), is again natural. With the use of A, B, P and Q, in the place of $A^{(0)}$, $A^{(1)}$, $P^{(0)}$ and $Q^{(0)}$, and with a superscribed dot signifying a differentiation with respect to x, we have then

$$\ddot{w} + \lambda^2 Q(x, \lambda)w = 0, \tag{10a}$$

and

$$\zeta = Aw + \frac{1}{\lambda}B\dot{w}. \tag{11a}$$

This latter relation leads, in the manner described in §4, to the formulas

$$\zeta' = [A' - \lambda BQx']w + \left[Ax' + \frac{1}{\lambda}B'\right]\dot{w},$$

$$(12a)\quad \zeta'' = [A'' - \lambda BQx'' - \lambda B\dot{Q}x'^2 - 2\lambda B'Qx' - \lambda^2 AQx'^2]w + \left[2A'x' + Ax'' - \lambda BQx'^2 + \frac{1}{\lambda}B''\right]\dot{w}.$$

Therefrom follows the evaluation

$$(13a)\quad \begin{aligned} \mathcal{L}(\zeta) \equiv \lambda^2 &\left\{(P - Qx'^2)A - \frac{1}{\lambda}(2Qx'B' + \dot{Q}x'^2B + Qx''B) + \frac{1}{\lambda^2}A''\right\}w \\ &+ \lambda\left\{(P - Qx'^2)B + \frac{1}{\lambda}(2x'A' + x''A) + \frac{1}{\lambda^2}B''\right\}\dot{w}, \end{aligned}$$

wherein each brace encloses an expression that is, in fact, a power series in $1/\lambda$.

The leading terms drop from both of the braced expressions in (13a) if $x(z)$ is chosen so that

$$(14)\quad p_0(z) - q_0(x)x'^2 = 0.$$

In an integrated form this relation can be taken to be

$$(15)\quad \int_0^x q_0^{\frac{1}{2}}(x)\,dx = \int_0^z p_0^{\frac{1}{2}}(z)\,dz,$$

it being supposed that the origin of z is chosen within the z-region, and that the origin of x is assigned to it. At this point the requirement that $x(z)$ be analytic clearly imposes upon $q_0(x)$ the specification that it vanish at every x that corresponds to a zero of $p_0(z)$, and that it vanish there to the same order as $p_0(z)$.

The terms in $1/\lambda$ drop from the braces of (13a) if the equations

$$\begin{aligned} (p_0 - q_0x'^2)a_1 + (p_1 - q_1x'^2)a_0 - (2q_0x'b_0' + \dot{q}_0x'^2b_0 + q_0x''b_0) &= 0, \\ (p_0 - q_0x'^2)b_1 + (p_1 - q_1x'^2)b_0 + (2x'a_0' + x''a_0) &= 0, \end{aligned}$$

are fulfilled. By virtue of the prior determination (14) these equations reduce to

$$(16)\quad \begin{aligned} -2\sqrt{q_0x'}(\sqrt{q_0x'}b_0)' + (p_1 - q_1x'^2)a_0 &= 0, \\ 2\sqrt{x'}(\sqrt{x'}a_0)' + (p_1 - q_1x'^2)b_0 &= 0. \end{aligned}$$

The system is integrable by quadratures. The sum of a_0 times the second and $-b_0$ times the first can be written as

$$(x'a_0^2 + q_0x'b_0^2)' = 0.$$

This is fulfilled by the choice

$$(17)\quad (a_0^2 + q_0b_0^2)x' \equiv 1.$$

With the use of this relation we may eliminate a_0 from the first equation (16) and b_0 from the second one. They thereupon integrate directly to give

$$
(18)\qquad \begin{aligned} b_0(z) &= \left(\frac{x'}{p_0}\right)^{\frac{1}{2}} \sin \int_0^z \frac{p_1 - q_1 x'^2}{2p_0^{\frac{1}{2}}}\, dz, \\ a_0(z) &= x'^{\frac{1}{2}} \cos \int_0^z \frac{p_1 - q_1 x'^2}{2p_0^{\frac{1}{2}}}\, dz. \end{aligned}
$$

The terms in $(1/\lambda)^{\mu+1}$ drop from the braces of (13a) if the equations

$$
(19)\qquad \begin{aligned} -2\sqrt{q_0 x'}\,(\sqrt{q_0 x'}\,b_\mu)' + (p_1 - q_1 x'^2)a_\mu + f_\mu &= 0, \\ 2\sqrt{x'}\,(\sqrt{x'}\,a_\mu)' + (p_1 - q_1 x'^2)b_\mu + \phi_\mu &= 0, \end{aligned}
$$

are fulfilled, the terms f_μ and ϕ_μ being constructed of the functions $a_\rho(z), b_\rho(z)$ with $\rho < \mu$. To integrate this system let the equations (16) and (19) be respectively multiplied by $-a_\mu/\sqrt{q_0}$, $-b_\mu\sqrt{q_0}$, $a_0/\sqrt{q_0}$, $b_0\sqrt{q_0}$ and added, and again let them be respectively multiplied by $-b_\mu$, a_μ, $-b_0$ and a_0 and added. The resulting equations are expressible as

$$
\begin{aligned} 2[\sqrt{q_0 x'}\,b_0\sqrt{x'}\,a_\mu - \sqrt{x'}\,a_0\sqrt{q_0 x'}\,b_\mu]' + \frac{a_0 f_\mu}{\sqrt{q_0}} + b_0\,\phi_\mu\sqrt{q_0} &= 0, \\ 2[\sqrt{x'}\,a_0\sqrt{x'}\,a_\mu + \sqrt{q_0 x'}\,b_0\sqrt{q_0 x'}\,b_\mu]' - b_0 f_\mu + a_0\,\phi_\mu &= 0. \end{aligned}
$$

These can be integrated and then solved for a_μ and b_μ to give

$$
(20)\qquad \begin{aligned} a_\mu(z) &= \frac{a_0(z)}{2}\int_0^z (b_0 f_\mu - a_0\,\phi_\mu)\,dt - \frac{\sqrt{q_0}\,b_0(z)}{2}\int_0^z \left(\frac{a_0 f_\mu}{\sqrt{q_0}} + b_0\,\phi_\mu\sqrt{q_0}\right) dt, \\ b_\mu(z) &= \frac{a_0(z)}{2\sqrt{q_0}}\int_0^z \left(\frac{a_0 f_\mu}{\sqrt{q_0}} + b_0\,\phi_\mu\sqrt{q_0}\right) dt + \frac{b_0(z)}{2}\int_0^z (b_0 f_\mu - a_0\,\phi_\mu)\,dt. \end{aligned}
$$

These formulas are successively applicable for $\mu = 1, 2, 3, \cdots$.

6. Some specific asymptotic theories

It is of no little interest to observe how the formulas of §5 have been found adaptable to important classes of differential equations (1a), the adaptation clearing the way, in each instance, for the construction of an asymptotic theory for the equations of the class in question. One such class, incidentally, is that of the equations having *no* turning-point. The algorithm of §4 is thus shown to be of a wider scope than that of §2, since it applies with no less effect whenever the latter does so. The decisive element, as was remarked in §4, inheres in the choice of an equation for the role (10a). This equation must have known solution forms, and must yield analytic formulas for $x(z)$ and the coefficients $a_\mu(z)$ and $b_\mu(z)$.

Class (i): *Equations* (1a) *in z-regions containing no turning-point.* For these equations the function $p_0(z)$ has no zero in the z-region. The choice $Q(x, \lambda) \equiv 1$, places the solvable equation $\ddot{w} + \lambda^2 w = 0$ into the role (10a), and gives to the relation (15) the form

$$x = \int_0^z p_0^{\frac{1}{2}}(z)\, dz.$$

This determination of $x(z)$ is analytic, and no singularities are seen to be involved in the determinations (18) or (20).

Class (ii): *Equations* (1a) *in regions which include a turning-point that is a simple zero of* $p_0(z)$. Let the origin be taken at the turning-point. The choice $Q(x, \lambda) \equiv x$ gives the role (10a) to the equation $\ddot{w} + \lambda^2 xw = 0$. This is a transformed Bessel equation, whose solutions are

$$x^{\frac{1}{2}}\mathfrak{C}(\tfrac{2}{3}\lambda x^{\frac{3}{2}}),$$

with $\mathfrak{C}$ signifying any cylinder (Bessel) function of the order $\frac{1}{3}$. The forms of these functions are known. The relation (15) can be put into the form

$$x = \left[\frac{3}{2}\int_0^z p_0^{\frac{1}{2}}(z)\, dz\right]^{\frac{2}{3}},$$

and this (with proper definition at $z = 0$) defines $x(z)$ to be analytic. The integrals in the formulas (18) and (20) are improper but convergent. The formulas give analytic determinations if the removable singularities in the expressions for the $b_\mu(z)$, $\mu = 0, 1, 2, \cdots$ are appropriately removed [2], [3].

Class (iii): *Equations* (1a) *in the region about a second order zero of* $p_0(z)$. The choice $Q(x, \lambda) = x^2 + \dfrac{4i}{\lambda}K(\lambda)$, in which K is an unspecified power series in $1/\lambda$ with constant coefficients, assigns the role (10a) to the differential equation

$$\ddot{w} + [\lambda^2x^2 + 4i\lambda K(\lambda)]w = 0.$$

This is a confluent form of the hypergeometric equation whose solutions are of known forms and are commonly symbolized by

$$x^{-\frac{1}{2}}M_{K,\pm\frac{1}{4}}(i\lambda x^2).$$

The relation (15) is effectively

$$x = \left[2\int_0^z p_0^{\frac{1}{2}}(z)\, dz\right]^{\frac{1}{2}},$$

which gives an analytic determination of $x(z)$. The coefficients k_μ of the series $K(\lambda)$ may now, and must, be chosen, in the instance of any given equation (1a), to make the integrals in the formulas (18) and (20) convergent. Thus, when k_0 (which is q_1) is chosen so as to give the function $(p_1 - q_1x'^2)$ a zero at the origin, the integrals in the formula (18) are proper, and the determinations (18) of $a_0(z)$ and $b_0(z)$ are analytic. By the successive appropriate choices of $k_{\mu-1}$, analyticity may be assured to the determinations of $a_\mu(z)$ and $b_\mu(z)$. This theory was given by R. W. McKelvey [4].

Class (iv): *Equations* (1a) *in regions containing two simple turning points, say at* $z = \alpha$ *and* $z = \beta$. The choice $Q(x, \lambda) \equiv c^2(1 - x^2) + \dfrac{1}{\lambda}K(\lambda)$, gives the role (10a) to a transformed Weber equation. With the choice

$$c = \frac{2}{\pi} \int_\alpha^\beta p_0^{\frac{1}{2}}(z)\, dz,$$

the relation (14) gives

$$c \int_{-1}^{x} (1 - x^2)^{\frac{1}{2}}\, dx = \int_\alpha^z p_0^{\frac{1}{2}}(z)\, dz$$

and hereby $x(z)$ is determined to be analytic, even at the points $z = \alpha$ and $z = \beta$. With proper choices of the constants k_μ the determinations (18) and (20) are analytic. This theory was elaborated from [5] by N. D. Kazarinoff [6].

7. The classification of differential equations, and general theory

For a differential equation (1) of the second order, the auxiliary equation (5) has just two roots, and if the equation is normalized these roots can coincide only at the value 0. For such an equation, therefore, the only feature in which turning points can differ is the order to which the (single) auxiliary root difference vanishes. In the classes (i), (ii) and (iii) of §6 that order is respectively 0, $\frac{1}{2}$, and 1.

In the case of a differential equation (1) of higher order, there is a correspondingly greater number of auxiliary roots, and therefore a plurality of root-differences. At a turning-point these latter may vanish to various orders, and the values at which the roots coincide may also be various. Turning-points therefore exist in greater variety, in as many kinds as there are configurations of the auxiliary roots. A salient matter is the number of roots that are involved in coincidences, for some of them may not do so but remain distinct. To focus the attention upon those that do, we shall designate the configuration of them—leaving the simple roots, if there are such, aside—as the *coincidence* pattern of the differential equation at the turning point.

A coincidence pattern involving k roots may present itself in a differential equation of the order k. It may, however, also present itself in an equation of any higher order n. The total root configuration is, of course, simplest in the former case, because the coincidence pattern appears then unaccompanied by additional distinct roots. An asymptotic theory applies to the differential equations (1), of all orders, which have a common coincidence pattern. Different coincidence patterns, on the other hand, distinguish equations whose asymptotic theories must be expected to be distinct. These assertions are, in effect, verified by a known general result, which is the following ([7]):

If the formal solutions are derivable, by any means, for the differential equations of the order k with a certain coincidence pattern involving the k roots, then they can be derived also, by the application of a specific algorithm, for any differential equation, whatever its order, having that same coincidence pattern.

For instance, the formal solutions can be obtained for any differential equation for which just one auxiliary root difference vanishes, and does so to the order

$\frac{1}{2}$ or 1. For its coincidence pattern is then that of the class (ii) or (iii) of §6. This was done for equations of the third order in [3] and by L. R. Bragg [8].

8. Asymptotic theories for equations with coincidence patterns involving more than two roots

An asymptotic theory may be expected to be the more intricate the larger the number of roots that are involved in its determinative coincidence pattern. For the metamorphoses which the differential equation's solutions may undergo at the turning point are then more intricate. As of the present, the asymptotic theories for differential equations (1) having more extensive coincidence patterns comprise a field that has only begun to be cultivated, the vast expanse of which awaits development by further research. The theories that have been developed are few. We shall outline them briefly.

The coincidence pattern upon three auxiliary roots, one root being identically zero and the root differences vanishing to the order $\frac{1}{2}$ is presented in third order differential equations (1) by those in which $p_0^{(2)}(z) \equiv p_0^{(0)}(z) \equiv 0$ and $p_0^{(1)}(z)$ has a simple zero. This latter is the turning-point. The form of the differential equation is

$$u''' + E^{(2)}(z, \lambda)u'' + \lambda^2\{P^{(1)}(z, \lambda)u' + E^{(0)}(z, \lambda)u\} = 0, \tag{21}$$

and of its auxiliary equation

$$\theta^3 + p_0^{(1)}(z)\theta = 0.$$

It was found in this case that the role of the equation (10) could be appropriately assigned to the equation

$$\frac{d^3w}{dx^3} + \lambda^2\left\{x\frac{dw}{dx} + K(\lambda)w\right\} = 0. \tag{22}$$

in which $K(\lambda)$ is an unspecified power series in $1/\lambda$ with constant coefficients [9]. The solutions of this equation were not known, but were completely determinable because of the relative simplicity of the coefficients, [10]. The series $K(\lambda)$ must be adjusted to the differential equation (21) that is given. The extensibility of this theory to differential equations (1) of higher order having the same coincidence pattern follows, of course, by the general result referred to in §7.

In the theory of hydrodynamic stability the so-called Orr-Sommerfeld equation

$$\frac{d^4\psi}{dy^4} - 2\alpha^2\frac{d^2\psi}{dy^2} + \alpha^4\psi - i\alpha R\left\{[\omega(y) - c]\left[\frac{d^2\psi}{dy^2} - \alpha^2\psi\right] - \frac{d^2\omega}{dy^2}\right\} = 0,$$

is prominent, and must be considered on an interval of y which includes a zero of the function $[\omega(y) - c]$. The parameter α is of moderate magnitude, but αR is large. The equation is of the general form

$$u'''' + \lambda^2\{P^{(2)}(z, \lambda)u'' + E^{(1)}(z, \lambda)u' + E^{(0)}(z, \lambda)u\} = 0, \tag{23}$$

in a region which encloses a simple zero of the function $p_0^{(2)}(z)$, namely with a turning-point. This has been the direct or indirect source of incentive for the development of asymptotic theory for the equation (23). The auxiliary equation in the case of the latter is

$$\theta^4 + p_0^{(2)}(z)\theta^2 = 0.$$

The coincidence pattern thus involves four auxiliary roots, with one root difference vanishing identically and the others vanishing to the order $\frac{1}{2}$ at the turning-point.

Asymptotic theory for the equation (23) has been developed, effectively by the method of §4, though still with substantial differences in procedure by C. C. Lin and A. L. Rabenstein and by myself. Of these derivations that of Lin and Rabenstein [11] holds quite scrupulously to the algorithm of §4. It assigns the role (10) to the differential equation

$$w'''' + \lambda^2\{xw'' + K^{(1)}(\lambda)w' + K^{(2)}(\lambda)w\} = 0, \tag{24}$$

in which $K^{(1)}(\lambda)$ and $K^{(2)}(\lambda)$ are both power series in $1/\lambda$, which must be adjusted to the given equation (23). The solution forms of the equation (24) have to be determined, to give explicitness to the formula (11). That was done by the method of Laplace transforms.

My own attack [12], [13], upon the problem of the equation (23) seizes first upon the fact that every such equation admits at least one formal solution which is a power series in $1/\lambda$ with coefficients that are analytic in z. By the use of this fact it is possible to so modify the method as to base the application of the algorithm of §4 upon the differential equation (22) of the third order, whose solution forms are known since they were determined in connection with the prior construction of the theory for the equation (21), rather than upon the equation (24) for which the determination of the solution forms remained as a quite substantial task. An incidental and fortunate feature of this method is that the function determinations that are required by the algortithm are all found to be explicitly possible in terms of quadratures.

9. Summary

It will be noted that in essence the method that has been set forth assigns differential equations (1) to various classes, for each of which the asymptotic theory can be expected to take its own form. The feature which determines the assignment at a turning-point is the coincidence pattern of the auxiliary roots. Each class includes equations of all orders equal to or greater than the number of roots in the pattern.

For the determination of the formal solutions of an equation (1) of a certain class, the algorithm of §4 refers the problem to a particular member of the class, namely the equation (10). This member is of the minimum order for the class and is likely to have optimally simple coefficients. If the solution forms of this equation are available in the mathematical literature, as they are for some important classes whose coincidence patterns involve just two roots, the problem at hand is as good as solved. In the alternative, the solution forms for this equation (10) must be determined. While that, in essence, recalls the original problem, it does so only as that is presented by a specific and optimally simple exemplar.

It is true that the choice of a differential equation for the role of (10) cannot be explicitly prescribed. Ingenuity and insight into the problem have to be enlisted for it. That, however, is not unusual in the creation of mathematical theory.

Asymptotic theories have been constructed for a number of classes of equations (1), as has been shown. To regard the method as of only limited applicability because there remain many classes to which, at this time, it has not been extended, is, in effect, to subscribe the thesis that all possible advances have been made. It seems more sanguine to believe that only further research is needed, and that for such there is promise of rewards.

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